Fermionic coherent states in infinite dimensions

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Coherent states

A coherent state

- provides a good approximation to a classical state.
- minimizes joint uncertainty relations.
- arises trough “shifting” of a ground state.
- diagonalizes annihilation operators.
- evolves to a coherent state.

Coherent states

- span a dense subspace of the state space.
- satisfy completeness relations.
- factorize correlation functions.
- have reproducing properties.

(Not all of these need apply!)

Bosonic coherent states

Let L be a the linear **phase space** of a simple **classical system**. L carries an anti-symmetric **symplectic form** $\omega : L \times L \to \mathbb{R}$. To **quantize** the system we make L into a **complex Hilbert space** with inner product $\{\cdot, \cdot\}$ so that its imaginary part coincides with 2ω.

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For each $\xi \in L$ there is a **coherent state** $K_\xi \in \mathcal{H}$ with wave function,

$$K_\xi(\phi) = \exp\left(\frac{1}{2}\{\xi,\phi\}\right).$$

K_0 is the ground state or **vacuum**.

Bosonic coherent states

These have properties,

$$\langle K_\xi, \psi \rangle = \psi(\xi) \quad \text{(reproducing property)},$$

$$\langle K_\xi, K_{\xi'} \rangle = \exp\left(\frac{1}{2}\{\xi', \xi\}\right) \quad \text{(inner product)},$$

$$\langle \psi, \eta \rangle = \int_{\hat{L}} \langle \psi, K_\xi \rangle \langle K_\xi, \eta \rangle \, \mathrm{d}\nu(\xi) \quad \text{(completeness)},$$

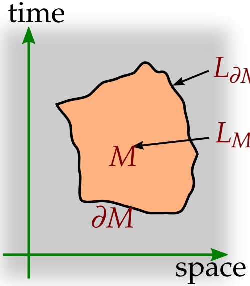
$$a_\phi K_\xi = \frac{1}{\sqrt{2}}\{\xi, \phi\} K_\xi \quad \text{(diagonalizing)},$$

$$\exp\left(\frac{1}{\sqrt{2}} a_\xi^\dagger\right) K_0 = K_\xi \quad \text{(shift of vacuum)}.$$

If $\xi(t)$ describes the **classical evolution** in phase space and $U(t_1, t_2)$ the **quantum evolution operator** from time t_1 to time t_2, then

$$U(t_1, t_2) K_{\xi(t_1)} = K_{\xi(t_2)}.$$

Amplitude formula for coherent states



Linear field theory

- interior solutions L_M
- boundary solutions $L_{\partial M}$
- $J : L_{\partial M} \to L_{\partial M}$ complex structure
- $L_{\partial M} = L_M \oplus J L_M$

For $\xi \in L_{\partial M}$ decompose $\xi = \xi^{\mathrm{c}} + \xi^{\mathrm{n}}$ where $\xi^{\mathrm{c}} \in L_M$ **classically allowed** and $\xi^{\mathrm{n}} \in J L_M$ **classically forbidden**.

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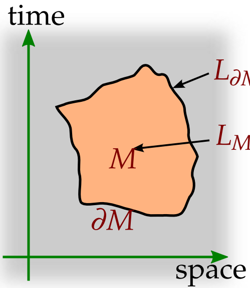

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The amplitude for the associated normalized coherent state $\tilde{K}_\xi$ is:

$$\rho_M(\tilde{K}_\xi) = \exp\left(\mathrm{i}\,\omega_{\partial M}(\xi^n, \xi^c) - \frac{1}{2} g_{\partial M}(\xi^n, \xi^n)\right)$$

[RO 2010] This has a simple and compelling physical interpretation.

No analogous fermionic coherent states

An analogous construction in terms of shift operators for fermions leads to **Grassmann coherent states**, parametrized by **Grassmann numbers**. These are useful for formal manipulations with the fermionic path integral.

But, they are elements of an **extension** of the Hilbert space with Grassmann numbers. They are not **states**.

Group theoretic approach to coherent states

[Gilmore, Perelomov]

Suppose there is a **"dynamical group"** G acting **unitarily** on the Hilbert space $\mathcal{H}$ of states. Choose a **special state** $\psi_0 \in \mathcal{H}$, e.g., the ground state. The **coherent states** are then the states generated from ψ_0 by application of G. Write:

$$\psi_g := g \triangleright \psi_0 \qquad \text{for } g \in G.$$

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$$\psi_g := g \triangleright \psi_0 \qquad \text{for } g \in G.$$

Let $H \subseteq G$ be the subgroup that maps ψ_0 to a multiple of itself. Then the space of coherent states is (projectively) equivalent to the **homogeneous space** G/H.

$$\psi_{gh} = \lambda_h \psi_g \qquad \text{for } g \in G,\ h \in H.$$

This yields a rich **geometric structure** for coherent states. If $\mathcal{H}$ is finite-dimensional, G is usually a **Lie group**. Often G arises by **exponentiation** of a **Lie algebra** $\mathfrak{g}$.

The CAR-algebra

Start with a complex Hilbert space L with inner product $\{\cdot,\cdot\}$.
("one-particle Hilbert space" or "space of classical solutions")

The state space of **fermionic quantum field theory** is the fermionic **Fock space** $\mathcal{F}$ over L. For each $\xi \in L$ there is an

- **annihilation operator** a_ξ and a
- **creation operator** $a^\dagger_\xi$ acting on $\mathcal{F}$.

These generate the unital **canonical anti-commutation relation (CAR)** algebra $\mathcal{A}$ with

- linearity relations

$$a_{\xi+\tau} = a_\xi + a_\tau, \quad a_{\lambda\xi} = \lambda a_\xi,$$

- and anti-commutation relations

$$a_\xi a_\tau + a_\tau a_\xi = 0, \quad a^\dagger_\xi a_\tau + a_\tau a^\dagger_\xi = \{\xi,\tau\}\mathbf{1}.$$

The CAR-algebra: grading

The CAR-algebra is a $\mathbb{Z}$-**graded** (integer-graded) algebra by declaring

- an **annihilation operator** to have **degree** -1,
- a **creation operator** to have **degree** $+1$.

We denote the subalgebras of $\mathcal{A}$

- of elements of degree 0 by $\mathcal{A}_0$,
- of elements of even degree by $\mathcal{A}_e$.

We denote the algebras completed in the operator norm topology by $\mathcal{A}', \mathcal{A}'_0, \mathcal{A}'_e$.

The dynamical Lie algebras: degree 0

Let $\mathcal{T}(L)$ be the algebra of **trace class operators** on L. Given $\lambda \in \mathcal{T}(L)$ define the operator $\hat{\lambda} : \mathcal{F} \to \mathcal{F}$ by,

$$\hat{\lambda} := \frac{1}{2} \sum_{i \in I} \left(a^\dagger_{\zeta_i} a_{\lambda(\zeta_i)} - a_{\lambda(\zeta_i)} a^\dagger_{\zeta_i} \right) = \sum_{i \in I} a^\dagger_{\zeta_i} a_{\lambda(\zeta_i)} - \frac{1}{2} \operatorname{tr}_L(\lambda) \mathbf{1},$$

where $\{\zeta_i\}_{i \in I}$ is an ON-basis of L. This is also called a **fermionic current** operator.

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$$[\hat{\lambda}, \hat{\lambda}'] = \hat{\lambda}'', \qquad \text{with} \quad \lambda'' = \lambda'\lambda - \lambda\lambda'.$$

Restricting to **skew-adjoint** operators $\hat{\lambda}$ generated by **skew-adjoint** $\lambda \in \mathcal{T}(L)$ yields a **real Lie algebra** $\mathfrak{h}$. If $\dim L = n$, then $\mathfrak{h} = \mathfrak{u}(n;\mathbb{R})$, $\mathfrak{h}^{\mathbb{C}} = \mathfrak{u}(n;\mathbb{C})$.

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Proposition

$\mathcal{A}'_0 = \mathrm{U}'(\mathfrak{h}^{\mathbb{C}}) = \mathrm{U}'(\mathfrak{h})^{\mathbb{C}}$.

The dynamical Lie algebras: even degree

A **real linear** map $\Lambda : L \to L$ is **anti-symmetric** iff for all $\xi, \tau \in L$,

$$\{\xi, \Lambda\tau\} = -\{\tau, \Lambda\xi\}.$$

Then Λ is **conjugate linear**, not complex linear.

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Then Λ is **conjugate linear**, not complex linear. Moreover, Λ^2 is **complex linear**, **self-adjoint**, and **(strictly) negative**,

$$\{\xi, \Lambda^2\tau\} = -\{\Lambda\tau, \Lambda\xi\} = \{\Lambda^2\xi, \tau\},$$
$$\{\xi, \Lambda^2\xi\} = -\{\Lambda\xi, \Lambda\xi\} \leq 0.$$

Let $\Xi(L)$ be the vector space of **anti-symmetric maps** $\Lambda : L \to L$ such that Λ^2 is **trace class**.

The dynamical Lie algebras: even degree

Given $\Lambda \in \Xi(L)$ define the operator $\hat{\Lambda} : \mathcal{F} \to \mathcal{F}$,

$$\hat{\Lambda} := \frac{1}{2} \sum_{i \in I} a_{\zeta_i} a_{\Lambda(\zeta_i)}.$$

These operators form a complex **abelian Lie algebra** $\mathfrak{m}_+$.

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These operators form a complex **abelian Lie algebra** $\mathfrak{m}_+$. The adjoint of $\hat{\Lambda}$ is,

$$\hat{\Lambda}^\dagger = \frac{1}{2} \sum_{i \in I} a^\dagger_{\Lambda(\zeta_i)} a^\dagger_{\zeta_i}.$$

These operators form a complex **abelian Lie algebra** $\mathfrak{m}_-$. Set $\mathfrak{m}^\mathbb{C} := \mathfrak{m}_+ \oplus \mathfrak{m}_-$. Let $\mathfrak{m}$ be the real subspace of **skew-adjoint** operators of the form $\hat{\Lambda} - \hat{\Lambda}^\dagger$.

The dynamical Lie algebras: even degree

Combining $\mathfrak{g}_e^{\mathbb{C}} := \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ yields a complex **Lie algebra** with additional relations,

$$[\hat{\lambda}, \hat{\Lambda}] = \hat{\Lambda}', \qquad \text{with} \quad \Lambda' = -\lambda\Lambda - \Lambda\lambda^*$$

$$[\hat{\lambda}, \hat{\Lambda}^\dagger] = \hat{\Lambda}'^\dagger, \qquad \text{with} \quad \Lambda' = \lambda^*\Lambda + \Lambda\lambda$$

$$[\hat{\Lambda}', \hat{\Lambda}^\dagger] = \hat{\lambda}, \qquad \text{with} \quad \lambda = \Lambda'\Lambda.$$

$\mathfrak{g}_e := \mathfrak{h} \oplus \mathfrak{m}$ is the real Lie subalgebra of skew-adjoint operators.

If $\dim L = n$, then $\mathfrak{g}_e = \mathfrak{so}(2n; \mathbb{R})$, $\mathfrak{g}_e^{\mathbb{C}} = \mathfrak{so}(2n; \mathbb{C})$.

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Proposition

$\mathcal{A}'_{\mathrm{e}} = \mathrm{U}'(\mathfrak{g}_{\mathrm{e}}^{\mathbb{C}}) = \mathrm{U}'(\mathfrak{g}_{\mathrm{e}})^{\mathbb{C}}$.

The dynamical Lie algebras: full

$\mathfrak{n}_+$ be the complex vector space spanned by the **annihilation operators**, $\mathfrak{n}_-$ be the complex vector space spanned by the **creation operators**. Set $\mathfrak{n}^{\mathbb{C}} := \mathfrak{n}_+ \oplus \mathfrak{n}_-$. Define $\hat{\xi} := 1/\sqrt{2}\, a_\xi$ and note $\hat{\xi}^\dagger = 1/\sqrt{2}\, a_\xi^\dagger$. Set $\mathfrak{n}$ to be the **real subspace** of **skew-adjoint** operators of the form $\hat{\xi} - \hat{\xi}^\dagger$.

The direct sum $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{n}$ is a **real Lie algebra** of **skew-adjoint** operators and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{n}^{\mathbb{C}}$ is its **complexification**.

If $\dim L = n$, then $\mathfrak{g} = \mathfrak{so}(2n+1;\mathbb{R})$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2n+1;\mathbb{C})$.

The dynamical Lie algebras: full

The additional relations are,

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Proposition

$\mathcal{A}' = \mathrm{U}'(\mathfrak{g}^{\mathbb{C}}) = \mathrm{U}'(\mathfrak{g})^{\mathbb{C}}$.

The dynamical group: degree 0

Let $H^{\mathbb{C}}$ be the exponential of $\mathfrak{h}^{\mathbb{C}}$, i.e, the set of operators on $\mathcal{F}$ of the form $\exp(\hat{\lambda})$ for $\lambda \in \mathcal{T}(L)$. Denote by H the restriction to λ **skew-adjoint**. H consists of **unitary** operators.

Proposition

$H^{\mathbb{C}}$ is a group: Given $\lambda_1, \lambda_2 \in \mathcal{T}(L)$ there is $\lambda_3 \in \mathcal{T}(L)$ s.t., $\exp(\hat{\lambda}_1) \exp(\hat{\lambda}_2) = \exp(\hat{\lambda}_3)$. Similarly, H is a group.

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Here, H is the **compact Lie group** $U(n)$ if L is finite-dimensional of dimension n.

The dynamical group

Define $G^{\mathbb{C}}_{\text{alg}}$ to be the group consisting of all **finite products of exponentials** of elements of $\mathfrak{g}^{\mathbb{C}}$ as operators on $\mathcal{F}$. Let $G'^{\mathbb{C}}_{\text{alg}}$ be the completion of $G^{\mathbb{C}}_{\text{alg}}$ in the operator norm topology. Define $G^{\mathbb{C}}$ as the intersection of $G'^{\mathbb{C}}_{\text{alg}}$ with the **invertible operators** on $\mathcal{F}$.

Proposition

$G^{\mathbb{C}}$ is a group and the subgroup $G^{\mathbb{C}}_{\text{alg}} \subseteq G^{\mathbb{C}}$ is dense.

Define G to the subgroup of $G^{\mathbb{C}}$ consisting of **unitary** operators.

If L is finite-dimensional of dimension n, G is isomorphic to the compact Lie group $SO(2n+1)$.

The dynamical group: special subgroups

Set $\mathfrak{p}_+ := \mathfrak{m}_+ \oplus \mathfrak{n}_+$ and $\mathfrak{p}_- := \mathfrak{m}_- \oplus \mathfrak{n}_-$. Define P_-, P_+ to be the image of the **exponential map** applied to $\mathfrak{p}_-$ and $\mathfrak{p}_+$ respectively.

Lemma

Let $\xi_1, \xi_2 \in L$ and $\Lambda_1, \Lambda_2 \in \Xi(L)$. Define $\Lambda' := \xi_2\{\cdot, \xi_1\} - \xi_1\{\cdot, \xi_2\}$. Then $\Lambda' \in \Xi(L)$ and,

$$\exp\left(\hat{\Lambda}_1 + \hat{\xi}_1\right) \exp\left(\hat{\Lambda}_2 + \hat{\xi}_2\right) = \exp\left(\hat{\Lambda}_1 + \hat{\Lambda}_2 + \frac{1}{2}\hat{\Lambda}' + \hat{\xi}_1 + \hat{\xi}_2\right).$$

Proposition

P_-, P_+ are subgroups of $G^{\mathbb{C}}$.

Decomposition of the dynamical group

The dynamical Lie algebra decomposes as $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}_- \oplus \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{p}_+$. This has an analogue in the group.

Decomposition Theorem

The map

$$P_- \times H^{\mathbb{C}} \times P_+ \to G^{\mathbb{C}}$$

given by the product in $G^{\mathbb{C}}$ is injective and its image dense in the operator norm topology.

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given by the product in $G^{\mathbb{C}}$ is injective and its image dense in the operator norm topology.

For $g \in G^{\mathbb{C}}$ and $\epsilon > 0$ there is

$$g' = \exp\left(\hat{\Lambda}^{\dagger} + \hat{\xi}^{\dagger}\right) \exp\left(\hat{\lambda}\right) \exp\left(\hat{\Lambda}' + \hat{\xi}'\right).$$

with $||g' - g|| < \epsilon$.

Denote the image of the map by $G^{\mathbb{C}}_{\text{dec}}$, its intersection with G by G_{dec}.

Action on vacuum state

Let $\psi_0 \in \mathcal{F}$ be the standard **vacuum state**. The Decomposition Theorem almost implements **normal ordering**, adapted to evaluate the action of $G^{\mathbb{C}}$ on ψ_0. The subgroup $P_+ \subset G^{\mathbb{C}}$ acts trivially, while $H^{\mathbb{C}}$ acts by multiplication with a scalar. Explicitly,

$$\exp\left(\hat{\Lambda}^\dagger + \hat{\xi}^\dagger\right)\exp\left(\hat{\lambda}\right)\exp\left(\hat{\Lambda}' + \hat{\xi}'\right)\psi_0 \\ = \exp\left(-\frac{1}{2}\operatorname{tr}_L(\lambda)\right)\exp\left(\hat{\Lambda}^\dagger + \hat{\xi}^\dagger\right)\psi_0.$$

The states so obtained are the **coherent states**.

Holomorphic approach

Start with the complexified dynamical group $G^{\mathbb{C}}$, even though it does not act unitarily. As we have seen, the subgroup mapping ψ_0 to a multiple of itself is generated by P_+ and $H^{\mathbb{C}}$. Call this subgroup X_+. Then, **coherent states** are in correspondence with elements of the **homogeneous space** $G^{\mathbb{C}}/X_+$.

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However, there is a **phase ambiguity** in identifying an element of $G^{\mathbb{C}}/X_+$ with a coherent state due to the action of X_+ in terms of phase factors. To fix this we choose a **representative** in each equivalence class. From the complex Decomposition Theorem we can do this by simply restricting to the subgroup $P_- \subset G^{\mathbb{C}}_{\text{dec}}$.

Parametrization of coherent states

We have identified coherent states with elements of P_-. P_- arises by **exponentiation** from the complex Lie algebra $\mathfrak{p}_-$. Use the latter to **parametrize coherent states**. Given $\Lambda \in \Xi(L)$ and $\xi \in L$, set

$$K : \mathfrak{p}_- \to \mathcal{F}, \quad (\Lambda, \xi) \mapsto \exp\left(\hat{\Lambda}^\dagger + \hat{\xi}^\dagger\right) \psi_0.$$

Properties of $\mathfrak{p}_-$

Since G is compact (at least in the finite-dimensional case), its Lie algebra $\mathfrak{g}$ carries a **Killing form** yielding a positive definite real inner product. This extends to a positive definite sesquilinear inner product on the complexification $\mathfrak{g}^{\mathbb{C}}$, making it into a **complex Hilbert space**. Explicitly, this **inner product** is,

$$\langle\langle \hat{\lambda}_1 + \hat{\Lambda}_1{}^\dagger + \hat{\Lambda}'_1 + \hat{\xi}_1{}^\dagger + \hat{\xi}'_1, \hat{\lambda}_2 + \hat{\Lambda}_2{}^\dagger + \hat{\Lambda}'_2 + \hat{\xi}_2{}^\dagger + \hat{\xi}'_2 \rangle\rangle \\ = 2\,\mathrm{tr}_L(\lambda_1^* \lambda_2) - \mathrm{tr}_L(\Lambda'_2 \Lambda'_1) - \mathrm{tr}_L(\Lambda_1 \Lambda_2) + 2\{\xi_2, \xi_1\} + 2\{\xi'_1, \xi'_2\}.$$

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Proposition

The map K is continuous, holomorphic and injective.

Normalization

The norm of a coherent state can be expressed using the **Fredholm determinant**.

Proposition

$$\|K(\Lambda,\xi)\|_{\mathcal{F}} = \left(1+\frac{1}{2}b\right)^{\frac{1}{2}} \det\left(\mathbf{1}_L - \Lambda^2\right)^{\frac{1}{4}}, \qquad b := \{\xi, (\mathbf{1}_L - \Lambda^2)^{-1}\xi\},$$

$$1 \le \|K(\Lambda,\xi)\|_{\mathcal{F}} \le \exp\left(\frac{1}{4}\|(\Lambda,\xi)\|_{\mathfrak{p}_-}\right).$$

Inner product

Proposition

Let $\Lambda_1, \Lambda_2 \in \Xi(L)$ and $\xi_1, \xi_2 \in L$. If $\mathbf{1}_L - \Lambda_1\Lambda_2$ is invertible set

$$b := \{\xi_2, (\mathbf{1}_L - \Lambda_1\Lambda_2)^{-1}\xi_1\}.$$

Then,

$$\langle K(\Lambda_1, \xi_1), K(\Lambda_2, \xi_2)\rangle = \left(1 + \frac{1}{2}b\right)\det(\mathbf{1}_L - \Lambda_1\Lambda_2)^{\frac{1}{2}}.$$

The correct branch of the square root is obtained by analytic continuation from $\Lambda_1 = \Lambda_2$. If $\mathbf{1}_L - \Lambda_1\Lambda_2$ is not invertible, the inner product vanishes.

Denseness and reproducing property

The coherent states really cover the whole Fock space:

Proposition

The image of K spans a dense subspace of $\mathcal{F}$.

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We can associate to a state a **holomorphic wave function**. It automatically satisfies the **reproducing property**.

Proposition

Given $\psi \in \mathcal{F}$ define the function

$$f_\psi : \mathfrak{p}_- \to \mathbb{C} \quad \text{by} \quad f_\psi(\Lambda, \xi) := \langle K(\Lambda, \xi), \psi \rangle. \tag{1}$$

Then, f_ψ is continuous and anti-holomorphic.

A reproducing kernel Hilbert space

Let $\mathrm{Hol}(\mathfrak{p}_-)$ denote the complex vector space of **continuous** and **anti-holomorphic** functions on $\mathfrak{p}_-$.

Theorem

The complex linear map

$$f : \mathcal{F} \to \mathrm{Hol}(\mathfrak{p}_-) \quad \text{given by} \quad \psi \mapsto f_\psi$$

is injective and thus realizes the Fock space $\mathcal{F}$ as a **reproducing kernel Hilbert space** of continuous anti-holomorphic functions on the Hilbert space $\mathfrak{p}_-$.

Conclusions and Outlook

Fermionic coherent states share many properties with their bosonic counterparts,

- denseness
- holomorphicity properties
- reproducing property
- generated by "shift operator" (element of P_-)
- coherence is preserved under evolution (not shown here)

Important differences:

- Parametrized not by L but by a larger space $L \oplus \Xi(L)$
- not semiclassical?

Outlook:

- measure, completeness?
- factorization of correlation functions?

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