Higher order squeezing of noncommutative q-photon-added coherent states

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Coherent States and their Applications: A Contemporary Panorama, November 14, 2016

S. Dey, V. Hussin; Phys. Rev. A 93, 053824 (2016)

Classical-like quantum states

Coherent states:

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad \text{or} \quad |\alpha\rangle = D(\alpha)|0\rangle, \quad D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

$$\Rightarrow \qquad |\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \qquad \alpha \in \mathbb{C}$$

Resemble classical object:

- Bridges quantum & classical optics
- Refers to a state of quantized electromagnetic field
- Time evolution is concentrated along the classical trajectories

Properties:

- Minimum uncertainty state (like $|0\rangle$): $\Delta x \Delta p = \hbar/2$
- Intelligent state: $\Delta x = \Delta p = \sqrt{\hbar/2}$
- Produces equal amount of noise in optical communication as $|0\rangle$

Nonclassical states

Coherent states in quantum information processing:

$$|\alpha\rangle + |-\alpha\rangle \quad \rightarrow \quad |0_L\rangle$$
$$|\alpha\rangle - |-\alpha\rangle \quad \rightarrow \quad |1_L\rangle$$

Qubit states: $|0_L\rangle,\ |1_L\rangle \Rightarrow$ Nonclassical

Some other nonclassical states constructed from coherent states:

- Squeezed states
- Photon-added coherent states
- Pair coherent states
- Photon-subtracted squeezed vacuum states

Nonclassicality versus entanglement

Glauber-Sudarshan's P representation:

$$\hat{\rho} = \int P(z)|z\rangle\langle z| d^2 z$$

- $P(z) \geq 0 \Rightarrow P(z)$ is classical probability density $\Rightarrow |z\rangle$ is classical
- $P(z) < 0 \;\; \Rightarrow \;\; |z\rangle$ is nonclassical

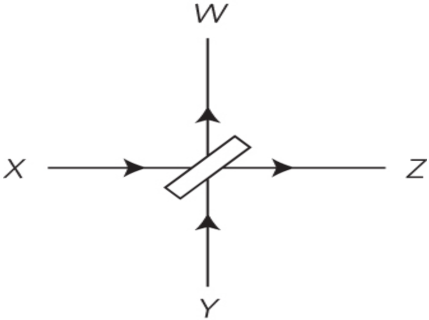
For a bipartite state $|a, b\rangle = |a\rangle \otimes |b\rangle$:

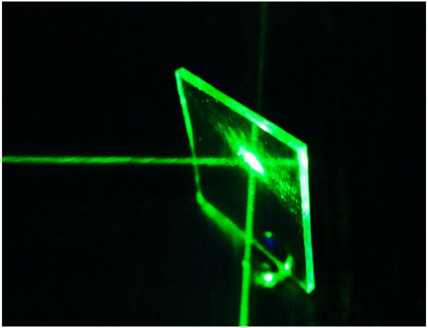
$$\hat{\rho} = \int P(a, b)|a, b\rangle\langle a, b| d^2 a \; d^2 b, \quad |a\rangle \in \mathcal{H}_A, |b\rangle \in \mathcal{H}_B$$

- $P(a, b)$ is a classical probability $\Rightarrow |a, b\rangle$ is separable
- $P(a, b) < 0 \Rightarrow |a, b\rangle$ is entangled

Nonclassicality is prosperous in quantum information theory!

Nonclassicality produces entanglement





Input: $X \rightarrow a, \; Y \rightarrow b,$
Output: $W : \; c \rightarrow \mathcal{B}a\mathcal{B}^{\dagger}, \; Z : \; d \rightarrow \mathcal{B}b\mathcal{B}^{\dagger}, \; [c, c^{\dagger}] = [d, d^{\dagger}] = 1$

$$\mathcal{B} = e^{\frac{\theta}{2}(a^{\dagger}be^{i\phi} - ab^{\dagger}e^{-i\phi})} \Leftarrow \text{ Beam splitter operator}$$

Output states are entangled, when at least one of the input states is nonclassical

Test of nonclassicality via squeezing

Quadrature squeezing

Define quadratures (dimensionless observables):

$$x = \frac{1}{2}(a + a^\dagger), \qquad y = \frac{1}{2i}(a - a^\dagger)$$

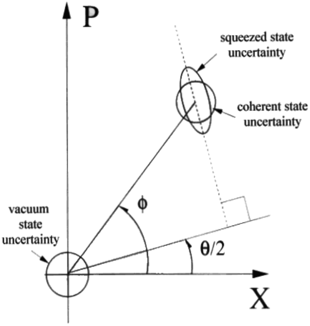
* $\Delta x = \Delta y = 1/\sqrt{2} \;\Rightarrow$ no squeezing ($\Delta x \Delta y = 1/2$)
* $\Delta x < \Delta y \;\Rightarrow$ squeezed in x, $\Delta x > \Delta y \;\Rightarrow$ squeezed in y
  - $\Delta x \Delta y > 1/2$
  - $\Delta x \Delta y = 1/2 \;\Rightarrow$ ideal squeezed states [S. Dey, A. Fring and V. Hussin, Int. J. Mod. Phys. B 30, 1650248 (2016)]

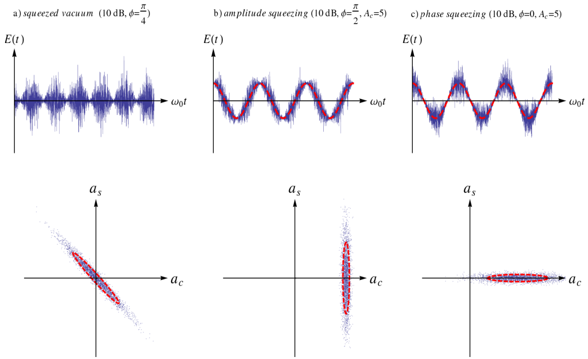
Photon number squeezing: photon number distribution is narrower than the average number of photons $(\Delta n)^2 < \langle n \rangle$

$$\text{Mandel parameter: } Q = \frac{(\Delta n)^2}{\langle n \rangle} - 1$$

$Q \geq 0 \;\Rightarrow$ No squeezing, $Q < 0 \;\Rightarrow$ Photon number squeezed

Squeezing in optical communication





Classical-like states ($|0\rangle$, $|\alpha\rangle$)

- Are separable
- Optical noise is equal to $|0\rangle$

Nonclassical states:

- Entangled
- Optical noise is lower than $|0\rangle$

Generalization

Harmonic oscillator

- Coherent states $\Rightarrow$ no quadrature or number squeezing
- Nonclassical states $\Rightarrow$ quadrature and number squeezed

What about other models!

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What about other models!

Generalized ladder operators:

$$A_q|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q, \quad A_q^\dagger|n\rangle_q = \sqrt{[n+1]_q}|n+1\rangle_q,$$

defined in the q-deformed Fock space:

$$|n\rangle_q = \frac{\left(A^\dagger\right)^n}{\sqrt{[n]_q!}}|0\rangle_q, \quad {}_q\langle 0|0\rangle_q = 1, \quad A|0\rangle_q = 0, \quad [n]_q! = \prod_{k=1}^{n}[k]_q\,.$$

An example

q-deformed oscillator algebra:

$$A_q A_q^\dagger - q^2 A_q^\dagger A_q = 1, \ |q| < 1 \quad \Rightarrow \quad [n]_q = \frac{1 - q^{2n}}{1 - q^2}$$

An example

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Physical reality!

Self-adjoint representation:

$$A_q = \frac{i}{\sqrt{1-q^2}}\left(e^{-i\check{x}} - e^{-i\check{x}/2}e^{2\tau\check{p}}\right),\ A_q^\dagger = \frac{-i}{\sqrt{1-q^2}}\left(e^{i\check{x}} - e^{2\tau\check{p}}e^{i\check{x}/2}\right)$$

with $\check{x} = x\sqrt{m\omega/\hbar}$ and $\check{p} = p/\sqrt{m\omega\hbar}$, $[x,p] = i\hbar$

Observables:

$$X = \gamma(A_q^\dagger + A_q), \quad P = i\delta(A_q^\dagger - A_q), \quad X^\dagger = X,\ P^\dagger = P$$

S. Dey, A. Fring, L. Gouba, P. G. Castro; Phys. Rev. D 87, 084033 (2013)

Noncommutativity

Commutation relation:

$$[X,P] = \frac{4i\gamma\delta}{1+q^2}\left[1 + \frac{q^2-1}{4}\left(\frac{X^2}{\gamma^2} + \frac{P^2}{\delta^2}\right)\right]$$

Constraints $\Rightarrow \gamma = \frac{\hbar}{2\delta},\ q = e^{2\tau\delta^2},\ \tau \in \mathbb{R}^+$, Non-trivial limit $\delta \to 0$

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Generalised uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B] \rangle\right| \geq \frac{\hbar}{2}\left[1 + \tau\left(\Delta P\right)^2 + \tau \langle P \rangle^2\right] \text{(NC case)}$$

- Heisenberg's uncertainty relation: $[A, B] = i\hbar$; give up knowledge about B, for $\Delta A = 0$
- Noncommutative case: $[A, B] \approx B^2$; give up knowledge also about B, for $\Delta A \neq 0$

Minimal lengths, areas and volumes

- In $1D$, $[X, P] = i\hbar\left(1 + \tau P^2\right)$:

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left[1 + \tau\left(\Delta P\right)^2 + \tau\langle P\rangle^2\right]$$

$\Rightarrow$ minimal length

$$\Delta X_{min} = \hbar\sqrt{\tau}\sqrt{1 + \tau\langle P^2\rangle},$$

from minimizing with $\left(\Delta A\right)^2 = \langle A^2\rangle - \langle A\rangle^2$
[B.Bagchi, A. Fring; Phys. Lett. A 373, 4307–4310 (2009)]

- $2D$&$3D$-versions are more complicated and lead to "minimal areas" and "minimal volumes" [S.Dey, A. Fring, L. Gouba; J. Phys. A: Math. Theor. 45, 385302 (2012)]

$1D$ perturbative noncommutative harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 - \hbar\omega\left(\frac{1}{2} + \frac{\tau}{4}\right),$$

defined on the noncommutative space

$$[X, P] = i\hbar\left(1 + \check{\tau} P^2\right), \qquad X = (1 + \check{\tau} p^2) x, \qquad P = p$$

Reality of spectrum, $h = \eta H \eta^{-1}$, with $\eta = (1 + \check{\tau} p^2)^{-1/2}$

$$h = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{\omega\tau}{4\hbar}(x^2 p^2 + p^2 x^2 + 2xp^2 x) - \hbar\omega\left(\frac{1}{2} + \frac{\tau}{4}\right) + \mathcal{O}(\tau^2)$$

Eigenvalues and eigenfunctions:

$$E_n = \hbar\omega e_n = \hbar\omega\left(An + Bn^2\right) + \mathcal{O}(\tau^2), \quad A = 1 + \frac{\tau}{2}, B = \frac{\tau}{2}$$

$$|\phi_n\rangle = |n\rangle - \frac{\tau}{16}\sqrt{(n-3)_4}\,|n-4\rangle + \frac{\tau}{16}\sqrt{(n+1)_4}\,|n+4\rangle + \mathcal{O}(\tau^2)$$

Pochhammer function $(x)_n := \Gamma(x+n)/\Gamma(x)$

A. Mostafazadeh, J. Math. Phys. 43, 2814 (2002)

Generalized q-deformed coherent states

Generalised ladder operators:

$$A_q|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q, \quad A_q^\dagger|n\rangle_q = \sqrt{[n+1]_q}|n+1\rangle_q,$$

$$A_q|\alpha\rangle_q = \alpha|\alpha\rangle_q$$

$$\Downarrow$$

$$|\alpha\rangle_q = \frac{1}{\mathcal{N}(\alpha,q)}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]_q!}}|n\rangle_q, \quad \mathcal{N}(\alpha,q) = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{[n]_q!}$$

Equivalent to nonlinear generalization for $[n]_q = nf^2(n)$

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Equivalent to nonlinear generalization for $[n]_q = nf^2(n)$

In the noncommutative case:

$$(\Delta X)^2\Big|_{|\alpha\rangle_q} = (\Delta Y)^2\Big|_{|\alpha\rangle_q} = \frac{1}{2}\Big|{}_q\langle\alpha|[X,Y]|\alpha\rangle_q\Big| = \frac{1}{4}\left\{1+(q^2-1)|\alpha|^2\right\}$$

S. Dey; Phys. Rev. D 91, 044024 (2015)

Photon-added coherent state

- Harmonic oscillator PACS:

$$|\alpha, m\rangle = \frac{1}{\mathcal{N}(\alpha, m)} a^{\dagger m} |\alpha\rangle, \quad \mathcal{N}^2(\alpha, m) = \langle\alpha| a^m a^{\dagger m} |\alpha\rangle.$$

  m is the number of photons added to the coherent state $|\alpha\rangle$
  [G. S. Agarwal and K. Tara; Phys. Rev. A 43, 492 (1991)]

- q-deformed PACS:

$$\begin{aligned} |\alpha, m\rangle_q &= \frac{1}{\mathcal{N}(\alpha, m, q)} A_q^{\dagger m} |\alpha\rangle_q \\ &= \frac{1}{\mathcal{N}(\alpha, m, q)\mathcal{N}(\alpha, q)} \sum_{n=0}^{\infty} \frac{\alpha^n}{[n]_q!} \sqrt{[n+m]_q!} |n+m\rangle_q, \end{aligned}$$

Hillery-type higher-order quadrature squeezing

Quadratures

$$Y_N(\phi) = \frac{1}{2}\left(A_q^N e^{-iN\phi} + A_q^{\dagger N} e^{iN\phi}\right),$$

are said to be squeezed if

$$_q\langle\alpha, m|[\Delta Y_N(\phi)]^2|\alpha, m\rangle_q < \frac{1}{4}\,_q\langle\alpha, m|[A_q^N, A_q^{\dagger N}]|\alpha, m\rangle_q.$$

Or, equivalently if the squeezing coefficient $S_H < 0$

$$\begin{aligned} S_H &= \frac{4\,_q\langle[\Delta Y_N(\phi)]^2\rangle_q - {}_q\langle[A_q^N, A_q^{\dagger N}]\rangle_q}{_q\langle[A_q^N, A_q^{\dagger N}]\rangle_q} \\ &= 2\frac{\mathrm{Re}\left[\left(_q\langle A_q^{2N}\rangle_q - {}_q\langle A_q^N\rangle_q^2\right)e^{-2iN\phi}\right] - \left|_q\langle A_q^N\rangle_q\right|^2 + {}_q\langle A_q^{\dagger N} A_q^N\rangle_q}{_q\langle A_q^N A_q^{\dagger N}\rangle_q - {}_q\langle A_q^{\dagger N} A_q^N\rangle_q} \end{aligned}$$

(General expression: applicable to any q-deformed models)

For photon-added coherent states, we compute

$$
{}_q\langle A_q^{\dagger N} A_q^{L}\rangle_q = \begin{cases} \frac{\alpha^{*(N-L)}}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m]_q![n+m+N-L]_q!}{[n]_q![n+N-L]_q![n+m-L]_q!} & \text{if} \quad N > L \\ \frac{\alpha^{L-N}}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m]_q![n+m+L-N]_q!}{[n]_q![n+L-N]_q![n+m-N]_q!} & \text{if} \quad L > N, \end{cases}
$$

$$
{}_q\langle A_q^{N} A_q^{\dagger L}\rangle_q = \begin{cases} \frac{\alpha^{N-L}}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m+N]_q!}{[n]_q![n+N-L]_q!} & \text{if} \quad N > L \\ \frac{\alpha^{*(L-N)}}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m+L]_q!}{[n]_q![n+L-N]_q!} & \text{if} \quad L > N, \end{cases}
$$

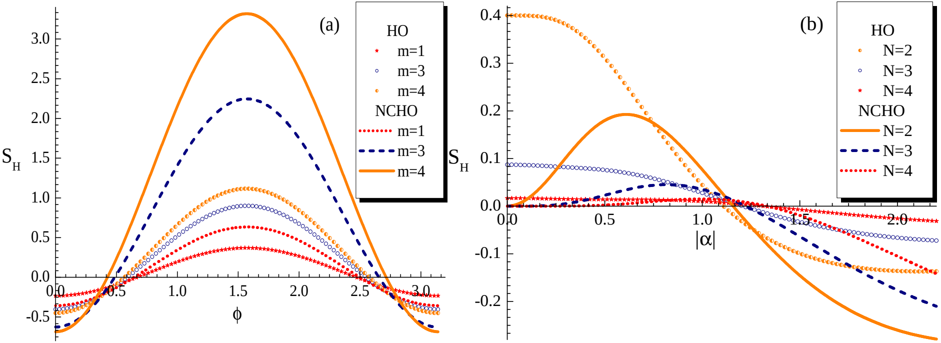
For $N = L$ we have

$$
{}_q\langle A_q^{\dagger N} A_q^{N}\rangle_q = \frac{1}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}([n+m]_q!)^2}{([n]_q!)^2[n+m-N]_q!},
$$

$$
{}_q\langle A_q^{N} A_q^{\dagger N}\rangle_q = \frac{1}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m+N]_q!}{([n]_q!)^2}.
$$

(Can be used for PACS for any q-deformed models)

Ordinary HO versus noncommutative HO



$q = 0.9$

Hong–Mandel-type higher-order quadrature squeezing

The quadrature

$$Y(\phi) = \frac{1}{2}\left(A_q e^{-i\phi} + A_q^\dagger e^{i\phi}\right)$$

is squeezed if

$$_q\langle \alpha, m|(\Delta Y(\phi))^{2N}|\alpha, m\rangle_q < (2N-1)!!\,\frac{[A_q, A_q^\dagger]^N}{4^N},$$

where $(2N-1)!! = 1 \cdot 3 \cdot 5 \cdots (2N-1)$.

Squeezing coefficient

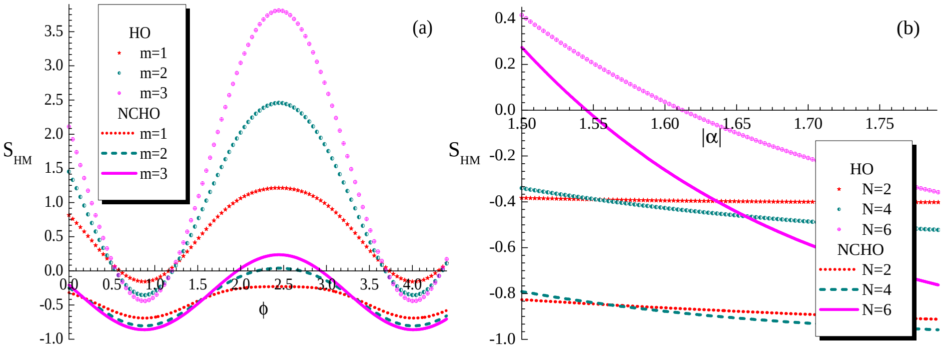
$$S_{HM} = \frac{2^{2N}{}_q\langle(\Delta Y(\phi))^{2N}\rangle_q - (2N-1)!![A_q, A_q^\dagger]^N}{(2N-1)!![A_q, A_q^\dagger]^N},$$

Ordinary HO versus noncommutative HO

We calculate

$$[A_q, A_q^\dagger]^N = \left[1 + (q^2-1)A_q^\dagger A_q\right]^N = \sum_{k=0}^{N} \binom{N}{k} (q^2-1)^k (A_q^\dagger A_q)^k,$$

$$_q\langle Y(\phi)\rangle_q^k = \sum_{s=0}^{k} \binom{k}{s} 2^{-k} e^{i\phi(2s-k)}\, {}_q\langle A_q\rangle_q^{k-s}\, {}_q\langle A_q^\dagger\rangle_q^s$$



Higher-order sub-Poissonian photon statistics

Higher-order correlation function

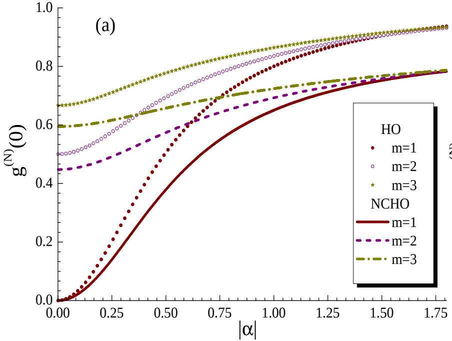
Squeezing: $g^{(N)}(0) = \frac{{}_q\langle(\Delta M)^N\rangle_q - {}_q\langle M\rangle_q}{{}_q\langle M\rangle_q^N} + 1 \quad < 1$

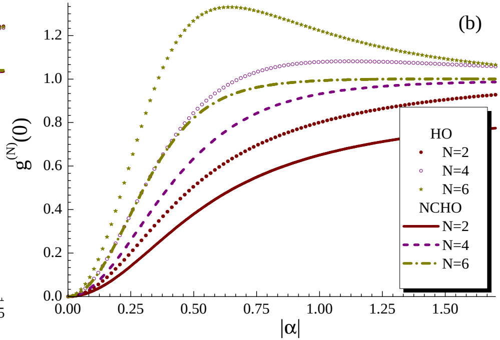
Higher-order Mandel parameter

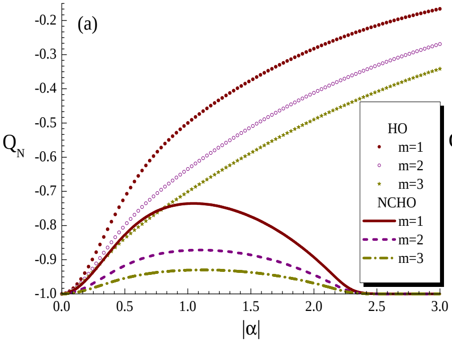
Squeezing: $Q_N = \frac{{}_q\langle(\Delta M)^N\rangle_q}{{}_q\langle M\rangle_q} - 1 \quad < 0$

$g^{(N)}(0)$ and Q_N are computed by using

$$
\begin{aligned}
{}_q\langle(\Delta M)^N\rangle_q &= \sum_{k=0}^{N} \binom{N}{k} (-1)^k {}_q\langle(A_q^\dagger A_q)^{N-k}\rangle_q {}_q\langle A_q^\dagger A_q\rangle_q^k \\
{}_q\langle(A_q^\dagger A_q)^N\rangle_q &= \frac{1}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m]_q!}{([n]_q!)^2} [n+m]_q^N
\end{aligned}
$$


(a)
$g^{(N)}(0)$
$|\alpha|$
HO
m=1
m=2
m=3
NCHO
m=1
m=2
m=3


(b)
$g^{(N)}(0)$
$|\alpha|$
HO
N=2
N=4
N=6
NCHO
N=2
N=4
N=6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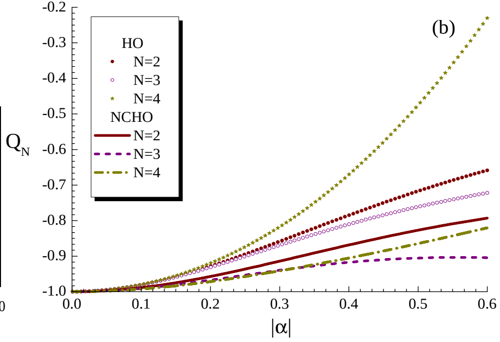
(b)
Q_N
$|\alpha|$
HO
N=2
N=3
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NCHO
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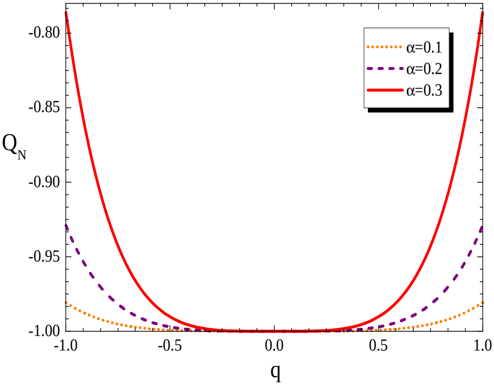
Further control on squeezing through q



Supporting investigations

Similar type of conclusions for other nonclassical models in noncommutative spaces:

- Schrödinger cat states for q-deformed oscillator
  [S. Dey; Phys. Rev. D 91, 044024 (2015)]
- Squeezed states
  [S. Dey, V. Hussin; Phys. Rev. D 91, 124017 (2015)]
- Cat states for perturbative harmonic oscillator
  [S. Dey, A. Fring, V. Hussin; Int. J. Mod. Phys. B 30, 1650248 (2016)]

Conclusions

- Constructed q-deformed photon-added coherent states in noncommutative space.
- Various nonclassical properties are analysed in arbitrary orders.
- Provided generic expressions in higher order for Hillery-type and Hong–Mandel-type squeezing coefficients as well as for Mandel parameter and correlation function.
- Possibilities of obtaining improved degree of nonclassicality are explored.
- An extra degree of freedom on nonclassicality may be obtained through the NC parameter q.

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Outlook

- One may construct many other quantum optical models using this noncommutative structure.
- There are many other versions of the noncommutative structure, which are worth exploring.
- Most exciting is to understand the models in real life experiments.