Representations of Infinite Coherent States*

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CMP '16

CIRM 14-18/11/16

Coherent States

Schrödinger '26
Glauber '60's
Klauder '60's

- Quantum States behaving in many respects Classically

- Particularly suited for 1/2-Classical Analysis, Quantum Optics, Signal Processing, Quantum Information,...

- Photons radiated by classical current sources are in Coherent States, for each mode.

- E-M field produced by LASERS are Coherent States

Goals

1) Construct a reservoir of infinitely many coherent states

- Infinite vol. limit, fixed particle density "à la" Araki-Woods '63

   A. Fixed number N of modes

   B. Continuous density distribution $\rho(k)$ of modes

   => implies random phases

2) Representations of the Infinite Coherent States (ICS)

3) Analyze the dynamics of small systems in contact with ICS

Setup

Non-interacting particles in a box $\Lambda = [-L/2, L/2]^d \subset \mathbb{R}^d$

- Single particle: $L^2(\Lambda, \mathrm{d}x)$ with Period. Bdry. Cond.

- Fourier: $f \in L^2(\Lambda, \mathrm{d}x) \xrightarrow{\mathfrak{F}} l^2(\frac{2\pi}{L}\mathbb{Z}^d) \ni \widehat{f}$

$$\widehat{f}_k = L^{-d/2} \int_\Lambda \mathrm{e}^{-\mathrm{i}kx} f(x) \mathrm{d}x \quad , \qquad f(x) = L^{-d/2} \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}kx} \widehat{f}_k$$

- Fock space:

$$\widehat{\mathcal{F}} = \bigoplus_{n \geq 0} \left(l^2(\tfrac{2\pi}{L}\mathbb{Z}^d) \right)^{\otimes^n_{\mathrm{symm}}} \qquad \text{with vacuum } \widehat{\Omega}$$

Setup

continued

- Creation/annihilation op's

$$a^*(\widehat{f}) = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} \widehat{f}_k a_k^* \quad , \quad \text{where} \quad a_k^* = \mathfrak{F}\, L^{-d/2}\, a^*(\mathrm{e}^{\mathrm{i}kx})\, \mathfrak{F}^{-1}$$

- Field & Weyl op's

$$\Phi(\widehat{f}) = \tfrac{1}{\sqrt{2}} \textstyle\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} (\widehat{f}_k a_k^* + \overline{\widehat{f}_k} a_k) \quad \& \quad W(\widehat{f}) = \mathrm{e}^{\mathrm{i}\Phi(\widehat{f})}$$

s.t.

$$W(\widehat{f})W(\widehat{g}) \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}\langle\widehat{f},\widehat{g}\rangle} W(\widehat{f}+\widehat{g})$$

- C*-algebra gen. by $\{W(\widehat{f}), \widehat{f}\}$ <=> algebra gen. by $\{a^{\sharp}(\widehat{f}), \widehat{f}\}$

Setup

continued

- A state η :"Observables" $\longrightarrow$ $\mathbb{C}$ is charact. by

$$\boxed{E(f) = \eta(W(\widehat{f})), \text{ all } f \in L^2(\Lambda, \mathrm{d}x)}$$

Expectation fctl.

- Any $E : L^2(\Lambda, \mathrm{d}x) \to \mathbb{C}$ s.t.

$$E(0) = 1$$

$$\overline{E(f)} = E(-f)$$

$$\sum_{k,k'=1}^{K} z_k \overline{z_{k'}} \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}\,\langle \widehat{f_k}, \widehat{f_{k'}} \rangle} E(f_k - f_{k'}) \geq 0,$$

$$\forall \ K \geq 1, z_k \in \mathbb{C}, f_k \in L^2(\Lambda, \mathrm{d}x)$$

determines a regular state on the Weyl C*-algebra

- Example: $E_{\mathrm{Fock}}(f) = \langle \Omega, W(\widehat{f})\Omega \rangle = \mathrm{e}^{-\frac{1}{4}\|f\|_2^2}$

N-mode Coherent States

- Pick N modes $k'_1, \ldots, k'_N \in \frac{2\pi}{L}\mathbb{Z}^d$, N $\mathbb{C}$-numbers $\alpha_1, \ldots, \alpha_N$
- Finite volume coherent state

$$\boxed{\widehat{\Psi} = \mathrm{e}^{\sum_{j=1}^{N} \alpha_j a^*_{k'_j} - \bar{\alpha}_j a_{k'_j}} \widehat{\Omega}}$$

displacement op.

$a^*_{k'_j} a_{k'_j}$ number op. of mode k'_j

s.t. $\left\langle \widehat{\Psi}, a^*_{k'_j} a_{k'_j} \widehat{\Psi} \right\rangle = |\alpha_j|^2$ # particles in mode k'_j

Remarks:

displacement op. is a Weyl op.

$$\left\langle \widehat{\Psi}, W(\widehat{f}) \widehat{\Psi} \right\rangle = E_{\mathrm{Fock}}(f)\, \mathrm{e}^{\mathrm{i}\sqrt{2}\mathrm{Re} \sum_{j=1}^{N} \bar{\alpha}_j \widehat{f}_{k'_j}}$$

Expectation fctl.

N-mode Infinite Vol. Limit

- Scaling:

Let $k_1, \ldots, k_N \in \mathbb{R}^d$ be fixed modes, and $n_j = n_j(L) \in \mathbb{Z}^d$

s.t. $k_j'(L) = 2\pi n_j(L)/L \quad \xrightarrow{L\to\infty} \quad k_j,\ j = 1, \ldots, N$

Let $\rho_j = |\alpha_j|^2/L^d$ be fixed densities of part. in mode k_j'

i.e. $\alpha_j(L) = L^{d/2}\sqrt{\rho_j}\ \mathrm{e}^{\mathrm{i}\theta_j}$ with θ_j a phase

Theorem:

$\forall\ f \in L^1(\mathbb{R}^d, \mathrm{d}x) \cap L^2(\mathbb{R}^d, \mathrm{d}x)$ with $\widehat{f}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}kx} f(x)\mathrm{d}x$

$$\boxed{\lim_{L\to\infty} \left\langle \widehat{\Psi}, W(\widehat{f})\widehat{\Psi} \right\rangle = E_N(f) = \mathrm{e}^{-\frac{1}{4}\|f\|^2}\ \mathrm{e}^{\mathrm{i}\,\mathrm{Re}\sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\theta_j} \sqrt{2\rho_j}\ \widehat{f}(k_j)}}$$

Continuous Density of Modes

Strategy: take N to infinity after infinite vol. limit

- Recall expectation fnctl:

$$E_N(f) = \mathrm{e}^{-\frac{1}{4}\|f\|^2}\ \mathrm{e}^{\mathrm{i}\,\mathrm{Re}\sum_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\theta_j}\sqrt{2\rho_j}\ \widehat{f}(k_j)}$$

- Let $\rho(k)$ be a given density of modes support. in $[-R, R]^d$

  s.t. $\rho(k)\,\mathrm{d}k$ spatial density of part. with momenta in $\mathrm{d}k$

- Discretization

$$k_j = (-R + j_1\frac{2R}{N}, \ldots, -R + j_d\frac{2R}{N}) \in \mathbb{R}^d$$

$$j_1, \ldots, j_d \in \{1, 2, \ldots, N\}$$

=> $$\Delta k_j = (2R/N)^d$$

=> $$\rho_j = \rho(k_j)\,\Delta k_j$$

Continuous Density of Modes

continued

Sum in $E_N(f)$: with $\theta(k)$ a phase function

$$\sum_{j\in\{1,\ldots,N\}^d} \mathrm{e}^{-\mathrm{i}\theta_j}\sqrt{2\rho_j}\,\widehat{f}(k_j) = (2R/N)^{d/2} \sum_{j\in\{1,\ldots,N\}^d} \mathrm{e}^{-\mathrm{i}\theta(k_j)}\sqrt{2\rho(k_j)}\,\widehat{f}(k_j)$$

$$\approx \quad (N/2R)^{d/2}\int_{[-R,R]^d} \mathrm{e}^{-\mathrm{i}\theta(k)}\sqrt{2\rho(k)}\,\widehat{f}(k)\mathrm{d}k \quad \xrightarrow{N\to\infty} \quad \infty$$

$\Delta k_j = (2R/N)^d$

Cure:

Take random phases

$\theta_j \;=\; \theta_j(\omega)$ i.i.d. over $[0,2\pi]$, distrib. μ

$$E_{N,\omega}(f) = \mathrm{e}^{-\frac{1}{4}\|f\|^2}\; e^{\mathrm{i}N^{-d/2}\sum_{j\in\{1,\ldots,N\}^d}\xi_j(\omega)}$$

$$\xi_j(\omega) = (2R)^{d/2}\sqrt{2\rho(k_j)}\,\mathrm{Re}\,\mathrm{e}^{-\mathrm{i}\theta_j(\omega)}\,\widehat{f}(k_j)$$

N^d indep. rand. var.

=> calls for CLT

Continuous Density of Modes
continued

Fact: If $\widehat{\mu}(1) = 0$ where $\widehat{\mu}(n) := \int_0^{2\pi} \mathrm{d}\mu(\theta)\mathrm{e}^{-\mathrm{i}n\theta}$

$$N^{-d/2} \sum_{j\in\{1,\ldots,N\}^d} \xi_j(\omega) \xrightarrow{\mathcal{D}} \mathcal{N}_\omega(0, \sigma_\mu(f)^2) \quad \text{as} \quad N \to \infty$$ CLT

where $\sigma_\mu(f)^2 := \int_{\mathbb{R}^d} \rho(k)\Big(|\widehat{f}(k)|^2 + \mathrm{Re}\,\{\widehat{\mu}(2)\widehat{f}(k)^2\}\Big)\mathrm{d}k$

Thm.

$\exists$ "nice" $\chi_\omega(f)$ s.t. — Itô

$\mathrm{Re}\,\chi_\omega(f) \sim \mathcal{N}(0, \sigma_\mu(f)^2)$

$E_{N,\omega}(f) \xrightarrow{\mathcal{D}} E_\omega(f) = \mathrm{e}^{-\frac{1}{4}\|f\|^2}\mathrm{e}^{\mathrm{i}\mathrm{Re}\,\chi_\omega(f)}$ — Exp. Fctnl.

Representation

Infinitely many particles => out of original Fock space

- C*-algebra gen. by $\{W(f), f\}$ <=> algebra gen. by $\{a^\sharp(f), f\}$
- GNS:

Given a state η on a C*-alg. $\mathfrak{A}$, $\exists$ $(\mathcal{H}, \pi, \Psi)$

$$\left.\begin{array}{ll} \text{a Hilbert sp.} & \mathcal{H} \\ \text{a normalized} & \Psi \in \mathcal{H} \\ \text{a *-hom} & \pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}) \end{array}\right\}$$

$$\boxed{\begin{array}{l} \text{s.t.} \quad \forall \; A \in \mathfrak{A} \\ \eta(A) = \langle \Psi, \pi(A)\Psi \rangle \end{array}}$$

- Infinite vol. limit:

$$E(f) = \underbrace{\eta(W(f))}_{\text{known}} = \underbrace{\langle \Psi, \pi(W(f))\Psi \rangle}_{\text{to be found}} \quad \text{on } \mathcal{H}$$

Random phases GNS representation

$$E_\omega(f) = \mathrm{e}^{-\frac{1}{4}\|f\|^2}\,\mathrm{e}^{\mathrm{i}\mathrm{Re}\,\chi_\omega(f)} \qquad \text{with} \qquad \mathrm{Re}\,\chi_\omega(f) \sim \mathcal{N}(0,\sigma_\mu(f)^2)$$

Theorem:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{\mathcal{D}} \subseteq \mathcal{F}(L^2(\mathbb{R}^d, \mathrm{d}x)) \\ \pi_\omega(W(f)) &= W_{\mathrm{Fock}}(f)\mathrm{e}^{\mathrm{i}\mathrm{Re}\,\chi_\omega(f)} \\ \Psi &= \Omega_{\mathrm{Fock}} \end{aligned}$$

- Represented random field & creation op's

$$\boxed{\begin{aligned} \Phi_\omega(f) &= \Phi_{\mathrm{Fock}}(f) + \mathrm{Re}\,\chi_\omega(f) \\ a^*_\omega(f) &= a^*_{\mathrm{Fock}}(f) + \frac{1}{\sqrt{2}}\,\chi_\omega(f). \end{aligned}}$$

- In particular:

$$\left\langle \Omega_{\mathrm{Fock}}\,,\, a^*_\omega(f) a_\omega(f)\; \Omega_{\mathrm{Fock}} \right\rangle = \boxed{\frac{1}{2}\left|\chi_\omega(f)\right|^2}$$

Chi squared law

N-level system coupled to a random ICS

Uniform distrib.: $\mathrm{d}\mu(\theta) = \frac{\mathrm{d}\theta}{2\pi}$

ICS Hamiltonian: $H_{\mathrm{R}} = \mathrm{d}\Gamma(\varepsilon)$ with $\varepsilon(k) = |k|$

N-Level system: $H_{\mathrm{S}} = \mathrm{diag}(e_1, \ldots, e_N)$ <-> $\{\varphi_j\}_{j=1}^N$

Free Hamiltonian:

modulo technicalities

$$H_0 = H_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}} + \mathbb{1}_{\mathrm{S}} \otimes H_{\mathrm{R}} \quad \text{“on”} \quad \mathbb{C}^N \otimes \mathcal{F}(L^2(\mathbb{R}^d, \mathrm{d}x))$$

Coupled Hamiltonian:

$$\boxed{H = H_0 + G \otimes \Phi_\omega(g) = H_0 + G \otimes (\Phi_{\mathrm{Fock}}(g) + \mathrm{Re}\chi_\omega(g))}$$

where $g \in L^2(\mathbb{R}^d, \mathrm{d}x)$ form fact.

$G = \mathrm{diag}(g_1, \ldots, g_N)$ “non-demolition”

N-level system coupled to a random ICS

continued

- Initial density matrix: $P_0 = \rho_{\mathrm{S}} \otimes |\Omega\rangle\langle\Omega|$

arbitrary DM

ICS with unif. random phases

- (Reduced) state at time t:

$$P(t) = \mathrm{e}^{-\mathrm{i}tH} P_0 \, \mathrm{e}^{\mathrm{i}tH} \quad \& \quad \rho_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{R}} P(t)$$

with entries $\rho_{k,l}(t) = \langle \varphi_k, \rho_{\mathrm{S}}(t)\varphi_l \rangle$

N-level system coupled to a random ICS

continued

- Exact reduced DM:

$$\rho_{k,l}(t) = \rho_{k,l}(0)\ \mathrm{e}^{-\mathrm{i}t(e_k - e_l)}$$

$$\times\, \mathrm{e}^{-\mathrm{i}t(g_k - g_l)\mathrm{Re}\chi_\omega(g)}\, \mathrm{e}^{\frac{\mathrm{i}}{2}(g_k^2 - g_l^2)\left\langle g, \frac{\sin(\varepsilon t) - \varepsilon t}{\varepsilon} g \right\rangle}\, \mathrm{e}^{-\frac{1}{2}(g_k - g_l)^2 \left\langle g, \frac{1-\cos(\varepsilon t)}{\varepsilon^2} g \right\rangle}$$

- Random phase

$$\boxed{\mathbb{E}\left[\mathrm{e}^{-\mathrm{i}t(g_k - g_l)\mathrm{Re}\chi_\omega(g)}\right] = \mathrm{e}^{-\frac{t^2}{2}(g_k - g_l)^2 \|\sqrt{\rho} g\|_2^2}}$$

- Averaged decoherence

$$\Gamma(t) = 2\left\langle g, \frac{\sin^2(\varepsilon t/2)}{\varepsilon^2} g \right\rangle$$

$$\left|\mathbb{E}[\rho_{k,l}(t)]\right| = \mathrm{e}^{-\frac{t^2}{2}(g_k - g_l)^2 \|\sqrt{\rho} g\|_2^2}\, \mathrm{e}^{-\frac{1}{2}(g_k - g_l)^2 \Gamma(t)}\, |\rho_{k,l}(0)|$$

$$\Gamma(t) \overset{t\to\infty}{\sim} \begin{cases} t & |g(k)| \overset{|k|\sim 0}{\sim} 1/|k| \\ \gamma & |g(k)| \overset{|k|\sim 0}{\sim} g_0 \end{cases}$$

$d = 3$

$\varepsilon(k) = |k|$

ICS => Gaussian decoherence strong classical behavior.

Closing remarks

- Expectation values of polynomials in $a(f), a^*(g)$
- Averaged quasifreeness of random ICS
- Gauge invariance iff $\widehat{\mu}(2) = 0$

Thank you !