Coherent states in a study of time problem

Przemysław Małkiewicz
National Centre for Nuclear Research, Warszawa

Coherent states and their applications:

A contemporary panorama

Nov 13-18, 2016, Marseille

Outline

GOAL OF THE TALK: study the concept of quantum dynamics in Hamiltonian constraint systems

1. Hamiltonian constraint and internal clocks
2. Toy model of gravity
3. Quantum evolution with internal clocks

HAMILTONIAN CONSTRAINT AND INTERNAL CLOCKS

Hamiltonian constraint formalism

Consider $2n$-dimensional phase space, $\omega = \mathrm{d}q_i \mathrm{d}p^i$. Physical states are constrained by the Hamiltonian $C(q_i, p^i) = 0$:

$$\frac{\partial O}{\partial \tau} = \{O, C\}, \quad C = 0$$

Now, consider the Hamiltonian $\mathcal{N} \cdot C(q_i, p^i)$, $\mathcal{N}(q_i, p^i, \tau) \neq 0$:

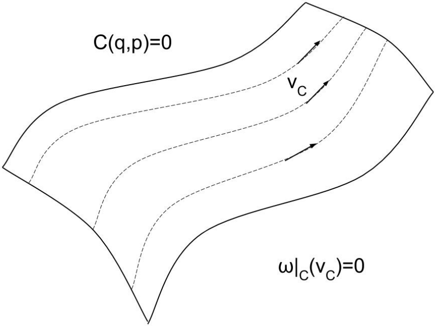
$$\frac{\partial O}{\partial \tau'} = \{O, \mathcal{N} \cdot C\}, \quad \mathcal{N} \cdot C = 0$$

$$\Rightarrow \quad \mathrm{d}\tau = \mathcal{N} \mathrm{d}\tau'$$

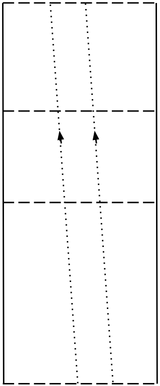
$\rightarrow$ Constraint formulation enforces time reparameterization - invariance. *Physically, it reflects the lack of fixed external time*
$\rightarrow$ Canonical relativity involves a Hamiltonian constraint that generates the dynamics of three-geometries. *There are ∞-many ways to slice a given spacetime into a family of three-geometries*

Geometric perspective

$\rightarrow$ $2n$ - dimensional extended phase space, $\omega = \mathrm{d}q_i \mathrm{d}p^i$
$\rightarrow$ $2n-1$ - dimensional constraint surface, $C(q_i, p^i) = 0$
$\rightarrow$ Physics encoded entirely in $\omega|_{C=0}$ induced from ω on the constraint surface $C = 0$ because $\omega|_{C=0}$ is degenerate and its null vector v_C is tangential to the physical motion



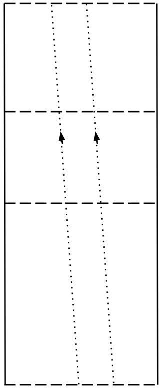
Reduction to unconstrained formalism



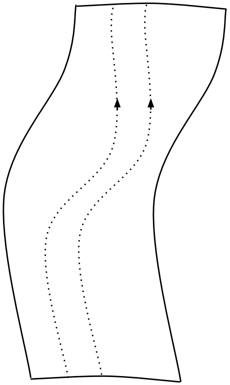
$(\mathbf{t}, q, p) \in \mathbf{R}^1$ x phase space

$\boldsymbol{\omega} = dq \wedge dp - dt \wedge dh$

Reduction to unconstrained formalism

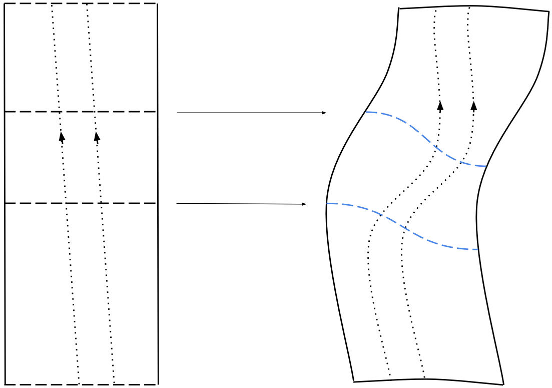


(**t**,q,p) $\in$ $\mathbf{R}^1$ x phase space
$\boldsymbol{\omega} = dq \wedge dp - dt \wedge dh$



constraint surface C=0
$\boldsymbol{\omega}|_{C=0}$

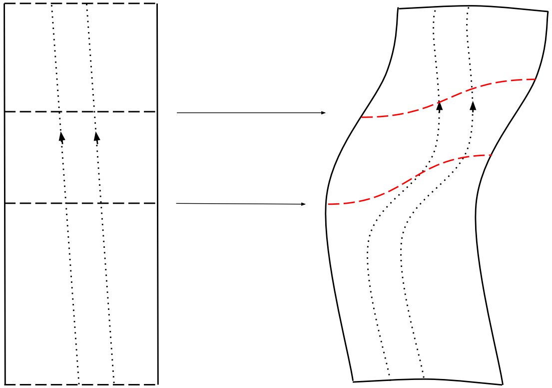
Reduction to unconstrained formalism



$(\mathbf{t},q,p) \in \mathbf{R^1}$ x phase space
$\boldsymbol{\omega} = dq \wedge dp - dt \wedge dh$

constraint surface C=0
$\boldsymbol{\omega}|_{C=0} = dq \wedge dp - dt \wedge dh$

Reduction to unconstrained formalism



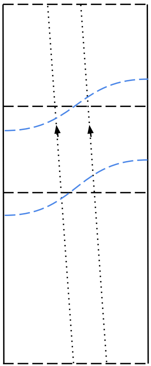
$(\mathbf{t}, q, p) \in \mathbf{R}^1 \times$ phase space
$\boldsymbol{\omega} = dq \wedge dp - dt \wedge dh$

constraint surface C=0
$\boldsymbol{\omega}|_{C=0} = dq \wedge dp - dt \wedge dh$

Clock transformations

Contact transformations:
$(q^I, p_I, t) \mapsto (\bar{q}^I, \bar{p}_I)$
such that:
$\omega_{\mathcal{C}} = \mathrm{d}q^I \mathrm{d}p_I - \mathrm{d}t\mathrm{d}h = \mathrm{d}\bar{q}^I \mathrm{d}\bar{p}_I - \mathrm{d}t\mathrm{d}\bar{h}$

Clock transformations:
$(q^I, p_I, t) \mapsto (\bar{q}^I, \bar{p}_I, \bar{t})$
such that:
$\omega_{\mathcal{C}} = \mathrm{d}q^I \mathrm{d}p_I - \mathrm{d}t\mathrm{d}h = \mathrm{d}\bar{q}^I \mathrm{d}\bar{p}_I - \mathrm{d}\bar{t}\mathrm{d}\bar{h}$

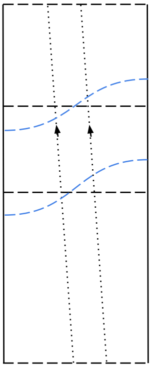


R[1] x phase space

Clock transformations

Contact transformations:
$(q^I, p_I, t) \mapsto (\bar{q}^I, \bar{p}_I)$
such that:
$\omega_{\mathcal{C}} = \mathrm{d}q^I \mathrm{d}p_I - \mathrm{d}t\mathrm{d}h = \mathrm{d}\bar{q}^I \mathrm{d}\bar{p}_I - \mathrm{d}t\mathrm{d}\bar{h}$

Clock transformations:
$(q^I, p_I, t) \mapsto (\bar{q}^I, \bar{p}_I, \bar{t})$
such that:
$\omega_{\mathcal{C}} = \mathrm{d}q^I \mathrm{d}p_I - \mathrm{d}t\mathrm{d}h = \mathrm{d}\bar{q}^I \mathrm{d}\bar{p}_I - \mathrm{d}\bar{t}\mathrm{d}\bar{h}$



$\mathbf{R^1}$ x phase space

$\rightarrow$ Clock transformations form a group $\mathcal{G}_{clock}$ with contact transformations $\mathcal{G}_{can}$ as its normal subgroup $\Rightarrow$ fibre bundle $\pi: \ \mathcal{G}_{clock} \rightarrow T$ over a space of internal clocks T with contact transformations $\mathcal{G}_{can}$ as a fibre.

A particularly useful section in $\mathcal{G}_{clock}$

Consider a section:

$$\sigma : T \ni t \mapsto (q, p, t) \in \mathcal{G}_{clock}$$

such that

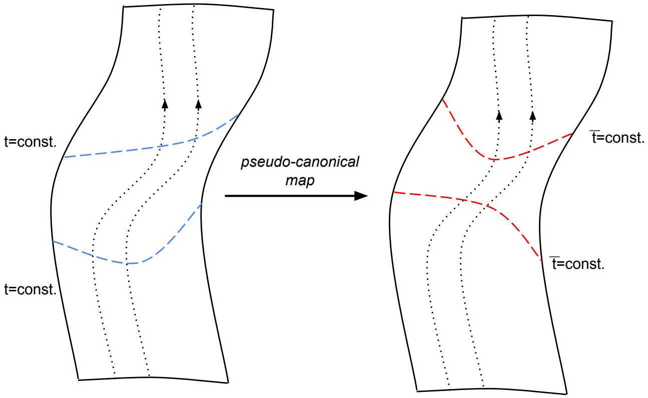
$$\mathrm{d}q\mathrm{d}p - \mathrm{d}t\mathrm{d}H(q, p) = \mathrm{d}\bar{q}\mathrm{d}\bar{p} - \mathrm{d}\bar{t}\mathrm{d}H(\bar{q}, \bar{p})$$

where $H(\cdot, \cdot)$ is preserved by solving $2n + 1$ algebraic equations:

$$\bar{T} = \bar{T}(T, q, p),\ \ C_I(T, q, p) = C_I(\bar{T}, \bar{q}, \bar{p}),\ \ I = 1, \ldots, 2n$$

$\rightarrow$ It is enough to quantize a single canonical framework and later switch to another interpretation of the basic variables to have a quantum theory in another clock. If $f(q, p) \mapsto \mathcal{A}_f(\mathbb{H})$, then $f(\bar{q}, \bar{p}) \mapsto \mathcal{A}_f(\mathbb{H})$
$\rightarrow$ Quantization is unique: constants of motion are given a unique quantum representation in all clocks. Any dissimilarities btw the quantum descriptions are due to different choices of clock.
Unitarity vs. pseudounitarity

Switching to another interpretation of the basic variables



TOY MODEL OF GRAVITY

Classical model

Consider a spacetime $\mathcal{M} = \Sigma \times \mathbb{R}$ equipped with

$$ds^2 = -N^2 dt^2 + q^{\alpha} \mathrm{d}x^i \mathrm{d}x^i$$

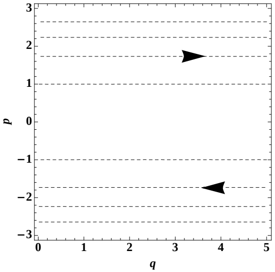
and filled with barotropic fluid "$\mathrm{p} = w\rho$". The Hamiltonian constraint reads

$$C = p_T + p^2$$

Solve the constraint wrt p_T:

$$\omega\Big|_{C=0} = \mathrm{d}q\mathrm{d}p + \mathrm{d}T\mathrm{d}p_T\Big|_{p_T=-p^2} = \mathrm{d}q\mathrm{d}p - \mathrm{d}T\mathrm{d}H_T$$

where $H_T = p^2$.



(1) T is an internal clock (2) dynamics is incomplete.

Affine quantization

The symmetry of the half-plane is the affine group.

$$(q,p)\cdot(q_0,p_0) = \left(qq_0, \frac{p_0}{q}+p\right),\ q\in\mathbb{R}_+^*,\ p\in\mathbb{R},$$

q - dilation, p - translation, $\mathrm{d}p\mathrm{d}q$ - left-invariant measure. There exists UIR.

Given a normalized vector $\psi_0 \in \mathcal{L}^2(\mathbb{R}_+^*, \mathrm{d}x)$, a continuous family of unit vectors are defined as

$$|q,p\rangle = U(q,p)|\psi_0\rangle\,, \quad \langle x|q,p\rangle = e^{i\mathfrak{p}x}\frac{1}{\sqrt{\mathfrak{q}}}\psi_0(x/\mathfrak{q})\,.$$

The resolution of unity guaranteed via Schur's Lemma

$$\int_{\Pi_+} \frac{\mathrm{d}q\mathrm{d}p}{2\pi a_P}|q,p\rangle\langle q,p| = c_{-1}\cdot 1$$

Quantization:

$$\boxed{f \mapsto A_f = \int_{\Pi_+} \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}}\, f(q,p)\,|q,p\rangle\langle q,p|}$$

Quantum model

The quantization of coordinate functions reads:

$$q \mapsto \frac{c_0(\psi_0)}{c_{-1}(\psi_0)} Q\,, \quad p \mapsto P\,,$$

where Q and P are position and momentum operators (on $\mathbb{R}_+^*$).

The quantization of the kinetic term reads:

$$p^2 \mapsto P^2 + \frac{K(\psi_0)}{Q^2}\,,$$

for $K \geqslant 3/4$ the above is self-adjoint.

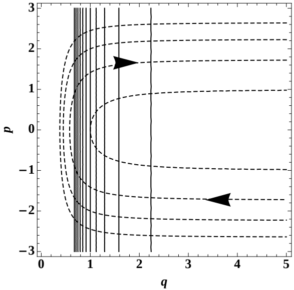
Phase space portrait

Minimization of the quantum action: $S_Q = \int \mathrm{d}T \langle \Psi | i \frac{\partial}{\partial T} - \hat{H}_T | \Psi \rangle$
Confinement to the 2-d submanifold $|\Psi\rangle = |q, p\rangle$
The approximate motion given by the Hamilton eqs with

$$\check{H}_T(q,p) := \langle q,p|\hat{H}_T|q,p\rangle = \int_{\Pi_+} \frac{\mathrm{d}p'\mathrm{d}q'}{2\pi c_{-1}} |\langle q,p|q',p'\rangle|^2 H_T(q',p')$$

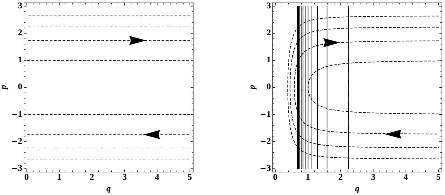
(For consistency $\check{q} = q$, $\check{p} = p$.) We find:

$$\check{H}_T = p^2 + \frac{K}{q^2}$$



QUANTUM EVOLUTION WITH INTERNAL CLOCKS

Switching between clocks



$$H_T = p^2 \qquad\qquad H_{sem} = p^2 + Kq^{-2}$$

Use the delay function D:

$$T \to \bar{T} = T + D(q,p)$$

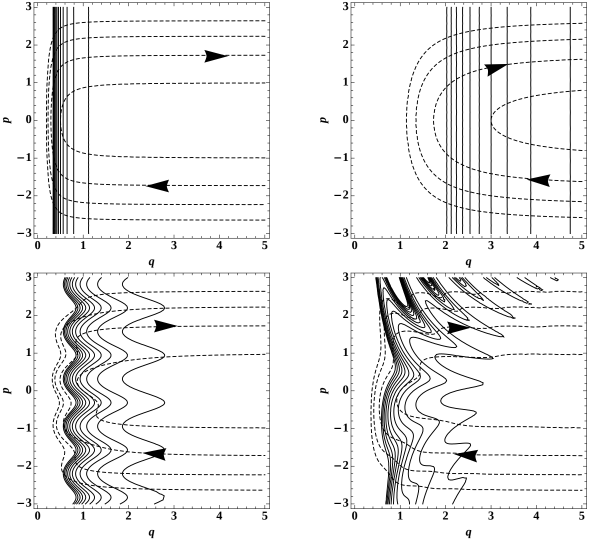
$$\bar{q} = q + 2pD, \quad \bar{p} = p$$

$$\mathrm{d}q\mathrm{d}p - \mathrm{d}T\mathrm{d}p^2 = \omega\big|_{C=0} = \mathrm{d}\bar{q}\mathrm{d}\bar{p} - \mathrm{d}\bar{T}\mathrm{d}\bar{p}^2$$

Semiclassical dissimilarities (as viewed in **the initial phase space):**

$$\bar{H}_{sem} = \bar{p}^2 + \frac{K}{\bar{q}^2} = p^2 + \frac{K}{(q+2pD)^2}$$

Clock effect



How to reconcile it with Quantum Mechanics

→ The idea is to have a set of *classical* internal variables which provide clocks for describing some *quantum* internal degrees of freedom. Clock transformations within the classical variables do not lead to the clock effect.
→ Consider a quantum Hamiltonian constraint system with (q_1, p_1) treated semiclassically with $|q_1, p_1\rangle$ and the rest given in $|\phi(q_2, \dots)\rangle$.

$$\begin{aligned}
\dot{q}_1 &= N \langle\phi|\, \partial_{p_1} \hat{\mathsf{H}}^{\mathrm{semi}}(q_1, p_1)\, |\phi\rangle \\
\dot{p}_1 &= -N \langle\phi|\, \partial_{q_1} \hat{\mathsf{H}}^{\mathrm{semi}}(q_1, p_1)\, |\phi\rangle \\
-i\, \partial_\tau\, |\phi\rangle &= N\, \hat{\mathsf{H}}^{\mathrm{semi}}(q_1, p_1)\, |\phi\rangle \\
0 &= \langle\phi|\, \hat{\mathsf{H}}^{\mathrm{semi}}(q_1, p_1)\, |\phi\rangle
\end{aligned}$$

$$N = N(\tau, q_1, p_1)$$

Conclusions

- Hamiltonian constraint systems involve internal clocks for describing evolution
- *Dynamical* predictions of the quantized models are tied to the choice of internal clock. In particular some predictions like the scale of the bounce, spectra of dynamical operators, ... appear unphysical in this light.
- The *asymptotic* semiclassical states provide a restricted domain for making clock-independent predictions. In particular the choice of clock should be irrelevant for making prediction for observables away from the bounce like primordial power spectra of cosmological perturbations.
- Suppose classical dynamical environments are assumed. *Conjecture:* Quantum mechanics can be obtaind from quantum models with internal clocks in this case.