Two dimensional de Sitter spinors and their SL (2,R) covariance

Etats cohérents et leurs applications:
un panorama contemporain

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Prelude: why I am here

- Crucial role of coset spaces G/H in constructing systems of coherent states
- In the second example in Perelomov's "Coherent States for Arbitrary Lie Group" CMP (1971) the coset space is the upper complex plane SL(2,R)/K (KAN)
- What is a simple geometric description of SL(2,R)/A ?

The first modification of GR

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1915)$$

The cosmological Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1917)$$

Lemaître's prophecy

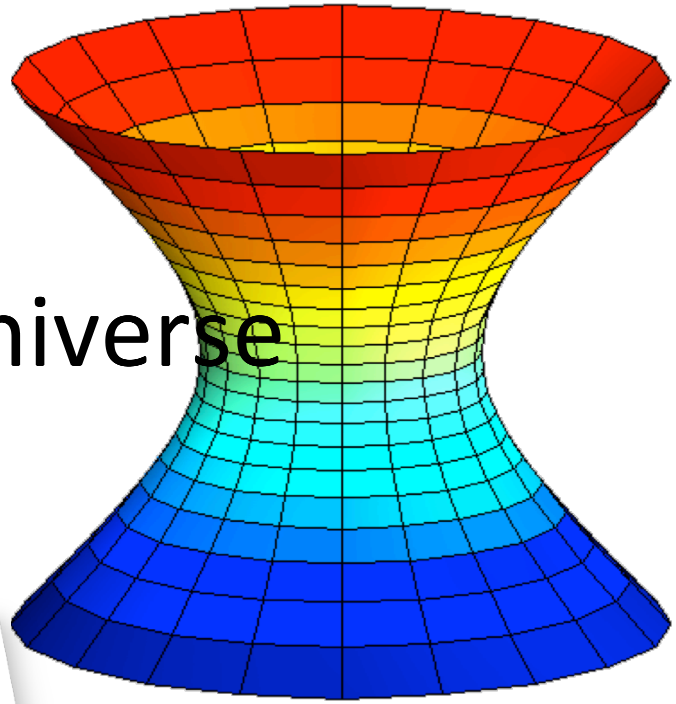
"The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory.

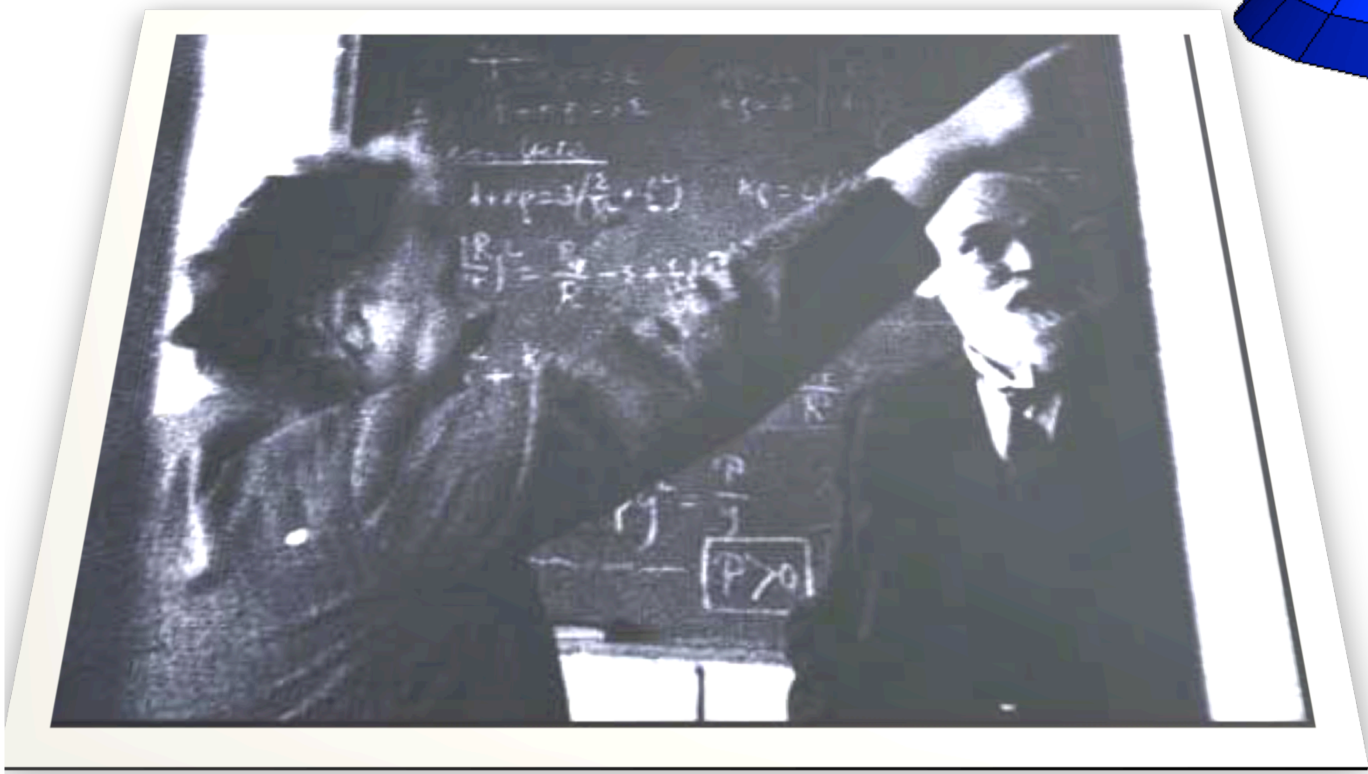
It may be that the discovery of the cosmological constant is such a case."

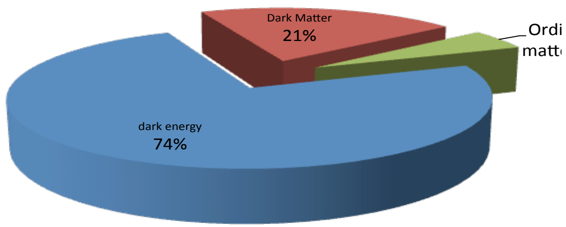
George E. Lemaître, article in the book
"Albert Einstein: Philosopher–Scientist", 1949

Seventy years after.

1997: The shape of the universe



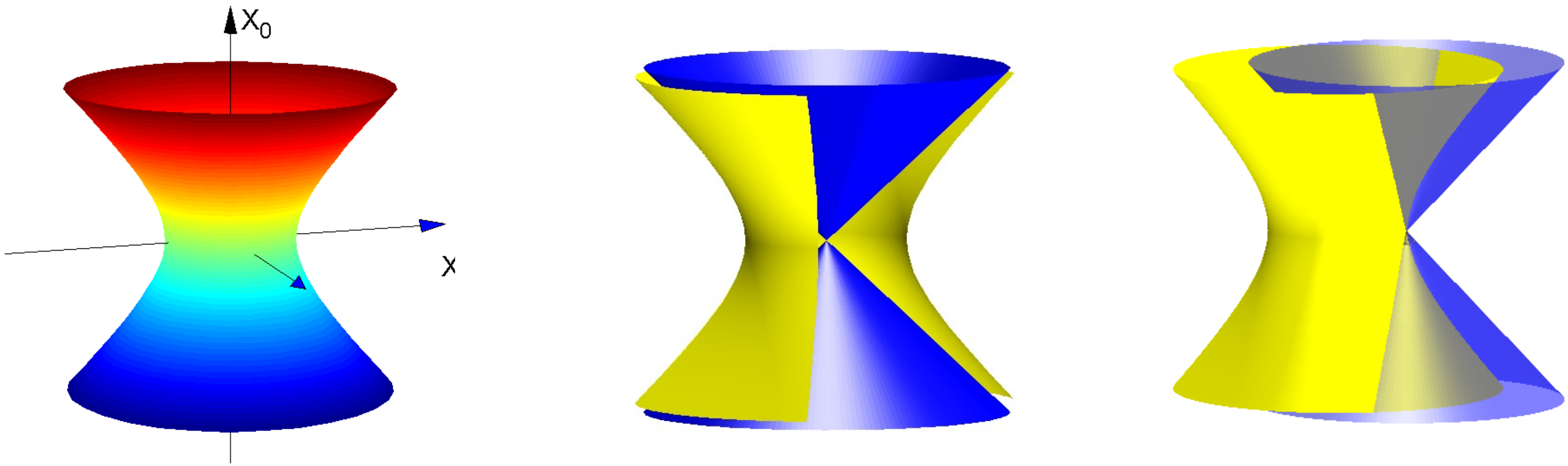




The de Sitter universe (1917)

$$M^{(1,4)}: \ \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1, -1)$$

$$\{X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = -R^2 = -3/\Lambda\}$$



$$\{\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 = 0\}$$

de Sitter Quantum Field Theory

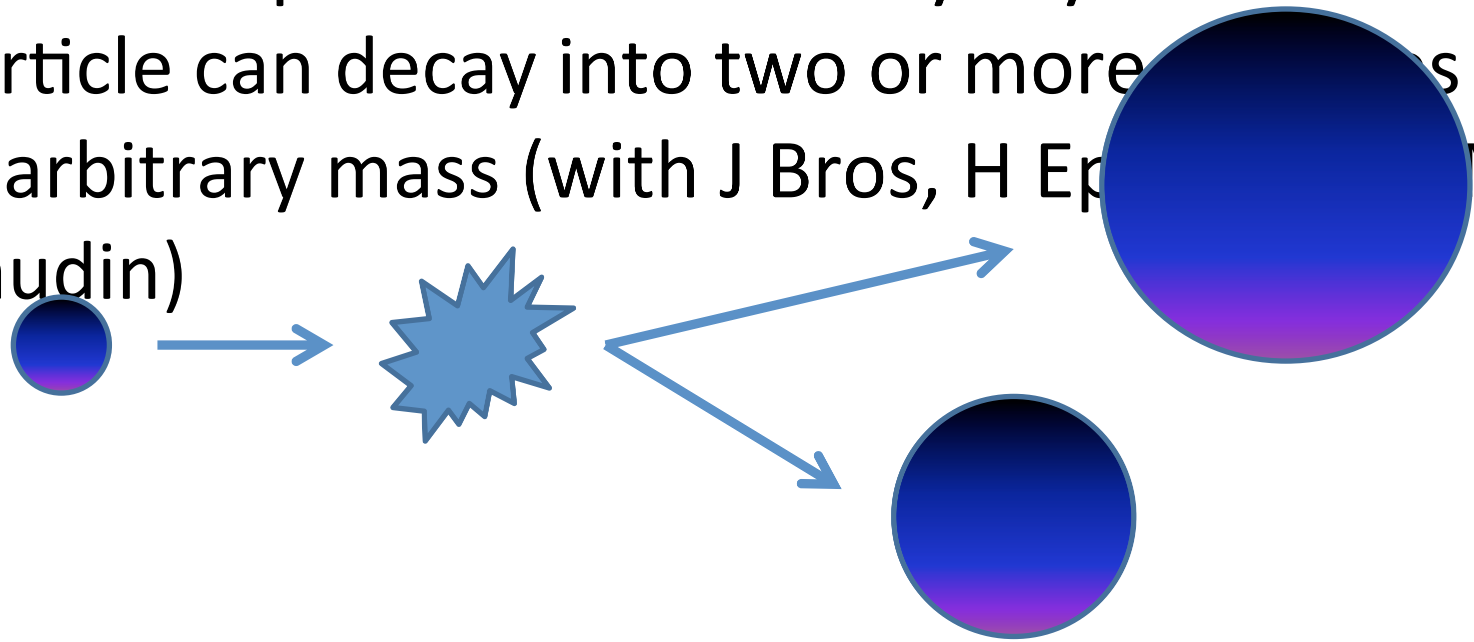
- W. Thirring. Quantum field theory in de Sitter space. Acta Physica Austriaca, suppl. IV, 1967, 269.
- Initially: a mathematical arena to test ideas of QFT on curved backgrounds
- The physical interest in dS QFT increased (exponentially!) starting from the eighties because of the inflationary paradigm.
- Today : dark age … -> dS -> FLRW -> dS (-> …)
- Naively believed to be a simple example of QFT on a curved spacetime while it is plagued by a very difficult infrared problem.

Example: particle decays

- There are no stable particle of mass

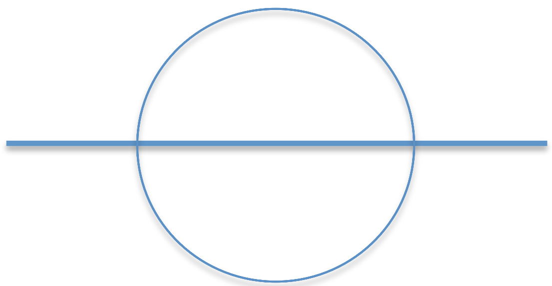
$$m^2 \geq \frac{9}{4R^2}$$

- Tree level perturbation theory says that such particle can decay into two or more [illegible]s of arbitrary mass (with J Bros, H Ep[illegible]M Gaudin)



Instabilities at two loops or more

- Sunset diagram in the ϕ^4 model in the Poincaré patch



$$
\begin{aligned}
\Delta_2 D^K(p\,|\eta_1,\eta_2) = \frac{\lambda^2}{6}\int \frac{d^{D-1}\vec{q}_1}{(2\pi)^{D-1}}\frac{d^{D-1}\vec{q}_2}{(2\pi)^{D-1}}\iint_{+\infty}^{0}\frac{d\eta_3 d\eta_4}{(\eta_3\eta_4)^D}\times \\
\times\Big[3\,D_0^K(p\,|\eta_1,\eta_3)\;D_0^K(q_1|\eta_3,\eta_4)\;D_0^K(q_2\,|\eta_3,\eta_4)\;D_0^A(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^A(p\,|\eta_4,\eta_2)- \\
-\frac{1}{4}\,D_0^K(p\,|\eta_1,\eta_3)\;D_0^A(q_1|\eta_3,\eta_4)\;D_0^A(q_2\,|\eta_3,\eta_4)\;D_0^A(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^A(p\,|\eta_4,\eta_2)- \\
-\frac{3}{4}\,D_0^R(p\,|\eta_1,\eta_3)\;D_0^K(q_1|\eta_3,\eta_4)\;D_0^A(q_2\,|\eta_3,\eta_4)\;D_0^A(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^A(p\,|\eta_4,\eta_2)+ \\
+D_0^R(p\,|\eta_1,\eta_3)\;D_0^K(q_1|\eta_3,\eta_4)\;D_0^K(q_2\,|\eta_3,\eta_4)\;D_0^K(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^A(p\,|\eta_4,\eta_2)- \\
-\frac{3}{4}\,D_0^R(p\,|\eta_1,\eta_3)\;D_0^K(q_1|\eta_3,\eta_4)\;D_0^R(q_2\,|\eta_3,\eta_4)\;D_0^R(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^A(p\,|\eta_4,\eta_2)- \\
-\frac{1}{4}\,D_0^R(p\,|\eta_1,\eta_3)\;D_0^R(q_1|\eta_3,\eta_4)\;D_0^R(q_2\,|\eta_3,\eta_4)\;D_0^R(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^K(p\,|\eta_4,\eta_2)+ \\
+3\,D_0^R(p\,|\eta_1,\eta_3)\;D_0^R(q_1|\eta_3,\eta_4)\;D_0^K(q_2\,|\eta_3,\eta_4)\;D_0^K(|\vec{p}-\vec{q}_1-\vec{q}_2|\;|\eta_3,\eta_4)\;D_0^K(p\,|\eta_4,\eta_2)\Big].
\end{aligned}
$$

- Contains secularly growing contributions
- (needed the Schwinger-Keldysh formalism (work in progress with E. Akhmedov)

What if such difficulties were artifacts of perturbation theory?

- There several approaches to nonperturbative QFT on flat space
- One is the study of exactly soluble two-dimensional models of QFT
- Why not to try and explore soluble (?) two-dimensional models in de Sitter?

Two historically important models

1) The Thirring Model (current-current)

$$i\gamma^\mu \partial_\mu \psi(x) = -gJ^\mu(x)\gamma_\mu \psi(x)$$

$$J^\mu(x) = \overline{\psi}(x)\gamma^\mu \psi(x) \qquad \partial_\mu J^\mu(x) = 0$$

2) Schwinger Model (2-DIM QED)

$$i\gamma^\mu \partial_\mu \psi(x) = -e\gamma^\mu A_\mu(x)\psi(x)$$

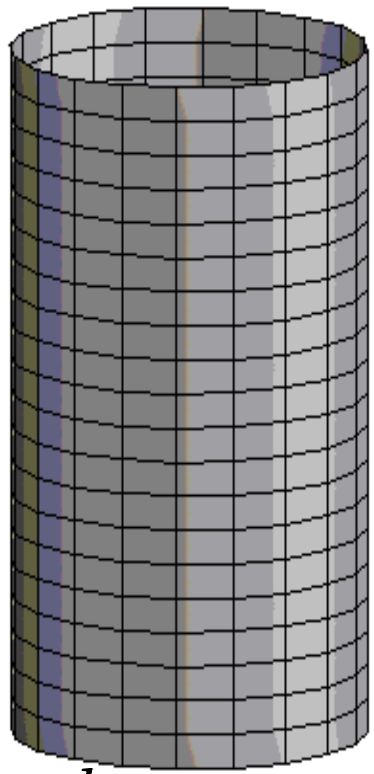
$$\partial^\nu F_{\mu\nu}(x) = -eJ_\mu(x) + \mathcal{A}_\mu(x)$$

3) Are their dS counterparts still soluble?
4) What can be learnt from the solutions?
5) Do they share the same difficulties of perturbation theory?

Two preliminary elementary questions

- What is going to replace the Dirac equation on the de Sitter manifold?
- What is the meaning of de Sitter covariance for spinor fields ?

Minkowskian cylinder



$$x^0 = t \in R, \qquad x^1 = \theta \in [0, 2\pi)$$

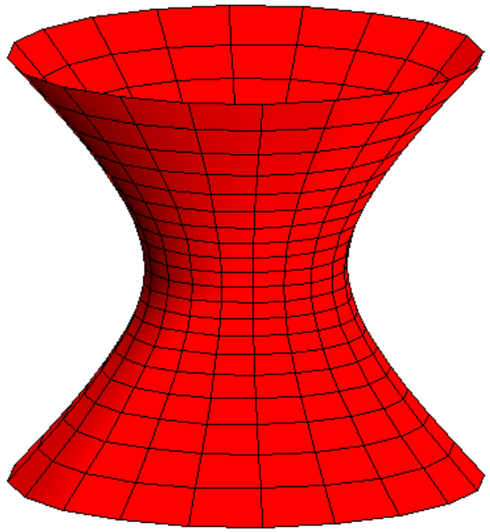
- Metric $$ds^2 = dt^2 - d\theta^2$$

- Clifford algebra $$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}\mathbb{I}.$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

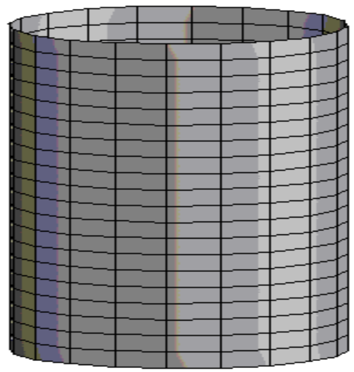
Conformal cylindrical coordinates

$$dS_2 = \{x \in \mathbb{R}^3 : \; (X^0)^2 - (X^1)^2 - (X^2)^2 = -1\}$$



$$\begin{cases} X^0 = r \tan t \\ X^1 = r \sin\theta / \cos t \\ X^2 = r \cos\theta / \cos t \end{cases}$$

$$r = R = 1$$



$$ds^2 = \frac{1}{\cos^2 t}(dt^2 - d\theta^2)$$

Conformal transformation of the spinors

- Given a massless Dirac (classical or quantum) spinor field on the cylinder $i\gamma^a \partial_a \psi = 0$

$$\phi = (\cos t)^{\frac{1}{2}} \psi$$

- ϕ is a (quantum) massless Dirac spinor field on the de Sitter manifold solving the Fock-Ivanenko massless equation:

$$i \cos t \ \gamma^a \partial_a \phi + \frac{i}{2} \sin t \ \gamma^0 \phi = 0$$

Spin bundles on the cylinder

- There are two inequivalent spin bundles.
- Two boundary conditions:
- 1) Periodic (Ramond)

$$\psi(t,\theta) = \psi(t,\theta + 2\pi)$$

- 2) Anti-periodic (Neveu-Schwarz)

$$\psi(t,\theta) = -\psi(t,\theta + 2\pi)$$

- In both cases observables are fully well defined on the cylinder (i.e. they are periodic)

Quantum Spinor field (Ramond)

- Two sets of ladder anticommuting operators acting in a Fock space:

$$\{a_j(p),\ a_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ , \quad \{b_j(p),\ b_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ . \quad j,k=1,2$$

$$\psi_1^{\mathrm{R}}(x) = \psi_1^{\mathrm{R}}(u) = \frac{1}{2\sqrt{\pi}}(a_1^*(0) + b_1(0)) + \frac{1}{\sqrt{2\pi}}\sum_{p>0}(a_1^*(p)e^{ipu} + b_1(p)e^{-ipu}) \qquad u = t + \theta$$

$$\psi_2^{\mathrm{R}}(x) = \psi_2^{\mathrm{R}}(v) = \frac{1}{2\sqrt{\pi}}(a_2^*(0) + b_2(0)) + \frac{1}{\sqrt{2\pi}}\sum_{p>0}(a_2^*(p)e^{ipv} + b_2(p)e^{-ipv}) \qquad v = t - \theta$$

$$w_1(x,\ y) = (\Omega,\ \psi_1(x)\psi_1^*(y)\Omega) = -\frac{i}{4\pi}\cot\left(\frac{u-u'-i0}{2}\right)$$

$$w_2(x,\ y) = (\Omega,\ \psi_2(x)\psi_2^*(y)\Omega) = -\frac{i}{4\pi}\cot\left(\frac{v-v'-i0}{2}\right)$$

Quantum Spinor field (Neveu-Schwarz)

- Same two sets of ladder anticommuting operators acting in a Fock space:

$$\{a_j(p),\ a_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ ,\quad \{b_j(p),\ b_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ . \qquad j,k = 1,2$$

$$\psi_1(x) = \frac{1}{\sqrt{2\pi}}\sum_{p\geq 0}(a_1^*(p)e^{ipu+iu/2} + b_1(p)e^{-ipu-iu/2}), \quad u = t+\theta$$

$$\psi_2(x) = \frac{1}{\sqrt{2\pi}}\sum_{p\geq 0}(a_2^*(p)e^{ipv+iv/2} + b_2(p)e^{-ipv-iv/2}), \quad v = t-\theta$$

$$w_1(x,\ y) = (\Omega,\ \psi_1(x)\psi_1^*(y)\Omega) = \frac{e^{iu/2}}{2\pi}\sum_{p\geq 0} e^{ipu} = \frac{i}{4\pi\sin(\frac{u-u'-i0}{2})}$$

$$w_2(x,\ y) = (\Omega,\ \psi_2(x)\psi_2^*(y)\Omega) = \frac{e^{iu/2}}{2\pi}\sum_{p\geq 0} e^{ipv} = \frac{i}{4\pi\sin(\frac{v-v'-i0}{2})}$$

Conformal transformation of the spinors

- Massless de Sitter Dirac (quantum) spinor field (either Ramond or Neveu-Schwarz)

$$\phi = (\cos t)^{\frac{1}{2}} \psi$$

- What about the de Sitter symmetry?
- There is a priori no reason to expect it. The spinors on the cylinder have less symmetries (space rotations + time translations)!

Another equation by Dirac

- Clifford algebra in the 3-D ambient spacetime

$$ds^2 = (dX^0)^2 - (dX^1)^2 - (dX^2)^2$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\not{X} = \gamma^\mu X_\mu = \begin{pmatrix} -iX_2 & X_0 - X_1 \\ X_0 + X_1 & iX_2 \end{pmatrix}$$

- Generators of the de Sitter (Lorentz) group

$$L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta} = -i(X_\alpha \partial_\beta - X_\beta \partial_\alpha) - \frac{i}{4}[\gamma_\alpha, \gamma_\beta]$$

The de Sitter (Casimir)-Dirac equation

$$Q = -\frac{1}{2} L^{\alpha\beta} L_{\alpha\beta} = \left(\frac{1}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} + i \right)^2 + \frac{1}{4}.$$

- Eigenvalues of the Casimir operator

$$Q\psi = \left(\nu^2 + \frac{1}{4} \right) \psi$$

- First order equation (Dirac 1935 4-dim)

$$\left(\frac{1}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} + i + \nu \right) \psi = 0$$

Solving the Dirac-Dirac equation

$$(iD + i + \nu)\,\psi = 0 \qquad\qquad iD = \frac{1}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta}$$

The crucial identity is

$$(D+1)D = \Box$$

$$(iD + i + \nu)\,(-iD + \nu)\,\chi = (\Box + \mu^2)\chi = 0$$

$$\mu^2 = \nu^2 + i\nu = (-1 + i\nu)(-i\nu)$$

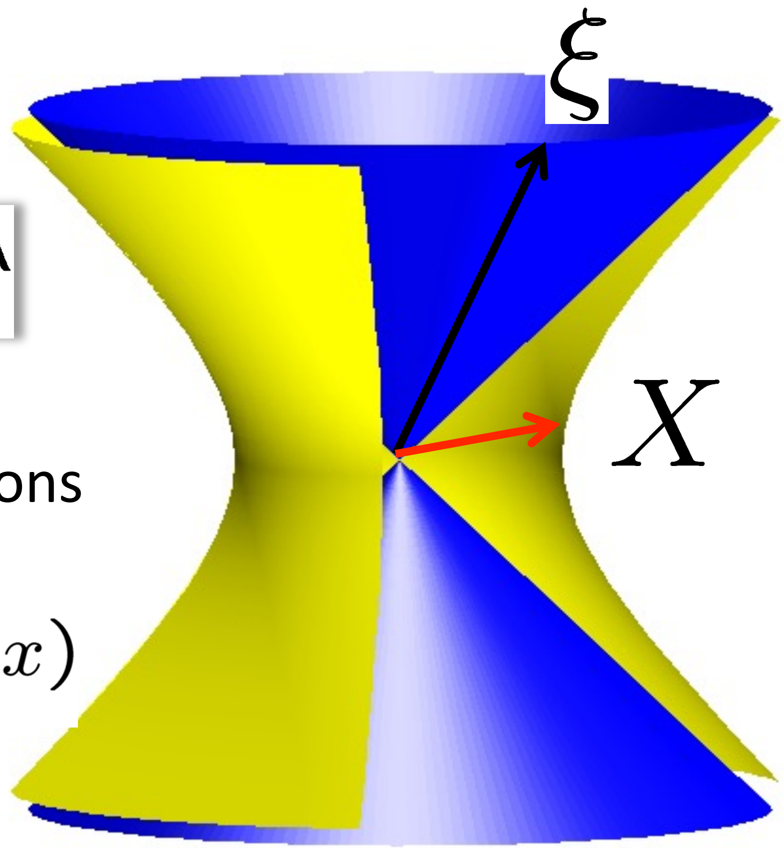
$$\psi = (-iD + \nu)\,\chi$$

de Sitter plane waves

$$X \cdot \xi = X_0\xi_0 - X_1\xi_1 - \ldots - X_d\xi_d$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$

$$\psi_\lambda(X,\xi) = (X \cdot \xi)^\lambda$$

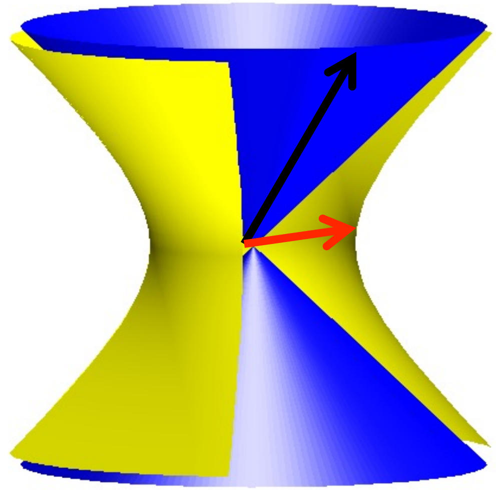


Plane waves are homogeneous functions

$$\psi(x,p) = e^{ip\cdot x} = e^{im(\hat{p}\cdot x)}$$

de Sitter plane waves

$$\Box(X\cdot\xi)^\lambda = \lambda(\lambda+d-1)(X\cdot\xi)^\lambda$$



Involution:

$$\lambda \longrightarrow \bar\lambda = -\lambda-(d-1)$$

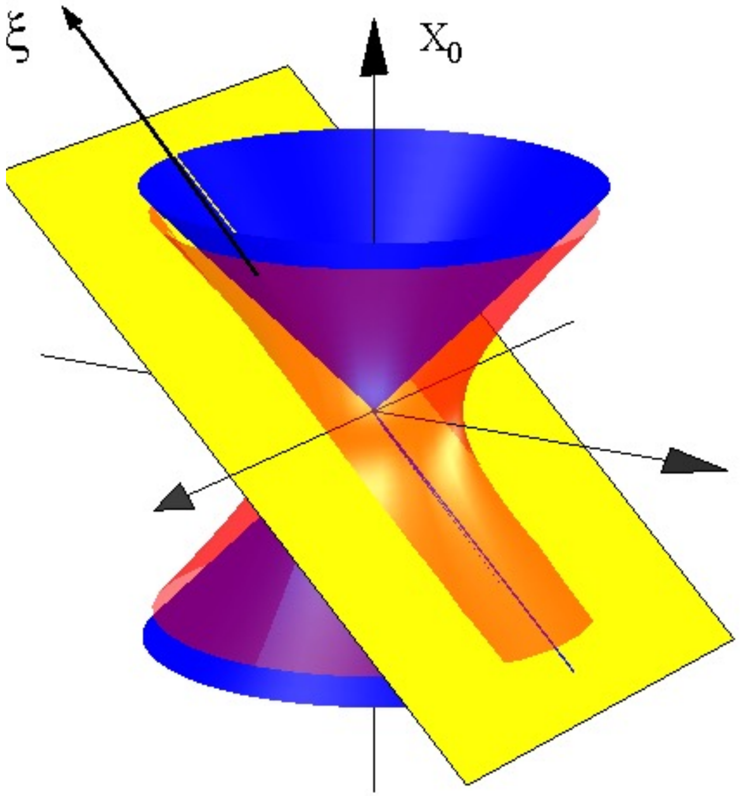
$$\lambda+\bar\lambda = -(d-1)$$

Scalar waves with (complex) squared mass:

$$m^2 = \lambda\bar\lambda$$

$$\left(\Box+\lambda\bar\lambda\right)(X\cdot\xi)^\lambda = 0, \qquad \left(\Box+\lambda\bar\lambda\right)(X\cdot\xi)^{\bar\lambda} = 0$$

The plane waves are however irregular



$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda$$

$$X \in dS : (X\cdot\xi) = 0$$

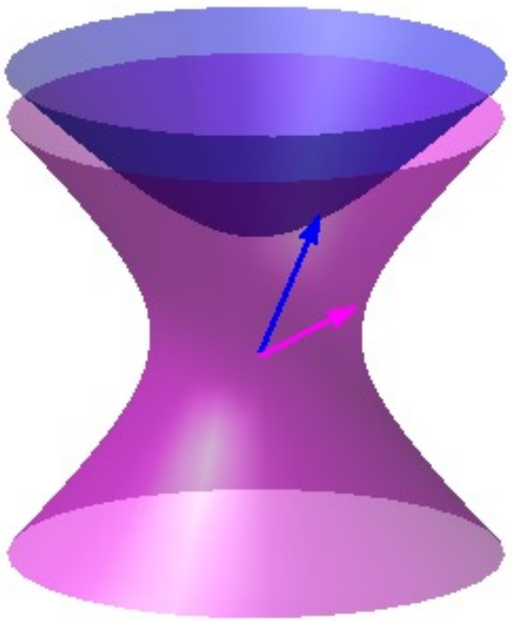
$$(X\cdot\xi)^\lambda \to |X\cdot\xi|^\lambda\,(a(\lambda)\theta(X\cdot\xi) + b(\lambda)\theta(-X\cdot\xi))$$

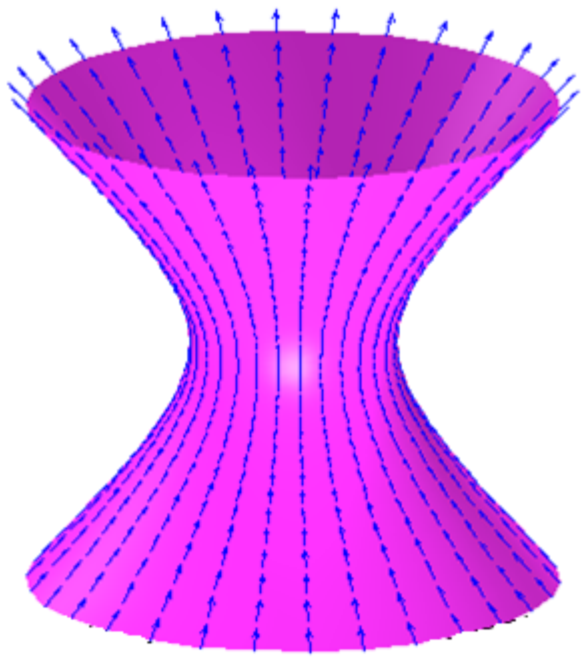
Geometry: de Sitter tubes

$$Z = X + iY, \quad X^2 - Y^2 = -R^2 \quad X \cdot Y = 0$$

$\mathcal{T}^+ = Y$ in the forward cone.

$\mathcal{T}^- = Y$ in the backward cone.



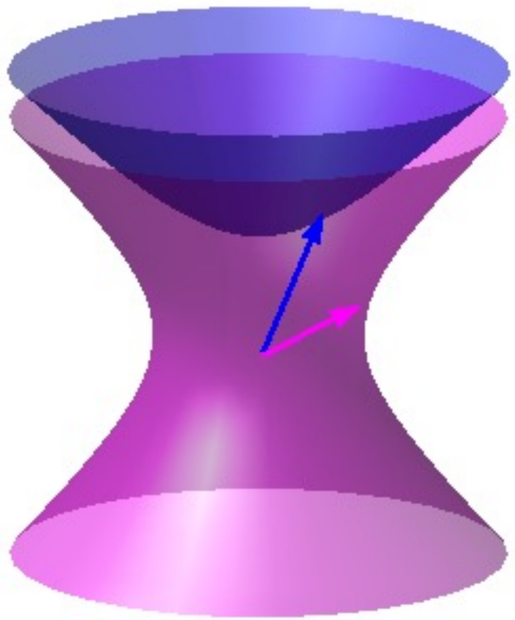


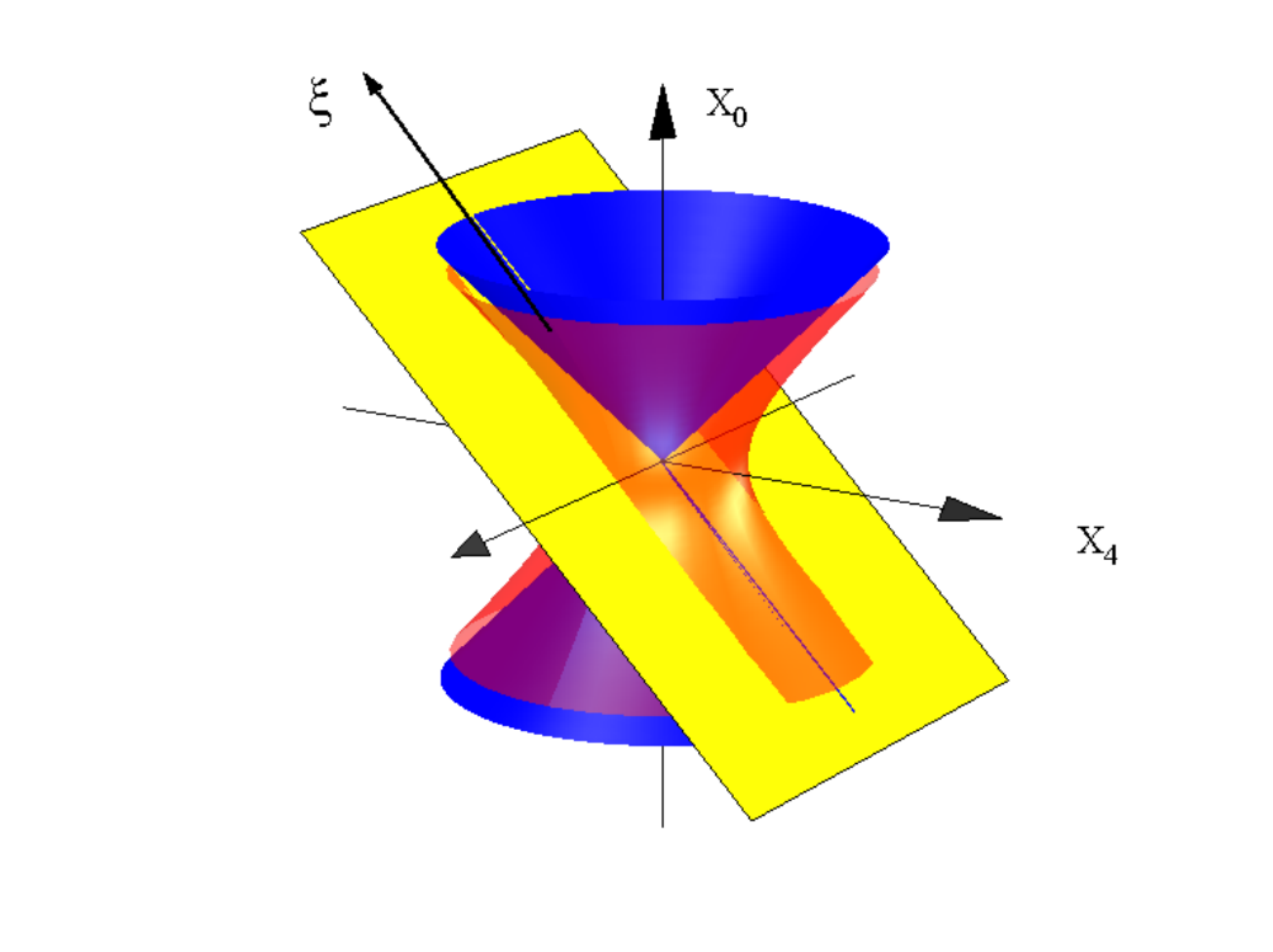
$$(Z \cdot \xi)^{\lambda} = (X \pm iY)^{\lambda}$$

$$Y^2 = (Y^0)^2 - (Y^1)^2 - (Y^2)^2 > 0, \qquad Y^0 > 0$$

$\Im Z \cdot \xi$ is positive for $Z \in \mathcal{T}^{+}$

$\Im Z \cdot \xi$ is negative for $Z \in \mathcal{T}^{-}$



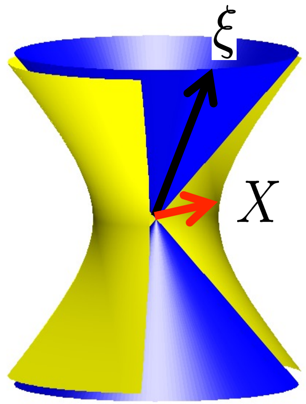


Boundary values on the reals:

$$(X \cdot \xi)_{\pm}^{\lambda} \to |X \cdot \xi|^{\lambda} \Big(\theta(X \cdot \xi) + e^{\pm i\pi\lambda}\theta(-X \cdot \xi)\Big)$$

Solving Dirac's Dirac equation

$$(iD + i + \nu)\,\psi = 0 \qquad iD = \frac{1}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta}$$

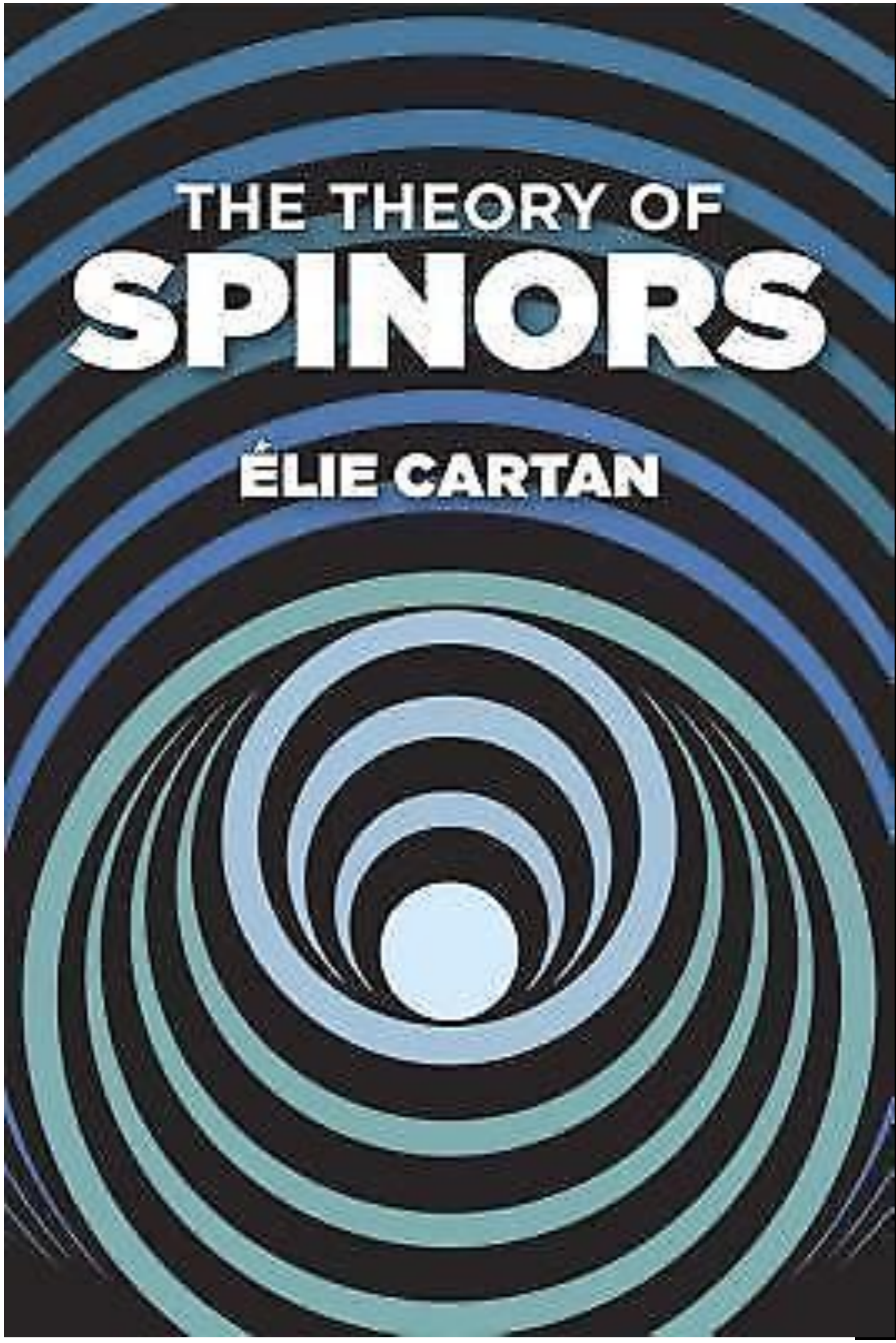


$$\psi(X;\xi) = (X\cdot\xi)^{-1+i\nu}u(\xi)$$

$$\not{\xi}\,u(\xi) = 0 \;= \begin{pmatrix} -i\xi_2 & \xi_0 - \xi_1 \\ \xi_0 + \xi_1 & i\xi_2 \end{pmatrix}\begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_1) \end{pmatrix}$$

$$u(\xi) = \frac{1}{\sqrt{2(\xi^0 - \xi^1)}}\begin{pmatrix} \xi^0 - \xi^1 \\ i\xi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\xi^0 - \xi^1} \\ i\sqrt{\xi^0 + \xi^1} \end{pmatrix}$$

Compare Cartan's definition of a spinor



CHAPTER III

SPINORS IN THREE-DIMENSIONAL SPACE

I. THE CONCEPT OF A SPINOR

52. Definition

Suppose the three-dimensional space E_3 is referred to a system of orthogonal co-ordinates; let (x_1, x_2, x_3) be an isotropic vector, i.e., have zero length. We can associate with this vector, the components of which satisfy

$$x_1^2 + x_2^2 + x_3^2 = 0,$$

two numbers ξ_0, ξ_1 given by

$$x_1 = \xi_0^2 - \xi_1^2,$$

$$x_2 = i(\xi_0^2 + \xi_1^2),$$

and

$$x_3 = -2\xi_0\xi_1.$$

These equations have two solutions given, for example, by the formulae

$$\xi_0 = \pm\sqrt{\frac{x_1 - ix_2}{2}} \quad \text{and} \quad \xi_1 = \pm\sqrt{\frac{-x_1 - ix_2}{2}}.$$

It is not possible to give a consistent choice of sign which will hold for all isotropic vectors in such a manner that the solution varies continuously with the vector. Thus, suppose there is such a choice; start with a definite isotropic vector and suppose it to be continuously rotated round $0x_3$ through an angle α: $x_1 - ix_2$ will be multiplied by $e^{-i\alpha}$, thus by continuity ξ_0 will be multiplied by $e^{-i\alpha/2}$. When the angle of rotation is 2π, the isotropic vector

The two-point function

- Defining the adjoint spinor as usual

$$\overline{u}(\xi) = u^*(\xi)\gamma^0 \qquad u(\xi) \otimes \overline{u}(\xi) = \frac{1}{2}\not{\xi}$$

$$W_\nu(X_1, X_2) = \frac{1}{2} c_\nu \int_\gamma (X_1 \cdot \xi)_-^{-1-i\nu} (X_2 \cdot \xi)_+^{-1+i\nu} \not{\xi} d\sigma(\xi)$$

- In the massless limit $\not{X} = \gamma^\mu X_\mu = \begin{pmatrix} -iX_2 & X_0 - X_1 \\ X_0 + X_1 & iX_2 \end{pmatrix}$

$$W_0(Z_1, Z_2) = \frac{1}{2\pi i} \frac{\not{Z}_1 - \not{Z}_2}{(Z_1 - Z_2)^2}$$

Spin group and de Sitter covariance

$$Sp(1,2) = \{g \in SL(2,C): \quad \gamma^0 g^\dagger \gamma^0 = g^{-1}\}.$$

$$g = \begin{pmatrix} a & i\,b \\ i\,c & d \end{pmatrix} \qquad ad + bc = 1$$

$$Sp(1,2) \text{ is conjugated to } SL(2,R) \text{ in } SL(2,C):$$

$$h = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{pmatrix} \qquad hgh^{-1} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$$

Covering

The covering projection $g \to \Lambda(g)$ of $Sp(1,2)$ onto $SO_0(1,2)$

$$g \to \Lambda(g)^{\alpha}{}_{\beta} = \frac{1}{2}\mathrm{tr}(\gamma^{\alpha} g \gamma_{\beta} g^{-1}) \qquad \Lambda(g) = \Lambda(-g)$$

$$\begin{pmatrix} a & ib \\ ic & d \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}\left(a^2+b^2+c^2+d^2\right) & \frac{1}{2}\left(-a^2+b^2-c^2+d^2\right) & cd-ab \\ \frac{1}{2}\left(-a^2-b^2+c^2+d^2\right) & \frac{1}{2}\left(a^2-b^2-c^2+d^2\right) & ab+cd \\ ac-bd & -ac-bd & ad-bc \end{pmatrix}$$

$Sp(1,2)$ acts on the de Sitter manifold by similarity

$$\not{X} = \gamma^{\mu} X_{\mu} = \begin{pmatrix} -iX_2 & X_0 - X_1 \\ X_0 + X_1 & iX_2 \end{pmatrix} \qquad \not{X}' = g \not{X} g^{-1}$$

$$\not{X}' = g \not{X} g^{-1} = \Lambda(g) X$$

dS covariance of Dirac's Dirac field

$$\psi'(Z) = g\psi\big(\Lambda^{-1}(g)Z\big)$$

$$gW_0(Z_1, Z_2)g^{-1} = \frac{1}{2\pi i}\frac{g\,\not{Z}_1 g^{-1} - g\,\not{Z}_2 g^{-1}}{(Z_1 - Z_2)^2} = \frac{1}{2\pi i}\frac{\not{\Lambda(g)Z_1} - \not{\Lambda(g)Z_2}}{(Z_1 - Z_2)^2}.$$

$$gW_\nu\big(\Lambda^{-1}(g)X_1, \Lambda^{-1}(g)X_2\big)g^{-1} =$$

$$= \frac{1}{2}c_\nu \int_\Gamma \big(\Lambda^{-1}(g)X_1 \cdot \xi\big)^{-1+i\nu}\big(\Lambda^{-1}(g)X_2 \cdot \xi\big)^{-1-i\nu} g\not{\xi}g^{-1} d\mu(\xi)$$

$$= \frac{1}{2}c_\nu \int_\Gamma \big(X_1 \cdot \Lambda(g)\xi\big)^{-1+i\nu}\big(X_2 \cdot \Lambda(g)\xi\big)^{-1-i\nu} \gamma^\alpha(\Lambda(g)\xi)_\alpha d\mu(\xi)$$

$$= W_\nu\big(X_1, X_2\big)$$

What is the relation between the two Dirac's equation?

$$\phi(t,\theta) = A(t,\theta)\psi(t,\theta)$$

Fock Ivanenko   A matrix   Dirac's Dirac

There are unexpected subtelties in the answer

The symmetric space Sp(1,2)/A

- Iwasawa decomposition $g = \begin{pmatrix} a & i\,b \\ i\,c & d \end{pmatrix} \qquad ad + bc = 1$

$$g = k(\zeta)\,n(\lambda)\,a(\chi) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{pmatrix} \begin{pmatrix} 1 & i\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}} \end{pmatrix};$$

$$\cos\frac{\zeta}{2} = \frac{a}{\sqrt{a^2+c^2}}, \quad \sin\frac{\zeta}{2} = \frac{c}{\sqrt{a^2+c^2}}, \quad \lambda = ab - cd, \quad e^{\frac{\chi}{2}} = \sqrt{a^2+c^2}$$

where $0 \le \zeta < 4\pi$ and λ and χ are real.

- Parametrization of the coset space Sp(1,2)/A

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

Group action

- Sp(1,2) acts on the coset space by left multiplication:

$$g\,\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

Rotation (K)

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

$$k(a) = \begin{pmatrix} \cos\frac{a}{2} & i\sin\frac{a}{2} \\ i\sin\frac{a}{2} & \cos\frac{a}{2} \end{pmatrix}$$

$$k(\alpha)\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\boxed{\lambda'(\alpha) = \lambda, \qquad \zeta'(\alpha) = \zeta + \alpha.}$$

Boost (A)

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

$$a(\kappa) = \begin{pmatrix} e^{\frac{\kappa}{2}} & 0 \\ 0 & e^{-\frac{\kappa}{2}} \end{pmatrix}$$

$$a(\kappa)\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\begin{cases} \lambda'(\kappa) = \lambda\cosh\kappa + \sinh\kappa(\lambda\cos\zeta + \sin\zeta), \\ \cot\frac{\zeta'(\kappa)}{2} = e^{\kappa}\cot\frac{\zeta}{2}. \end{cases}$$

Lightlike Boost (N)

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\, n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

$$n(\mu) = \begin{pmatrix} 1 & i\mu \\ 0 & 1 \end{pmatrix}$$

$$n(\mu)\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\begin{cases} \lambda'(\mu) = \lambda\left(1+\frac{1}{2}\mu^2\right) - \mu\left(\lambda+\frac{\mu}{2}\right)\sin\zeta + \mu\left(1-\frac{1}{2}\lambda\mu\right)\cos\zeta, \\ \cot\frac{\zeta'(\mu)}{2} = \cot\frac{\zeta}{2} - \mu. \end{cases}$$

Maureer-Cartan metric

- The Maureer-Cartan form $dg\, g^{-1}$ gives to the symmetric space Sp(1,2)/A a natural Lorentzian metric
- There exists a inner automorphism of Sp(1,2) that leaves A invariant

$$g \to \mu(g) = -\gamma^2 g \gamma^2 \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

- It may be used to construct a map from the coset space Sp(1,2)/A into the group Sp(1,2) and an induced Lorentzian metric on Sp(1,2)/A

$$g(\tilde{X}) = g\mu(g)^{-1} = -\tilde{X}\gamma^2\tilde{X}^{-1}\gamma^2.$$

$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\, g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

Maureer-Cartan metric

$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\, g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

1. The metric is invariant under the group left action
2. The curvature is constant (R=-2) and the Ricci tensor is proportional to the metric:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu} = 0$$

3. The map

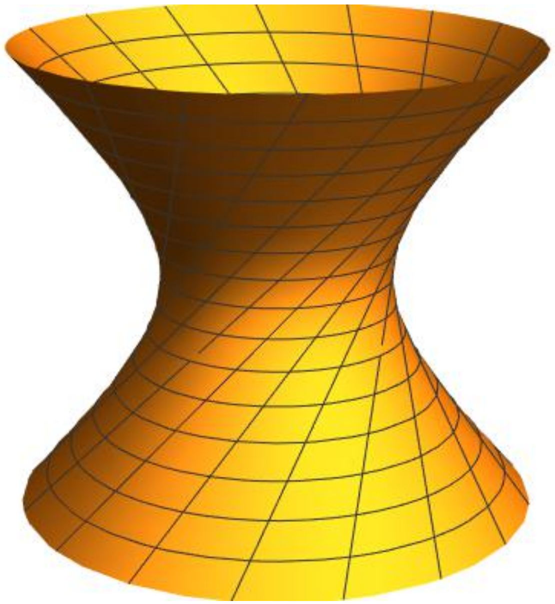
$$p : \tilde{X}(\lambda,\zeta) \to X(\lambda,\zeta) = \begin{cases} X^0 = -\lambda \\ X^1 = \lambda\cos\zeta + \sin\zeta \\ X^2 = \cos\zeta - \lambda\sin\zeta \end{cases}$$

is a covering map.

$$ds^2 = \left(dX^{0^2} - dX^{1^2} - dX^{2^2}\right)\Big|_{dS_2} = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

de Sitter

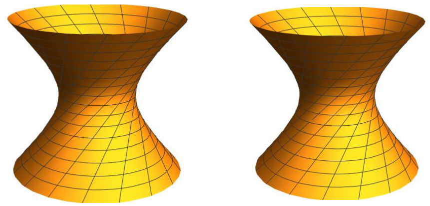
$$\begin{cases} X^0 = -\lambda \\ X^1 = \lambda \cos\zeta + \sin\zeta \\ X^2 = \cos\zeta - \lambda \sin\zeta \end{cases}$$



$$ds^2 = \left(dX^{0\,2} - dX^{1\,2} - dX^{2\,2}\right)\Big|_{dS_2} = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

Double covering of de Sitter

- In conclusion: the symmetric space



$$Sp(1,2)/A = \widetilde{dS_2}$$

may be identified with the double covering of the two dimensional de Sitter universe.

- The spin group Sp(1,2) acts directly on the covering space as a group of spacetime transformations:

$$\tilde{X} \to g\tilde{X}$$

- We were not able to find the above identification in the (enormous) literature on the group SL(2,R).

Gursey and Lee's trick

$$\{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = g^{ij}$$

- There should (more than one) matrix *S* such that

$$\alpha^i = S\beta^i S^{-1}$$

- The solution only exists on the covering manifold. The most convenient choice

$$S(t,\theta) = \frac{1}{\sqrt{\cos t}} \begin{pmatrix} \cos\frac{t-\theta}{2} & i\sin\frac{t-\theta}{2} \\ -i\sin\frac{t+\theta}{2} & \cos\frac{t+\theta}{2} \end{pmatrix}$$

Gursey and Lee's trick

- What is the relation between the two Dirac's equation? Introduce the matrices β:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\beta^\mu = \frac{\partial y^\mu}{\partial X^\nu}\gamma^\nu, \quad y^\mu = (t, \theta, r)$$

$$\begin{cases} X^0 = r\tan t \\ X^1 = r\sin\theta/\cos t \\ X^2 = r\cos\theta/\cos t \end{cases}$$

$$\{\beta^i, \beta^j\} = g^{ij}, \;\; i, j = 0, 1$$

$$\beta^2 = -\frac{\not{X}}{r}, \qquad \{\beta^i, \beta^2\} = 0$$

Dressing

$$S(t,\theta) = \frac{1}{\sqrt{\cos t}} \begin{pmatrix} \cos\frac{t-\theta}{2} & i\sin\frac{t-\theta}{2} \\ -i\sin\frac{t+\theta}{2} & \cos\frac{t+\theta}{2} \end{pmatrix}$$

- Given a solution Ψ of the Dirac Dirac Equation the dressed spinor

$$\phi(t,\theta) = \frac{1}{\sqrt{2}} f(t,\theta) S(t,\theta)(1 - \not{X})\Psi(t,\theta)$$

- Solves the Fock-Ivanenko-Dirac equation

$$i\alpha^t\left(\partial_t + \Gamma_t\right)\phi + i\alpha^\theta\left(\partial_\theta + \Gamma_\theta\right)\phi - i\alpha^i(\partial_i \ln f)\phi - \nu\phi = 0$$

Remarks

$$S(t,\theta) = \frac{1}{\sqrt{\cos t}} \begin{pmatrix} \cos\frac{t-\theta}{2} & i\sin\frac{t-\theta}{2} \\ -i\sin\frac{t+\theta}{2} & \cos\frac{t+\theta}{2} \end{pmatrix}$$

- The matrix S is anti-periodic well-defined only on the double covering of the de Sitter hyperboloid.

- The map $(t,\theta) \to S(t,\theta)$ is thus a map from the double covering of the de Sitter spacetime with values in the spin group Sp(1,2)

- The dressing changes periodicities: periodic (R) fields become anti-periodic (NS) and viceversa.

Answer to the second question

- Dress the DD field and get a quantum field solving the ϕ_ν standard Dirac-Fock-Iwanenko) equation.
- The dressed field has NS antiperiodicity and therefore well-defined only on the covering of the de Sitter manifold

$$\psi'(X) = g\psi\big(\Lambda^{-1}(g)X\big)$$

$$\phi'(\tilde{X}) = \Sigma(g, \tilde{X})\, \phi(g^{-1}\tilde{X}),$$

$$\Sigma(g, \tilde{X}) = S(\tilde{X})\, g\, S(g^{-1}\tilde{X})^{-1}$$

- $\Sigma(g, \tilde{X})$ is a nontrivial cocyle of Sp(1,2)

$$\Sigma(g_1, \tilde{X})\Sigma(g_2, g_1^{-1}\tilde{X}) = \Sigma(g_1 g_2, \tilde{X}).$$

Cocyclic de Sitter Covariance

- The de Sitter covariance of the de Sitter FI Dirac NS field is thus expressed in terms of a cocycle.

$$\phi'(\tilde{X}) = \Sigma(g, \tilde{X})\, \phi(g^{-1}\tilde{X}),$$

$$\Sigma(g_1, \tilde{X})\Sigma(g_2, g_1^{-1}\tilde{X}) = \Sigma(g_1 g_2, \tilde{X}).$$

- On the other hand there is no covariant Dirac field (in the above sense) in the Ramond sector.
- The following remarkable result play an important role in the construction of the de Sitter - Thirring model :

For any g in the spin group $Sp(1,2)$ the cocycle $\Sigma(g, \tilde{X})$ is diagonal.

In the end: massless NS Spinors have a hidden de Sitter symmetry

$$\frac{S(t,\theta)(\Omega, \Psi_0(t,\theta)\overline{\Psi}_0(t',\theta')\Omega)S(t',\theta')^{-1}}{\sqrt{\cos t}\sqrt{\cos t'}} =$$

$$= -\frac{i}{4\pi}\begin{pmatrix} 0 & \frac{1}{\sin\left(\frac{1}{2}(u-u')\right)} \\ \frac{1}{\sin\left(\frac{1}{2}(v-v')\right)} & 0 \end{pmatrix}$$

$$\Psi_0(t,\theta) = \sqrt{\cos t}\, S(t,\theta)^{-1}\psi^{\mathrm{NS}}(t,\theta).$$

In the very end: the Thirring field

- Field Equation $i\alpha^\mu(\partial_\mu + \Gamma_\mu)\psi = -g\alpha^\mu J_\mu \psi$
- Solution $\phi(x) = e^{i\chi^+(x)}\phi_0(x)e^{i\chi^-(x)}$.

$$\chi_1^\pm(x) = \alpha j^\pm(x) - \beta\tilde{j}^\pm(x) + a_1^\pm(x)Q_1 + b_1^\pm(x)Q_2 \ ,$$
$$\chi_2^\pm(x) = \alpha j^\pm(x) + \beta\tilde{j}^\pm(x) + a_2^\pm(x)Q_2 + b_2^\pm(x)Q_1 \ .$$

- Under certain conditions it is possible to find local and de Sitter covariant solutions

Perspectives

- Opens the way to he study of integrable QFT models on the de Sitter manifold
- Based on work in progress with Henri Epstein (IHES) and Emil Akhmedov (ITEP)