Classical and quantum Kummer shapes
in memory of S. Twareque Ali (1942-2016)

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Contents:

Motivation:

$$\begin{array}{ccccc} (M,\omega) & \xrightarrow{\ \mathcal{K}\ } & (\mathbb{CP}(\mathcal{H}),\omega_{FS}) & \xrightarrow{\ \iota\ } & (\mathcal{U}^1(\mathcal{H}),\{\cdot,\cdot\}_{LP}) \\ \Big\downarrow \Sigma & & \Big\downarrow U_\Sigma & & \Big\downarrow Ad^*_{U_\Sigma} \\ (M,\omega) & \xrightarrow{\ \mathcal{K}\ } & (\mathbb{CP}(\mathcal{H}),\omega_{FS}) & \xrightarrow{\ \iota\ } & (\mathcal{U}^1(\mathcal{H}),\{\cdot,\cdot\}_{LP}) \end{array} \qquad (1)$$

- (M,ω) - symplectic manifold, Σ - symplectomorphism of (M,ω),
- ω_{FS} - Fubini-Study form,
- $\mathcal{U}^1(\mathcal{H}) \ni \rho$ iff $\rho^+ = \rho$ and $||\rho||_1 := Tr|\rho| < \infty$,
- $\mathcal{U}^\infty(\mathcal{H}) \ni X$ iff $X^+ = -X$ and $||X||_\infty < \infty$,
- $\{F,G\}_{LP}(\rho) := iTr(\rho[DF(\rho),DG(\rho)])$, where $F,G \in C^\infty(\mathcal{U}^1(\mathcal{H}))$,
- $DF(\rho), DG(\rho) \in \mathcal{U}^1(\mathcal{H})^* \cong \mathcal{U}^\infty(\mathcal{H})$-Banach-Lie algebra.

See [5], [6] and [7].

Settings

- We assume $\Omega^{N+1} := \{(z_0, \dots, z_N)^T \in \mathbb{C}^{N+1} : |z_k| > 0, \text{ for } k = 0, 1, \dots, N\}$ as the phase space with the standard Poisson bracket

$$\{f, g\} = -i \sum_{n=0}^{N} \left(\frac{\partial f}{\partial z_n} \frac{\partial g}{\partial \bar{z}_n} - \frac{\partial g}{\partial z_n} \frac{\partial f}{\partial \bar{z}_n} \right), \tag{2}$$

of $f, g \in C^\infty(\Omega^{N+1})$ i.e. for coordinate function we have

$$\{z_k, \bar{z}_l\} = i\delta_{kl}, \quad \{z_k, z_l\} = 0, \quad \{\bar{z}_k, \bar{z}_l\} = 0.$$

• We will take

$$H = h_0(|z_0|^2, |z_1|^2, \ldots, |z_N|^2) + g_0(|z_0|^2, |z_1|^2, \ldots, |z_N|^2) z_0^{l_0} z_1^{l_1} \cdots z_N^{l_N} + \\ + g_0(|z_0|^2, |z_1|^2, \ldots, |z_N|^2) z_0^{-l_0} z_1^{-l_1} \cdots z_N^{-l_N} \quad (3)$$

as Hamiltonian for $(N+1)$-harmonic oscillators.

• In (3) the following convention is assumed

$$z_i^{l_i} = \begin{cases} z^{l_i} & \text{for} \quad l_i \geqslant 0 \\ \bar{z}^{|l_i|} & \text{for} \quad l_i < 0 \end{cases} \quad (4)$$

for $z_i \in \mathbb{C}$ and $l_i \in \mathbb{Z}$.

• In the Kummers paper [3] a Hamiltonian system, in which the interaction between harmonic oscillators is described by Hamiltonian (3), where h_0 is a polynomial of degree smaller than $|l_0| + \ldots + |l_N|$ and g_0 is a constant, was integrated.

Classical reduction

• In our approach we integrate the system given by Hamiltonian (3) passing to the new canonical coordinates

$$I_k := \sum_{j=0}^{N} \rho_{kj} |z_j|^2, \quad \psi_l := \sum_{j=0}^{N} \kappa_{jl} \phi_j, \tag{5}$$

where $z_j = |z_j| e^{i\phi_j}$, $k, l = 0, \ldots, N$ and the real $(N+1) \times (N+1)$ matrix $\rho = (\rho_{ij})$ satisfies resonance condition

$$\det \rho \neq 0 \quad \text{and} \quad \sum_{j=0}^{N} \rho_{ij} l_j = \delta_{0i}. \tag{6}$$

$\kappa = (\kappa_{ij})$ is the inverse of $\rho = (\rho_{ij})$.

- Ω^{N+1} is invariant with respect to the Hamiltonian flows

$$\sigma_r(t)(z_0, \dots, z_N) = (e^{i\rho_{r0}t} z_0, \dots, e^{i\rho_{rN}t} z_N), \tag{7}$$

generated by I_r, where $t \in \mathbb{R}$ and $r = 0, 1, \dots, N$.

- The resonance condition (6) implies that the flows σ_r are periodic

$$\sigma_r(t + T_r) = \sigma_r(t) \tag{8}$$

for $r = 1, 2, \dots, N$.

- We assume that $T_1, \dots, T_N$ are minimal periods.
- Expressing $\sigma_r(t)$ in the coordinates $(I_0, \dots, I_N, \psi_0, \dots, \psi_N)$ we find that

$$\sigma_r(t)(I_0, \dots, I_N, \psi_0, \dots, \psi_N) = (I_0, \dots, I_N, \psi_0, \dots, \psi_r + t, \dots, \psi_N). \tag{9}$$

- Because of (6) the variable ψ_0 depends on $\phi_0, \dots, \phi_N$ as follows

$$\psi_0 = \sum_{j=0}^{N} l_j \phi_j. \tag{10}$$

From the above it follows that one can assume

$$0 < \psi_r \leq T_r, \quad 2\pi \sum_{i \in N_n} l_i < \psi_0 \leq 2\pi \sum_{i \in N_p} l_i, \tag{11}$$

where $r = 1, 2, \ldots, N$, $N_n := \{0 \leq i \leq N : l_i < 0\}$ and $N_p := \{0 \leq i \leq N : l_i > 0\}$. The coordinates $(I_0, \ldots, I_N)$ belong to the cone $\Lambda^{N+1} \subset \mathbb{R}^{N+1}$ defined by inequalities

$$\begin{aligned} l_0 I_0 + \sum_{j=1}^{N} \kappa_{0j} I_j &> 0, \\ &\cdots \\ l_N I_0 + \sum_{j=1}^{N} \kappa_{Nj} I_j &> 0. \end{aligned} \tag{12}$$

In coordinates (5) the Poisson bracket (2) assumes the form

$$\{f, g\} = \sum_{n=0}^{N} \left(\frac{\partial f}{\partial I_n} \frac{\partial g}{\partial \psi_n} - \frac{\partial g}{\partial I_n} \frac{\partial f}{\partial \psi_n} \right) \tag{13}$$

so, one has

$$\{I_k, I_l\} = \{\psi_k, \psi_l\} = 0, \quad \{I_k, \psi_l\} = \delta_{kl}, \tag{14}$$

where $k, l = 0, \ldots, N$.

Hamiltonian (3) in coordinates (5) is given by

$$H = H_0(I_0, \dots, I_N) + 2\sqrt{\mathcal{G}_0(I_0, \dots, I_N)}\cos\psi_0, \tag{15}$$

where the functions $H_0(I_0, \dots, I_N)$ and $\mathcal{G}_0(I_0, \dots, I_N)$ are defined as the superposition of functions $h_0(|z_0|^2, \dots, |z_N|^2)$ and $|g_0(|z_0|^2, \dots, |z_N|^2)|^2(|z_0|^{2|l_0|} \dots |z_N|^{2|l_N|})$ with the linear map

$$|z_j|^2 = \sum_{k=0}^{N} \kappa_{jk} I_k, \tag{16}$$

i.e.

$$\mathcal{G}_0(I_0, \dots, I_N) := g_0\left(\sum_{j=0}^{N} \kappa_{0j} I_j, \dots, \sum_{j=0}^{N} \kappa_{Nj} I_j\right)^2 \times$$
$$\times \left(\sum_{j=0}^{N} \kappa_{0j} I_j\right)^{|l_0|} \dots \left(\sum_{j=0}^{N} \kappa_{Nj} I_j\right)^{|l_N|}. \tag{17}$$

• Since,

$$\{I_k, H\} = 0, \tag{18}$$

for $k = 1, \dots, N$, we will consider the integrals of motion $I_1, \dots, I_N$ as the components of the momentum map

$$\mathbf{J}(I_0, \dots, I_N, \psi_0, \dots, \psi_N) = \begin{pmatrix} I_1 \\ \vdots \\ I_N \end{pmatrix}, \tag{19}$$

where we identified $\mathbb{R}^N$ with the dual of Lie algebra of the N-dimensional torus $\mathbb{T}^N = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$.

• The momentum map $\mathbf{J} : \Omega^{N+1} \to \mathbb{R}^N$ is a submersion. So, the level set $\mathbf{J}^{-1}(c_1, \dots, c_N)$ of $(c_1, \dots, c_N)^T \in \mathbf{J}(\Omega^{N+1})$ is a real submanifold of Ω^{N+1}.

• Notice that

$$a < I_0 < b, \quad 0 \leq \psi_0 < 2\pi \tag{20}$$

where

$$a := \max_{i \in N_p} \left\{ -\frac{1}{l_i} \sum_{j=1}^{N} \kappa_{ij} c_j \right\}, \qquad b := \min_{i \in N_n} \left\{ -\frac{1}{l_i} \sum_{j=1}^{N} \kappa_{ij} c_j \right\}. \tag{21}$$

if $(I_0, I_1, \ldots, I_N, \psi_0, \psi_1, \ldots, \psi_N) \in \mathbf{J}^{-1}(c_1, \ldots, c_N)$.
• We have $\mathbf{J}^{-1}(c_1, \ldots, c_N)/\mathbb{T}^N \cong]a, b[\times \mathbb{S}^1$.
• $\mathbf{J}^{-1}(c_1, \ldots, c_N) \to \mathbf{J}^{-1}(c_1, \ldots, c_N)/\mathbb{T}^N$ is a trivial $\mathbb{T}^N$-pricipal bundle over the reduced symplectic manifold $\mathbf{J}^{-1}(c_1, \ldots, c_N)/\mathbb{T}^N$.

- In coordinates (I_0, ψ_0) on $\mathbf{J}^{-1}(c_1, \ldots, c_N)/\mathbb{T}^N$, the reduced Poisson bracket of $F, G \in C^\infty(\mathbf{J}^{-1}(c_1, \ldots, c_N)/\mathbb{T}^N)$ is given by

$$\{F, G\} = \frac{\partial F}{\partial I_0}\frac{\partial G}{\partial \psi_0} - \frac{\partial G}{\partial I_0}\frac{\partial F}{\partial \psi_0} \tag{22}$$

and Hamiltonian (15) reduces to

$$H_0(I_0, c_1, \ldots, c_N) + 2\sqrt{\mathcal{G}_0(I_0, c_1, \ldots, c_N)}\cos\psi_0 = E = const. \tag{23}$$

- Hamilton equations are

$$\frac{dI_0}{dt} = 2\sqrt{\mathcal{G}_0(I_0, c_1 \ldots, c_N)}\sin\psi_0, \tag{24}$$

$$\frac{d\psi_0}{dt} = \frac{\partial H_0}{\partial I_0}(I_0, c_1 \ldots, c_N) + \frac{\partial \mathcal{G}_0}{\partial I_0}(I_0, c_1 \ldots, c_N)\frac{\cos\psi_0}{\sqrt{\mathcal{G}_0(I_0, c_1, \ldots, c_N)}}, \tag{25}$$

and one can integrate them by quadratures. Namely, from (24) and (23) one obtains

$$\left(\frac{dI_0}{dt}(t)\right)^2 = 4\mathcal{G}_0(I_0(t), c_1, \ldots, c_N) - (E - H_0(I_0(t), c_1, \ldots, c_N))^2. \tag{26}$$

Substituting $I_0(t)$ into (25) we find $\psi_0(t)$. We find ψ_k integrating

$$\frac{d\psi_k}{dt} = \frac{\partial H_0}{\partial c_k}(I_0, c_1, \ldots, c_N) + \frac{\partial \mathcal{G}_0}{\partial c_k}(I_0, c_1, \ldots, c_N)\frac{\cos\psi_0}{\sqrt{\mathcal{G}_0(I_0, c_1, \ldots, c_N)}}. \tag{27}$$

Classical Kummer shape

• In order to visualize the geometry of the reduced symplectic manifold $\mathbf{J}^{-1}(c_1, \dots, c_N)/\mathbb{T}^N$ let us introduce a map $\mathcal{F} : \Omega^{N+1} \to \mathbb{C}$ given by

$$z = x + iy = \mathcal{F}(z_0, \dots, z_N) := g_0(|z_0|^2, |z_1|^2, \dots, |z_N|^2) z_0^{l_0} \cdots z_N^{l_N} = \\ = \sqrt{\mathcal{G}_0(I_0, \dots, I_N)} e^{i\psi_0}, \quad (28)$$

which is constant on the orbits of $\mathbb{T}^N$ and thus, can be considered as a function of arguments $I_0, \dots, I_N, \psi_0$.

• The variables $I_0, I_1, \dots, I_N, x$ and y are functionally closed with respect to the Poisson bracket, i.e. one has

$$\{I_0, x\} = -y, \quad \{I_0, y\} = x, \\ \{x, y\} = \frac{1}{2} \frac{\partial \mathcal{G}_0}{\partial I_0}(I_0, I_1, \dots I_N), \quad (29) \\ \{I_k, x\} = \{I_k, y\} = 0,$$

for $k, l = 1, 2, \dots, N$. So, they generate Poisson subalgebra $\mathcal{K}_{\mathcal{G}_0}(\Omega^{N+1})$ of the standard Poisson algebra $(C^\infty(\Omega^{N+1}), \{\cdot, \cdot\})$.

• Since functions $x, y, I_0 \in C^\infty(\Omega^{N+1})$ are invariants of $\mathbb{T}^N$, they define the corresponding functions on the reduced phase space $\mathbf{J}^{-1}(c_1, \ldots, c_N)/\mathbb{T}^N$. Hence, there is a map

$$\Phi_{c_1,\ldots,c_N}(I_0, \psi_0) := \begin{pmatrix} \sqrt{\mathcal{G}_0(I_0, c_1, \ldots, c_N)}\cos\psi_0 \\ \sqrt{\mathcal{G}_0(I_0, c_1, \ldots, c_N)}\sin\psi_0 \\ I_0 \end{pmatrix} \tag{30}$$

of $]a, b[\times\mathbb{S}^1$ onto the circularly symmetric surface $\mathcal{C}^{-1}(0)$ in $\mathbb{R}^2\times]a, b[$ given by the equation

$$\mathcal{C}(x, y, I_0) := -\frac{1}{2}(x^2 + y^2 - \mathcal{G}_0(I_0, c_1, \ldots, c_N)) = 0 \tag{31}$$

on $(x, y, I_0)^T \in \mathbb{R}^2\times]a, b[$.

• We call $\mathcal{C}^{-1}(0)$ the **Kummer shape**.

- Consider the Poisson algebra $(C^\infty(\mathbb{R}^3), \{\cdot,\cdot\}_\mathcal{C})$ with the Nambu bracket

$$\{f,g\}_\mathcal{C} := \det[\nabla\mathcal{C}, \nabla f, \nabla g] \tag{32}$$

as the Poisson bracket, where $f,g \in C^\infty(\mathbb{R}^3)$ and $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial I_0}\right)^T$.

- The Kummer shape $\mathcal{C}^{-1}(0)$ is a symplectic leaf and $\Phi_{c_1,\ldots,c_N} : \mathbf{J}^{-1}(c_1,\ldots,c_N)/\mathbb{T}^N \to \mathcal{C}^{-1}(0)$ is a symplectic $\sum_{i=0}^N |l_i|$-fold covering of $\mathcal{C}^{-1}(0)$.
- The functions $x, y, I_0 \in \mathcal{K}_{\mathcal{G}_0}(\Omega^{N+1})$ after reduction to $\mathbf{J}^{-1}(c_1,\ldots,c_N)/\mathbb{T}^N$ satisfy

$$\{I_0, x\} = -y, \quad \{I_0, y\} = x, \tag{33}$$

$$\{x, y\} = \frac{1}{2}\frac{\partial\mathcal{G}_0}{\partial I_0}(I_0, c_1, \ldots, c_N). \tag{34}$$

Thus they generate a Poisson algebra $\mathcal{K}_{\mathcal{G}_0}(c_1,\ldots,c_N)$ isomorphic to $(C^\infty(\mathbb{R}^3), \{\cdot,\cdot\}_\mathcal{C})$. This Poisson algebra is the reduction of the Poisson subalgebra $\mathcal{K}_{\mathcal{G}_0}(\Omega^{N+1}) \subset C^\infty(\Omega^{N+1})$.

- We shall call $\mathcal{K}_{\mathcal{G}_0}(c_1,\ldots,c_N)$ the **classical Kummer shape algebra**.

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Quantum system

Quantum Hamiltonian:

$$\mathbf{H} = h_0(a_0^* a_0, ..., a_N^* a_N) + g_0(a_0^* a_0, ..., a_N^* a_N) a_0^{l_0} ... a_N^{l_N} + \\ + a_0^{-l_0} ... a_N^{-l_N} g_0(a_0^* a_0, ..., a_N^* a_N), \quad (35)$$

where

$$a_i^{l_i} = \begin{cases} a_i^{l_i} & \text{if } l_i \geq 0 \\ (a_i^*)^{-l_i} & \text{if } l_i < 0 \end{cases} \quad (36)$$

and

$$[a_i, a_j^*] = \hbar \delta_{ij}, \qquad [a_i, a_j] = 0, \qquad [a_i^*, a_j^*] = 0. \quad (37)$$

Hamiltonians of such types model many physcial phenomena in nonlinear quantum optics, e.g. parametric amplification, parametric conversion, Kerr effect for a certain choice of $l_0, \ldots, l_N$ and functions g_0, h_0.

Quantum reduction

We introduce the operators

$$A := g_0(a_0^*a_0, ..., a_N^*a_N)a_0^{l_0}...a_N^{l_N}, \tag{38}$$

$$A_i := \sum_{j=0}^{N} \rho_{ij} a_j^* a_j, \tag{39}$$

where $i = 0, 1, ..., N$. They satisfy

$$\begin{aligned} [A_0, A] = -\hbar A, \qquad [A_0, A^*] = \hbar A^*, \\ [A, A_i] = [A^*, A_i] = [A_i, A_j] = 0, \\ AA^* = \mathcal{G}_\hbar(A_0, A_1, ..., A_N), \\ A^*A = \mathcal{G}_\hbar(A_0 - \hbar, A_1, ..., A_N), \end{aligned} \tag{40}$$

where $i = 1, 2, ..., N$, $j = 0, \ldots, N$.

The function $\mathcal{G}_\hbar$ is defined by

$$\mathcal{G}_\hbar(A_0, \dots, A_N) := g_0\left(\sum_{j=0}^{N} \kappa_{0j}A_j, ..., \sum_{j=0}^{N} \kappa_{Nj}A_j\right)^2 \mathcal{P}_{l_0}\left(\sum_{j=0}^{N} \kappa_{0j}A_j\right) ...\mathcal{P}_{l_N}\left(\sum_{j=0}^{N} \kappa_{Nj}A_j\right), \tag{41}$$

where

$$\mathcal{P}_{l_i}(x) := \begin{cases} (x+\hbar)...(x+l_i\hbar) & \text{if } l_i > 0 \\ 1 & \text{if } l_i = 0 \\ x(x-\hbar)...(x-(-l_i-1)\hbar) & \text{if } l_i < 0 \end{cases} \tag{42}$$

In terms of the operators $A_0, A_1, ..., A_N, A, A^*$ the Hamiltonian (35) is written as follows

$$\mathbf{H} = H_0(A_0, A_1, \dots, A_N) + A + A^*, \tag{43}$$

where the function H_0 is defined as the superposition of the function h_0 with the linear map inverse to (39).

• It is easy to see that

$$[A_i, \mathbf{H}] = 0 \tag{44}$$

for $i = 1, 2, \ldots, N$. So, we have commuting integrals of motion: $A_1, \ldots, A_N$, which also commute with A_0.

• Notice here that the operators $A_0, A_1, \ldots, A_N$ are diagonalized in the standard Fock basis

$$|n_0, n_1, \ldots, n_N\rangle := \frac{1}{\sqrt{n_0! \ldots n_N!}} \hbar^{-\frac{1}{2}(n_0+\ldots+n_N)} (a_0^*)^{n_0} \ldots (a_N^*)^{n_N} |0, \ldots, 0\rangle, \tag{45}$$

where $n_i \in \mathbb{Z}_+ \cup \{0\}$, with the eigenvalues $c_0, c_1, \ldots, c_N$ related to $n_0, n_1, \ldots, n_N$ by

$$c_i = \hbar \sum_{j=0}^{N} \rho_{ij} n_j, \qquad i = 0, 1, ..., N. \tag{46}$$

We will use them for a new parametrization $\{|c_0, c_1, \ldots, c_N\rangle\}$ of the Fock basis $\{|n_0, n_1, ..., n_N\rangle\}$.

• We can reduce the quantum system described by the Hamiltonian (35) to the Hilbert subspace $\mathcal{H}_{c_1,\dots,c_N} \subset \mathcal{H}$ spanned by the eigenvectors

$$\{|c_0 + \hbar n, c_1, \dots, c_N\rangle\}_{n=0}^{L}, \tag{47}$$

$L = \min_{i \in N_n}\{[-\frac{v_i}{l_i}]\}$, of A_0, with fixed eigenvalues $c_1, \dots, c_N$ of the operators $A_1, \dots, A_N$.

• Equivalently one can write (47) as follows

$$\{|v_0 + nl_0, \dots, v_N + nl_N\rangle\}_{n=0}^{L},$$

where

$$v_k = \frac{1}{\hbar}\sum_{j=0}^{N} \kappa_{kj} c_j. \tag{48}$$

• In the following we assume that c_0 satisfies

$$\mathcal{G}_\hbar(c_0 - \hbar, c_1, \dots, c_N) = 0, \tag{49}$$

which is equivalent to the assumption that $|c_0, c_1, \dots, c_N\rangle$ is a vacuum state of the annihilation operator $\mathbf{A}$, i.e. one has

$$\mathbf{A}|c_0, c_1, \dots, c_N\rangle = 0. \tag{50}$$

Quantum Kummer shape

- The operators A_0, A, A^* after reduction to $\mathcal{H}_{c_1,\dots,c_N}$ are given by

$$\mathbf{A}_0|c_0 + \hbar n, c_1, \dots, c_N\rangle = (c_0 + \hbar n)|c_0 + \hbar n, c_1, \dots, c_N\rangle \quad (51)$$

$$\mathbf{A}|c_0 + \hbar n, c_1, \dots, c_N\rangle = \\ = \sqrt{\mathcal{G}_\hbar(c_0 + \hbar(n-1), c_1, \dots, c_N)}|c_0 + \hbar(n-1), c_1, \dots, c_N\rangle \quad (52)$$

$$\mathbf{A}^*|c_0 + \hbar n, c_1, \dots, c_N\rangle = \sqrt{\mathcal{G}_\hbar(c_0 + \hbar n, c_1, \dots, c_N)}|c_0 + \hbar(n+1), c_1, \dots, c_N\rangle. \quad (53)$$

We denote by $\mathbf{Q}_{\mathcal{G}_\hbar}(\mathcal{H}_{c_1,\dots,c_N})$ the operator algebra generated by the reduced operators $\mathbf{A}, \mathbf{A}^*$ and $\mathbf{A}_0$.

- In accordance with the classical case, we will call this algebra a **quantum Kummer shape algebra**.

• The reduced operators $\mathbf{A}_0$, $\mathbf{A}$ and $\mathbf{A}^*$ satisfy

$$\begin{aligned} [\mathbf{A}_0, \mathbf{A}] = -\hbar \mathbf{A}, \qquad [\mathbf{A}_0, \mathbf{A}^*] &= \hbar \mathbf{A}^*, \\ \mathbf{A}^*\mathbf{A} &= \mathcal{G}_\hbar(\mathbf{A}_0 - \hbar, c_1, \ldots, c_N), \\ \mathbf{A}\mathbf{A}^* &= \mathcal{G}_\hbar(\mathbf{A}_0, c_1, \ldots, c_N). \end{aligned} \tag{54}$$

and Hamiltonian (35) is given by

$$\mathbf{H} = H_0(\mathbf{A}_0, c_1, \ldots, c_N) + \mathbf{A} + \mathbf{A}^*. \tag{55}$$

- One can assume that

$$\mathcal{G}_{\hbar}(\mathbf{A}_0, c_1, \ldots, c_N) = \mathcal{R}(q^{\frac{1}{\alpha}(\mathbf{A}_0 - c_0)}), \tag{56}$$

where $0 < q < 1$ and α is a constant which has action dimension. Taking the bounded operator

$$\mathbf{Q} := q^{\frac{1}{\alpha}(\mathbf{A}_0 - c_0)}, \tag{57}$$

instead of $\mathbf{A}_0$ we rewrite the relations (54) as follows

$$\mathbf{A}\mathbf{Q} = q^{\frac{\hbar}{\alpha}}\mathbf{Q}\mathbf{A}, \quad \mathbf{Q}\mathbf{A}^* = q^{\frac{\hbar}{\alpha}}\mathbf{A}^*\mathbf{Q}, \tag{58}$$

$$\mathbf{A}^*\mathbf{A} = \mathcal{R}(q^{-\frac{\hbar}{\alpha}}\mathbf{Q}), \quad \mathbf{A}\mathbf{A}^* = \mathcal{R}(\mathbf{Q}). \tag{59}$$

The operator C^*-algebras defined by the above relations were investigated in O.A., *Quantum Algebras and q-Special Functions Related to Coherent States Maps of the Disc*, Commun. Math. Phys. **192**, 183-215, 1998 .

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Covariant symbols and Moyal $*$-product

• Glauber coherent states for a system of $N+1$ non-interacting modes (harmonic oscillators):

$$|z_0,\dots,z_N\rangle := \sum_{n_0,\dots,n_N=0}^{\infty} \frac{z_0^{n_0}\cdots z_N^{n_N}}{\sqrt{n_0!\dots n_N!}}\hbar^{-\frac{1}{2}(n_0+\dots+n_N)}|n_0,\dots,n_N\rangle, \quad (60)$$

where $z_0,\dots,z_N \in \mathbb{C}$ and $|n_0,\dots,n_N\rangle$ are the elements of the Fock basis of the Hilbert space $\mathcal{H}$.

• The covariant symbol $\langle F\rangle : \mathbb{C}^{N+1} \to \mathbb{C}$ of an operator

$$F = \sum_{m_0,\dots,m_N,n_0,\dots,n_N=0}^{\infty} f_{\bar{m}_0,\dots,\bar{m}_N,n_0,\dots,n_N}(a_0^*)^{m_0}\dots(a_N^*)^{m_N}a_0^{n_0}\dots a_N^{n_N} \quad (61)$$

is defined by the mean value of F on the coherent states:

$$\langle F\rangle(\bar{z}_0,\dots,\bar{z}_N,z_0,\dots,z_N) := \frac{\langle z_0,\dots,z_N|F|z_0,\dots,z_N\rangle}{\langle z_0,\dots,z_N|z_0,\dots,z_N\rangle}$$

$$= \sum_{m_0,\dots,m_N,n_0,\dots,n_N=0}^{\infty} f_{\bar{m}_0,\dots,\bar{m}_N,n_0,\dots,n_N}(\bar{z}_0)^{m_0}\dots(\bar{z}_N)^{m_N}z_0^{n_0}\dots z_N^{n_N}. \quad (62)$$

The $*_\hbar$-product of covariant symbols $f, g \in C^\infty(\mathbb{C}^{N+1})$ of the operators F and G is defined in the following way:

$$(f *_\hbar g)(\bar{z}_0, ..., \bar{z}_N, z_0, ..., z_N) := \langle FG\rangle(\bar{z}_0, ..., \bar{z}_N, z_0, ..., z_N). \tag{63}$$

Using the resolution

$$\int_{\mathbb{C}^{N+1}} \frac{|w_0, ..., w_N\rangle\langle w_0, ..., w_N|}{\langle w_0, ..., w_N|w_0, ..., w_N\rangle} d\nu_\hbar(\bar{w}_0, ..., \bar{w}_N, w_0, ..., w_N) = \mathbb{1}, \tag{64}$$

of identity $\mathbb{1}$, where $d\nu_\hbar$ is the Liouville measure normalized by a factor, one obtains from (63) the standard formula for $*_\hbar$-product

$$\begin{aligned}(f *_\hbar g)(\bar{z}_0, ..., \bar{z}_N, z_0, ..., z_N) = \\ = \int_{\mathbb{C}^{N+1}} f(\bar{z}_0, \ldots, \bar{z}_N, w_0, \ldots, w_N) g(\bar{w}_0, \ldots, \bar{w}_N, z_0, \ldots, z_N) \times \\ \times e^{-\frac{1}{\hbar}(|z_0 - w_0|^2 + \ldots + |z_N - w_N|^2)} d\nu_\hbar(\bar{w}_0, ..., \bar{w}_N, w_0, ..., w_N) = \\ = \sum_{j_0, \ldots, j_N = 0}^{\infty} \frac{\hbar^{j_0 + \ldots + j_N}}{j_0! \ldots j_N!} \left(\frac{\partial^{j_0}}{\partial z_0^{j_0}} \cdots \frac{\partial^{j_N}}{\partial z_N^{j_N}}\right) f(\bar{z}_0, \ldots, \bar{z}_N, z_0, \ldots, z_N) \times \\ \times \left(\frac{\partial^{j_0}}{\partial \bar{z}_0^{j_0}} \cdots \frac{\partial^{j_N}}{\partial \bar{z}_N^{j_N}}\right) g(\bar{z}_0, \ldots, \bar{z}_N, z_0, \ldots, z_N).\end{aligned} \tag{65}$$

• Note here that

$$f *_\hbar g \xrightarrow[\hbar\to 0]{} f\cdot g \tag{66}$$

and

$$\lim_{\hbar\to 0}\frac{-i}{\hbar}(f *_\hbar g - g *_\hbar f) = \{f, g\}, \tag{67}$$

i.e. in the limit $\hbar \to 0$ we come back to the Poisson algebra of real smooth functions on $\mathbb{C}^{N+1}$.

• In particular case one obtains the correspondences

$$\langle A_k\rangle = I_k, \tag{68}$$

$$\langle A\rangle \xrightarrow[\hbar\to 0]{} z, \tag{69}$$

$$\langle \mathbf{H}\rangle \xrightarrow[\hbar\to 0]{} H. \tag{70}$$

• In the limit $\hbar \to 0$ commutation relations (40) expressed by their covariant symbols give the relations (29) for the corresponding classical quantities which define classical Kummer shape algebra.

• Summing up, the system of $N+1$ quantum harmonic oscillators converges in the classical limit $\hbar \to 0$ to its classical counterpart.

Reduced coherent states and reduced $*$-product

Now we will apply the classical and quantum reduction procedures to the construction of the reduced coherent state map

$$\mathcal{K}_{c_1,\ldots,c_N} : \mathbf{J}^{-1}(c_1,\ldots,c_N)/\mathbb{T}^N \to \mathbb{CP}(\mathcal{H}_{c_1,\ldots,c_N}). \tag{71}$$

Note that the Glauber coherent state map $K_G : \Omega^{N+1} \to \mathcal{H}$ has equivariance property

$$|e^{i\rho_{r0}t}z_0,\ldots,e^{i\rho_{rN}t}z_N\rangle = e^{\frac{it}{\hbar}A_r}|z_0,\ldots,z_N\rangle, \tag{72}$$

where $K_G(z_0,\ldots,z_N) = |z_0,\ldots,z_N\rangle$, $r = 1,\ldots,N$ and $t \in \mathbb{R}$.

- We also recall that $I_0, I_1, \dots, I_N$ are invariants of the Hamiltonian flows generated by them.
- Passing in (72) from the complex canonical coordinates $(z_0, \dots, z_N, \bar{z}_0, \dots, \bar{z}_N)$ to the real canonical coordinates $(I_0, I_1, \dots, I_N, \psi_0, \psi_1, \dots, \psi_N)$ we find that, for $r = 1, \dots, N$, one has

$$P_{c_1,\dots,c_N}|I_0, c_1, \dots, c_N, \psi_0, \dots, \psi_r + t, \dots, \psi_N\rangle = \\ = e^{\frac{i}{\hbar}c_r t} P_{c_1,\dots,c_N}|I_0, c_1, \dots, c_N, \psi_0, \dots, \psi_r, \dots, \psi_N\rangle, \quad (73)$$

if $(z_0, \dots, z_N, \bar{z}_0, \dots, \bar{z}_N)^T \in \mathbf{J}^{-1}(c_1, \dots, c_N)$, where $P_{c_1,\dots,c_N} : \mathcal{H} \to \mathcal{H}_{c_1,\dots,c_N}$ is the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}_{c_1,\dots,c_N}$.

Let us assume that g_0 is a constant and define the complex analytic map $K_{c_1,\dots,c_N}:\mathbb{C}\ni z\mapsto|z;c_1,\dots,c_N\rangle\in\mathcal{H}_{c_1,\dots,c_N}$ by

$$|z;c_1,\dots,c_N>:=$$

$$=\sum_{n=0}^{L}\frac{z^n}{(\hbar^{\frac{1}{2}(l_0+\dots+l_N)}g_0)^n\sqrt{(v_0+nl_0)!\dots(v_N+nl_N)!}}\times$$

$$\times\,|v_0+nl_0,\dots,v_N+nl_N\rangle, \quad (74)$$

where $L+1=\dim\mathcal{H}_{c_1,\dots,c_N}$ and $v_k=\frac{1}{\hbar}\sum_{j=0}^{N}\kappa_{kj}c_j$ for $k=0,\dots,N$,

Proposition

Suppose that $g_0 = const$, then

$$(i)\ P_{c_1,\ldots,c_N}|z_0,\ldots,z_N\rangle|_{\mathbf{J}^{-1}(c_1,\ldots,c_N)} = \frac{z_0^{v_0}\ldots z_N^{v_N}}{\sqrt{\hbar^{v_0+\ldots+v_N}}}|z;c_1,\ldots,c_N\rangle =$$

$$= \frac{e^{i\sum_{j=0}^{N}\frac{c_j}{\hbar}\psi_j}}{\sqrt{\hbar^{v_0+\ldots+v_N}}}\left(\kappa_{00}I_0+\sum_{j=1}^{N}\kappa_{0j}c_j\right)^{\frac{v_0}{2}}\ldots\left(\kappa_{N0}I_0+\sum_{j=1}^{N}\kappa_{Nj}c_j\right)^{\frac{v_N}{2}}\times$$

$$\times|z;c_1,\ldots,c_N\rangle, \qquad (75)$$

(ii) The map

$$z = g_0\frac{\prod_{i\in N_p} z_i^{l_i}}{\prod_{j\in N_n} z_j^{|l_j|}} = \frac{1}{\prod_{j\in N_n}\left(\kappa_{j0}I_0+\sum_{k=1}^{N}\kappa_{jk}c_k\right)^{|l_j|}}\times$$

$$\times\sqrt{\mathcal{G}_0(I_0,c_1,\ldots,c_N)}e^{i\psi_0} \qquad (76)$$

defines the isomorphism $]a,b[\times\mathbb{S}^1\cong\mathbb{C}\backslash\{0\}$.

• Notice that if $\dim_{\mathbb{C}} \mathcal{H}_{c_1,\ldots,c_N} = \infty$ and g_0 is any positive function, then the coherent states (74) can be generalized by

$$|z; c_1, \ldots, c_N\rangle = \frac{1}{\sqrt{v_0! \ldots v_N!}} \Big(|v_0, \ldots, v_N\rangle + \\ + \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\mathcal{G}_\hbar(0) \ldots \mathcal{G}_\hbar(n-1)}} |v_0 + nl_0, \ldots, v_N + nl_N\rangle \Big), \quad (77)$$

where $\mathcal{G}_\hbar(n) := \mathcal{G}_\hbar(c_0 + \hbar n, c_1, \ldots, c_N)$, what is equivalent to

$$\mathbf{A}|z; c_1, \ldots, c_N\rangle = z|z; c_1, \ldots, c_N\rangle. \quad (78)$$

• In the subsequent considerations we will postulate the existence of the resolution

$$\mathbb{1}_{c_1,\dots,c_N} = \int_{\mathbb{C}} |z; c_1, \dots, c_N\rangle\langle z; c_1, \dots, c_N| d\mu_{c_1,\dots,c_N}(\bar{z}, z) \quad (79)$$

for $\mathbb{1}_{c_1,\dots,c_N} = P_{c_1,\dots,c_N}|_{\mathcal{H}_{c_1,\dots,c_N}}$ with respect to some measure $d\mu_{c_1,\dots,c_N}(\bar{z}, z)$.

• We define the covariant symbol

$$\langle \mathbf{F} \rangle(\bar{z}, z) := \frac{\langle z; c_1, \dots, c_N|\mathbf{F}|z; c_1, \dots, c_N\rangle}{\langle z; c_1, \dots, c_N|z; c_1, \dots, c_N\rangle}, \quad (80)$$

for an operator $\mathbf{F} : \mathcal{D}(\mathbf{F}) \to \mathcal{H}_{c_1,\dots,c_N}$.

• Since one has one-to-one correspondence between the operators $\mathbf{F}, \mathbf{G}$ and their symbols, we can define the $*_\hbar$-product of covariant symbols

$$(\langle \mathbf{F} \rangle *_\hbar \langle \mathbf{G} \rangle)(\bar{z}, z) := \frac{\langle z; c_1, \dots, c_N|\mathbf{FG}|z; c_1, \dots, c_N\rangle}{\langle z; c_1, \dots, c_N|z; c_1, \dots, c_N\rangle}, \quad (81)$$

From (81) and (79) we obtain

$$(\langle \mathbf{F}\rangle *_\hbar \langle \mathbf{G}\rangle)(\bar{z}, z) = \int_{\mathbb{D}_{R_\hbar}} \frac{\langle z; c_1, \dots, c_N \mid \mathbf{F} \mid w; c_1, \dots, c_N\rangle}{\langle z; c_1, \dots, c_N \mid w; c_1, \dots, c_N\rangle} \times$$
$$\times \frac{\langle w; c_1, \dots, c_N \mid \mathbf{G} \mid z; c_1, \dots, c_N\rangle}{\langle w; c_1, \dots, c_N \mid z; c_1, \dots, c_N\rangle} |a_{c_1,\dots,c_N}(z, w)|^2 d\nu_{c_1,\dots,c_N}(\bar{w}, w), \tag{82}$$

where

$$a_{c_1,\dots,c_N}(z, w) := \frac{\langle z; c_1, \dots, c_N | w; c_1, \dots, c_N\rangle}{\langle z; c_1, \dots, c_N | z; c_1, \dots, c_N\rangle^{\frac{1}{2}} \langle w; c_1, \dots, c_N | w; c_1, \dots, c_N\rangle^{\frac{1}{2}}} \tag{83}$$

and

$$d\nu_{c_1,\dots,c_N}(\bar{w}, w) = \langle w; c_1, \dots, c_N | w; c_1, \dots, c_N\rangle d\mu_{c_1,\dots,c_N}(\bar{w}, w). \tag{84}$$

Proposition

The $*_\hbar$-product (81) of the covariant symbols $f(\bar{z},z)=\sum_{k,l=0}^{\infty} f_{\bar{k},l}\bar{z}^k z^l$ and $g(\bar{z},z)=\sum_{k,l=0}^{\infty} g_{\bar{k},l}\bar{z}^k z^l$ of operators $\mathbf{F}:=\sum_{k,l=0}^{\infty} f_{\bar{k},l}\mathbf{A}^{*k}\mathbf{A}^l$ and $\mathbf{G}:=\sum_{r,s=0}^{\infty} g_{\bar{r},s}\mathbf{A}^{*r}\mathbf{A}^s$, respectively, is given by

$$(f *_\hbar g)(\bar{z},z)=$$
$$=\frac{1}{\langle z;c_1,\ldots,c_N|z;c_1,\ldots,c_N\rangle} f(\bar{z},\bar{\partial}_{\mathcal{G}_\hbar})\left(g(\bar{z},z)\langle z;c_1,\ldots,c_N|z;c_1,\ldots,c_N\rangle\right), \tag{85}$$

where, by definition, the operator $\partial_{\mathcal{G}_\hbar}$ acts on the monomial z^n in the following way

$$\partial_{\mathcal{G}_\hbar} z^n := \mathcal{G}_\hbar(n-1)z^{n-1}, \tag{86}$$

if $n\geq 1$ and $\partial_{\mathcal{G}_\hbar} z^n=0$ if $n=0$. The operator $f(\bar{z},\bar{\partial}_{\mathcal{G}_\hbar})$ is defined by

$$f(\bar{z},\bar{\partial}_{\mathcal{G}_\hbar}) := \sum_{k,l=0}^{\infty} f_{\bar{k},l}\bar{z}^k\bar{\partial}_{\mathcal{G}_\hbar}^l \tag{87}$$

and acts on the complex coordinate $\bar{z}$ only.

• One defines the Lie bracket $\{f,g\}_{\mathcal{G}_0}$ of the covariant symbols f and g by

$$\{f,g\}_{\mathcal{G}_0} := \lim_{\hbar\to 0}\frac{-i}{\hbar}(f *_\hbar g - g *_\hbar f). \tag{88}$$

• When exponents $l_0,\ldots,l_N$ are nonnegative we find that the covariant symbols of $\mathbf{A},\mathbf{A}^*$ are $z,\bar{z}$,respectively. If additionally function $\mathcal{G}_\hbar$ is invertible as a function of A_0 (for example when g_0 is constant) the covariant symbol of $\mathbf{A}_0$ in the limit $\hbar\to 0$ gives I_0.

• We have

$$\{I_0,z\}_{\mathcal{G}_0} = iz, \tag{89}$$

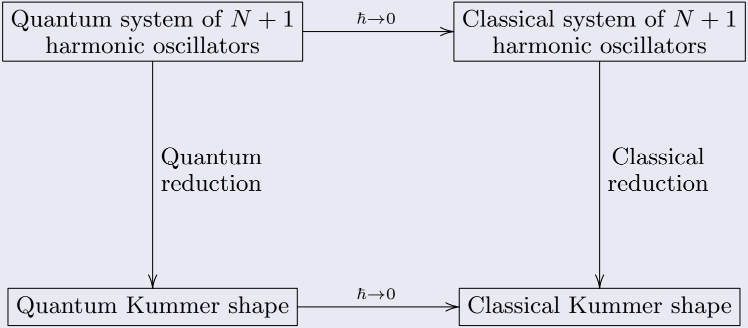
$$\{I_0,\bar{z}\}_{\mathcal{G}_0} = -i\bar{z}, \tag{90}$$

$$\{z,\bar{z}\}_{\mathcal{G}_0} = -i\frac{\partial\mathcal{G}_0}{\partial I_0}(I_0). \tag{91}$$

• In the classical limit $\hbar\to 0$ the quantum Kummer shape algebra, i.e. the operator algebra defined by the operators (54), corresponds to the classical Kummer shape (33-34) with Nambu bracket $\{\cdot,\cdot\}_{\mathcal{C}}$ defined by structural function $\mathcal{G}_0$.

Proposition

In the case when exponents $l_0, \dots, l_N$ are nonnegative and quantum structural function $\mathcal{G}_\hbar(\cdot, c_1, \dots, c_N)$ is invertible as a function of A_0 (e.g. when g_0 is constant), passing to the classical limit $\hbar \to 0$ intertwines both reduction procedures, quantum and classical, i.e.
Figure 2.



Reproducing measure for the reduced coherent states

Proposition

Suppose that g_0 is a constant. We have the following reproducing property:

$$K_{c_1,\dots,c_N}(\bar{v},w) = \int_{\mathbb{C}} K_{c_1,\dots,c_N}(\bar{v},z) K_{c_1,\dots,c_N}(\bar{z},w) d\mu_{c_1,\dots,c_N}(\bar{z},z), \qquad (92)$$

for reproducing kernel $K_{c_1,\dots,c_N}$ given by

$$K_{c_1,\dots,c_N}(\bar{z},w) = \langle z; c_1,\dots,c_N | w; c_1,\dots,c_N\rangle$$
$$= \frac{1}{v_0!\dots v_N!}\, {}_rF_s\left[\begin{matrix}\alpha_1, & \alpha_2, & \dots, & \alpha_r \\ \beta_1, & \beta_2, & \dots, & \beta_s\end{matrix}\,;\, \frac{\bar{z}w}{g_0^2 l_0^{l_0}\dots l_N^{l_N}\hbar^{l_0+\dots+l_N}}\right], \qquad (93)$$

for $r = 1 + \sum_{i\in N_n} |l_i|$ and $s = \sum_{i\in N_p} l_i$, where

$$(\alpha_1, \alpha_2, \dots, \alpha_r) =$$
$$= \left(1, \frac{v_{i_1}}{l_{i_1}}, \dots, \frac{v_{i_1} - (-l_{i_1} - 1)}{l_{i_1}}, \dots, \frac{v_{i_{r-1}}}{l_{i_{r-1}}}, \dots, \frac{v_{i_{r-1}} - (-l_{i_{r-1}} - 1)}{l_{i_{r-1}}}\right) \qquad (94)$$

Proposition continued

and

$$(\beta_1, \beta_2, \dots, \beta_s) = \\ = \left(\frac{v_{j_1}+1}{l_{j_1}}, \dots, \frac{v_{j_1}+l_{j_1}}{l_{j_1}}, \dots, \frac{v_{j_s}+1}{l_{j_s}}, \dots, \frac{v_{j_s}+l_{j_s}}{l_{j_s}}\right). \quad (95)$$

and reproducing measure $d\mu_{c_1,\dots,c_N}(\bar{z}, z) = \rho_{c_1,\dots,c_N}(|z|^2)d|z|^2 d\psi_0$, $z = |z|e^{i\psi_0}$, is defined by

$$\rho_{c_1,\dots,c_N}(|z|^2) := \frac{1}{2\pi l_0^2 \hbar^{N+1+v_0+\dots+v_N} g_0^{\frac{2v_0+2}{l_0}}} |z|^{2\left(\frac{v_0+1}{l_0}-1\right)} \times \\ \times \int_{[0,+\infty)^N} x_1^{v_1 - \frac{l_1(v_0+1)}{l_0}} \dots x_N^{v_N - \frac{l_N(v_0+1)}{l_0}} \times \\ \times e^{-\frac{1}{\hbar}\left(|z|^{\frac{2}{l_0}}(g_0^2 x_1^{l_1} \dots x_N^{l_N})^{-\frac{1}{l_0}} + x_1 + \dots + x_N\right)} dx_1 \dots dx_N. \quad (96)$$

Example

1.For $l_0 = l_1 = 1$ and $|0, v_1\rangle$ as a vacuum state, $v_1 \in \mathbb{N}$, the reproducing kernel takes the form

$$K_{c_1}(\bar{z}, w) = \frac{1}{v_1!}\ {}_0F_1\left[-; v_1 + 1; \frac{\bar{z}w}{\hbar^2}\right]. \tag{97}$$

The density function is given by

$$\rho_{c_1}(|z|^2) = \frac{1}{2\pi\hbar^2}\left(\frac{|z|^2}{\hbar^2}\right)^{\frac{v_1}{2}} K_{v_1}\left(2\frac{|z|}{\hbar}\right), \tag{98}$$

where K_{v_1} is the modified Bessel function of the second kind.
2. Assuming $l_0 = 1, l_1 = -1$ and choosing $|0, v_1\rangle$ as the vacuum state, one has

$$K_{c_1}(\bar{z}, w) = \frac{1}{v_1!}(1 + \bar{z}w)^{v_1}. \tag{99}$$

From (96) one has

$$\rho_{c_1}(|z|^2) = \frac{1}{2\pi}\frac{(v_1 + 1)!}{(1 + |z|^2)^{v_1+2}}. \tag{100}$$

Example

3. In the case $N = 2$ and $(l_0, l_1, l_2) = (1, -1, -1)$ function

$$\rho_{c_1,c_2}(x) = \frac{(v_1+1)!(v_2+1)!}{2\pi\hbar^{\frac{v_1+v_2+1}{2}}} g_0^{v_2+v_1+1} e^{\frac{g_0^2}{2\hbar x}} W_{-\frac{(v_1+v_2+3)}{2};\frac{v_1-v_2}{2}}\left(\frac{g_0^2}{\hbar x}\right), \tag{101}$$

where $W_{-\frac{(v_1+v_2+3)}{2};\frac{v_1-v_2}{2}}$ is a Whittaker function, defines the reproducing measure $d\nu_{c_1,c_2}(\bar{z}, z) = \rho(|z|^2)d|z|^2 d\psi$ for reproducing kernel

$$K_{c_1,c_2}(\bar{z}, w) = \sum_{n=0}^{L} \frac{(\bar{z}w\hbar)^n}{g_0^{2n} n!(v_1-n)!(v_2-n)!} =$$

$$= \frac{1}{v_1!v_2!}\ {}_2F_0\left[\begin{matrix} -v_1, & -v_2 \\ & - \end{matrix}\ ; \frac{\bar{z}w\hbar}{g_0^2}\right] \tag{102}$$

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