Coherent states for supersymmetric partners of solvable systems

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Outline

1. Introduction

For solvable quantum system with Hamiltonian H_0 having discrete energy energy spectrum (finite or infinite), we get **ladder operators** such that

$$a\psi_n(x) = \sqrt{k(n)}\psi_{n-1}(x), \quad a^\dagger\psi_n(x) = \sqrt{k(n+1)}\psi_{n+1}(x).$$

It leads to a generalized Heisenberg algebra ($N\psi_n(x) \equiv n\psi_n(x)$):

$$[a, N] = a, \quad [a^\dagger, N] = -a^\dagger, \quad [a, a^\dagger] = k(N+1) - k(N),$$

Different choices of $k(n)$:
- linearized: $k(n) = n$;
- factorization of H_0: $k(n) = \mathcal{E}(n)$, where $\mathcal{E}(n)$ is the shifted energy ($\mathcal{E}(0) = 0$).

1. Introduction

Different definitions of **coherent states** (equivalent for HO case):
-Eigenstates of a:

$$a\Psi_{\text{Ge}}(z;x) = z\Psi_{\text{Ge}}(z;x);$$

- Action of a displacement operator D(z):

$$D(z)\psi_0(x) = \exp(za^\dagger - z^* a)\psi_0(x) = \Psi_{\text{Ge}}(z;x);$$

- Minimum Heisenberg Uncertainty Relation:

$$\sigma_x\sigma_p = \frac{1}{2}, \quad \sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2.$$

1. Introduction

- Gaussian states: (ϕ_0, $n_0 \geq 0$ and $\sigma_0 > 0$ are real parameters)

$$\Psi_{\mathrm{G}}(n_0, \sigma_0, \phi_0; x, t) = \sum_{n=0}^{\infty} \frac{e^{-\frac{(n-n_0)^2}{4\sigma_0^2} - in\phi_0}}{\sqrt{N_{\mathrm{G}}(n_0, \sigma_0)}} e^{-i\omega\mathcal{E}(n)t} \psi_n(x),$$

- First definition of coherent states:

$$\Psi_{\mathrm{Ge}}(z; x, t) \equiv \frac{1}{\sqrt{N_{\mathrm{Ge}}(z)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e^{-i\omega\mathcal{E}(n)t} \psi_n(x).$$

(equivalence in HO case: $n_0 = z_0 - 1$ and $\sigma_0^2 = z_0/2$, $z_0 = |z|$.)

1. Introduction

Starting from a solvable Hamiltonian H_x, a **SUSY partner Hamiltonian** $\tilde{H}_x$ is obtained from the intertwining relations (Q_x and $Q_x^\dagger$ are called the intertwining or transformation operators)

$$\tilde{H}_x Q_x = Q_x H_x, \quad Q_x^\dagger \tilde{H}_x = H_x Q_x^\dagger.$$

The intertwining operators can be differential operators in x of **any order**.
We will consider in this talk differential operators in x of **second order**.

2. Infinite well, ladder operators and coherent states

2.1. The model

A particle of mass M is subject to a potential taken to be

$$V(x) = \begin{cases} 0, & 0 < x < \pi \\ \infty, & \text{otherwise.} \end{cases}$$

The stationary eigenstates and the discrete energies of this system are

$$\psi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \qquad E_n = \frac{\hbar^2}{2M} n^2, \quad n = 1, 2, \ldots$$

In the following, we will use dimensionless units, setting $\hbar = 1$, $M = 1/2$, such that the Hamiltonian is $H_x = -\frac{d^2}{dx^2} + V(x)$.

The GeCS (annihilation eigenstates) can be defined as long as the Hamiltonian H of the system has a non degenerate spectrum and admits a lowest energy equal to zero.
We thus work with the shifted Hamiltonian $\mathcal{H} \equiv H - E(0)\mathbb{I}$ instead of H. It has the same eigenstates as $\mathcal{H}$ and energy eigenvalues are

$$E(n) - E(0) = (n-1)(n+1) \equiv \mathcal{E}(n).$$

2. Infinite well, ladder operators and coherent states

2.1. The model

We can choose the ladder operators, known as **linearized operators**, such that

$$I\psi_n(x) = \sqrt{n-1}\ \psi_{n-1}(x), \quad I^\dagger\psi_n(x) = \sqrt{n}\ \psi_{n+1}(x).$$

The set $\{I, I^\dagger, N\}$ thus satisfies the usual Heisenberg algebra:

$$[I, N] = I, \quad [I^\dagger, N] = -I^\dagger, \quad [I, I^\dagger] = 1.$$

A realization of these ladder operators as differential operators of order 1 in x, with dependence in N, is known as:

$$I = \left[N\cos(x) - \sin(x)\frac{d}{dx}\right]\frac{\sqrt{N-1}}{N},$$

$$I^\dagger = \frac{\sqrt{N-1}}{N-1}\left[N\cos(x) + \sin(x)\frac{d}{dx}\right].$$

The GeCS can thus be defined as eigenstates of the **linearized annihilation operator**. We will call them **linearized coherent states**(LCS). They take the form, as usual:

$$\Psi_{\mathrm{Ge}}(z;x) \equiv \frac{1}{\sqrt{N_{\mathrm{Ge}}(z)}} \sum_{n=1}^{\infty} \frac{z^{(n-1)}}{\sqrt{(n-1)!}} \psi_n(x).$$

Such choice makes them identical to the displacement operator coherent states ($D(z)$ CS) as in the case of the harmonic oscillator. Indeed, we get

$$D(z)\psi_1(x) = \exp(zI^{\dagger} - z^* I)\psi_1(x) = \Psi_{\mathrm{Ge}}(z;x).$$

Starting from a solvable Hamiltonian $H_x = -\frac{d^2}{dx^2} + V(x)$, a **SUSY partner Hamiltonian** $\tilde{H}_x = -\frac{d^2}{dx^2} + \tilde{V}(x)$ is obtained from the intertwining relations

$$\tilde{H}_x Q_x = Q_x H_x, \quad Q_x^\dagger \tilde{H}_x = H_x Q_x^\dagger.$$

Note that when the intertwining operators are differential operators in x of **first order**, we get the SUSY partner as the trigonometric Pöschl-Teller system with $\tilde{V}_1(x) = \frac{2}{\sin^2 x}$.

The new sytem we will work with has been obtained from intertwining operators which are differential operators in x of **second order** in the so called confluent case:

$$Q_x = \frac{d^2}{dx^2} + \eta(x)\frac{d}{dx} + \epsilon + \frac{1}{2}(\eta^2(x) - \eta'(x))$$

and

$$Q_x^\dagger = \frac{d^2}{dx^2} - \eta(x)\frac{d}{dx} + \epsilon + \frac{1}{2}(\eta^2(x) - 3\eta'(x)),$$

where ϵ is an arbitrary constant. The function $\eta(x)$ satisfies:

$$2\eta(x)\eta''(x) - (\eta'(x))^2 - 4\eta^2(x)\eta'(x) + \eta^4(x) + 4\epsilon\ \eta^2(x) = 0$$

and the new potential is given as

$$\tilde{V}(x) = 2\eta'(x).$$

The resolution of

$$2\eta(x)\eta''(x) - (\eta'(x))^2 - 4\eta^2(x)\eta'(x) + \eta^4(x) + 4\epsilon\,\eta^2(x) = 0$$

leads to admissible solutions for $\epsilon = k^2$ with $k = 1, 2, \ldots$. We get

$$\eta(x; k, \omega) = \frac{4k\sin^2(kx)}{\sin(2kx) + 2k(\pi\omega - x)},$$

where ω is an arbitrary constant.

The corresponding family of potentials are given as

$$\tilde{V}(x;k,\omega)=\begin{cases}\frac{32k^2\sin(kx)[\sin(kx)+k(\pi\omega-x)\cos(kx)]}{[\sin(2kx)+2k(\pi\omega-x)]^2}, & 0<x<\pi\\ \infty, & \text{otherwise.}\end{cases}$$

These potentials are non singular if $\omega\in]-\infty,0[\,\cup\,]1,\infty[$.

2. Infinite well, ladder operators and coherent states

2.2. Supersymmetric partners

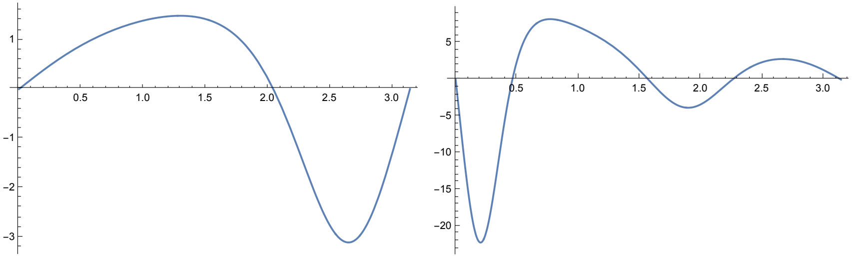

Figure 1 - SUSY potential $\tilde{V}(x; k = 1, \omega = 2)$ (left) and $\tilde{V}(x; k = 2, \omega = -0.5)$(right) as a function of x.

2. Infinite well, ladder operators and coherent states

2.2. Supersymmetric partners

The normalised SUSY eigenstates $\tilde{\psi}_n(x)$ are obtained as usual:

$$\tilde{\psi}_n(x; k, \omega) = (k^2 - E_n)^{-1} Q_x \psi_n(x), \quad (n \neq k).$$

For $n \neq k$, they are physical states, i.e. they are normalisable and such that $\tilde{\psi}_n(0; k, \omega) = \tilde{\psi}_n(\pi; k, \omega) = 0$, since $\eta(0; k, \omega) = \eta(\pi; k, \omega) = 0$. The corresponding energies are $E_n = n^2$ as in the original case.
For $n = k$, we have $Q_x \psi_k(x) = 0$. Solving $Q_x^\dagger \psi_k(x) = 0$ and $\tilde{H}_x \psi_k(x) = k^2 \psi_k(x)$, we get

$$\tilde{\psi}_k(x; k, \omega) = \sqrt{\frac{2}{\pi}} \sin(kx) \frac{2\pi k \sqrt{\omega(\omega - 1)}}{\sin(2kx) + 2k(\pi\omega - x)}.$$

It is normalisable and such that $\tilde{\psi}_k(0, k, \omega) = \tilde{\psi}_k(\pi, k, \omega) = 0$.
With this additional state the spectrum of $\tilde{H}_x$ is thus complete.

The annihilation operator of the new system may be written as $L_{S2} = Q_x l Q_x^\dagger$. It is now a differential operator in x of order 5. We thus get the action

$$L_{S2}\tilde{\psi}_n(x;k,\omega) = (n^2 - k^2)((n-1)^2 - k^2)\sqrt{n-1}\tilde{\psi}_{n-1}(x;k,\omega).$$

Again, we can use the linearized annihilation operator l_{S2} for which

$$l_{S2}\tilde{\psi}_n(x;k,\omega) = \sqrt{n-1}\tilde{\psi}_{n-1}(x;k,\omega).$$

The associated CS can be defined as eigenstates of the annihilation operator:

$$l_{S2}\tilde{\psi}(x;k,\omega;z) = z\tilde{\psi}(x;k,\omega;z).$$

Since $l_{S2}\tilde{\psi}_k(x;k,\omega) = 0$, we see that

$$l_{S2}\tilde{\psi}(x;k,\omega;z) = \sum_{n=2}^{k-1} c_n(z)\tilde{\psi}_{n-1}(x;k,\omega) + \sum_{n=k+2}^{k-1} c_n(z)\tilde{\psi}_{n-1}(x;k,\omega).$$

It means that the solution of the eigenstate equation for the CS is given by

$$\tilde{\psi}(x;k,\omega;z) = \sum_{n=k+1}^{\infty} c_n(z)\tilde{\psi}_{n-1}(x;k,\omega),$$

where the $c_n(z)$ are determined as usual.
The displacement operator $D_{I_{S2}}(z)$ definition of CS will help to recover the missing states. Indeed,
$D_{I_{S2}}(z)\tilde{\psi}_k(x;k,\omega) = \tilde{\psi}_k(x;k,\omega)$ and we thus get

$$\tilde{\psi}(x;k,\omega;z) = \sum_{n=2}^{k-1} c_n(z)\tilde{\psi}_{n-1} + c_k\tilde{\psi}_k + \sum_{n=k+1}^{\infty} c_n(z)\tilde{\psi}_{n-1}.$$

Case $k = 1$.

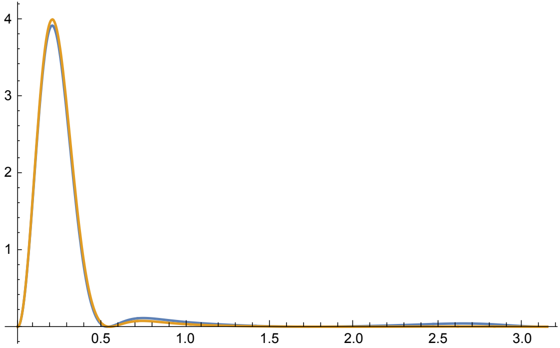

Figure 2 - Probability density of CS for the case $\tilde{V}(x; k = 1, \omega = 2)$.

3. Truncated oscillator, ladder operators and coherent states

3.1. The model

A quantum harmonic oscillator truncated at the origin by an infinite barrier is described by the Hamiltonian $H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x)$, where

$$V_0(x) = \begin{cases} \frac{x^2}{2} & \text{if } x > 0 \\ \infty & \text{if } x \leq 0 \end{cases} ,$$

is the potential of the system.

The energy eigenstates (satisfying the boundary conditions) are known as

$$\psi_k(x) = \left[\sqrt{\pi}\, 4^k (2k+1)!\right]^{-1/2} \mathrm{e}^{-x^2/2}\, H_{2k+1}(x)\,.$$

The corresponding eigenvalues are $E_k = 2k + \frac{3}{2}$, $k = 0, 1, \ldots$

3. Truncated oscillator, ladder operators and coherent states

3.1. The model

Moreover, the natural ladder operators are $l^{\pm} = (a^{\pm})^2$, where $a^{\pm}$ are the ladder operators for the standard harmonic oscillator. We thus get the commutation relations:

$$\left[H, l^{\pm}\right] = \pm\, 2 l^{\pm}\,, \qquad \left[l^{+}, l^{-}\right] = 4H$$

and their action on the energy eigenstates is

$$l^{-}\psi_k(x) = \sqrt{2k(2k+1)}\,\psi_{k-1}(x)\,,\; l^{+}\psi_{k-1}(x) = \sqrt{2k(2k+1)}\,|\psi_k(x)\,.$$

3. Truncated oscillator, ladder operators and coherent states

3.1. The model

We can again obtain a realization of the ladder operators as a differential operator of order 1 in x. Indeed, introducing the number operator N such that $N\psi_n(x) \equiv n\ \psi_n(x)$ and the fact that $\frac{d^2}{dx^2}\psi_n(x) = (x^2 - 4n - 3)\psi_n(x)$, we get:

$$I^- = x\frac{d}{dx} + x^2 - (2N+1),$$

$$I^+ = -x\frac{d}{dx} + x^2 - (2N+2),$$

After a supersymmetric transformation with Q_x and $Q_x^\dagger$ as differential operators in x of second order, we get the supersymmetric partner with potential:

$$V_{\mathrm{S}}(x) = \frac{x^2}{2} - \frac{2\left(16x^8 - 32x^6 + 24x^4 + 72x^2 + 9\right)}{\left(4x^4+3\right)^2}.$$

3. Truncated oscillator, ladder operators and coherent states

3.2.Supersymmetric partner

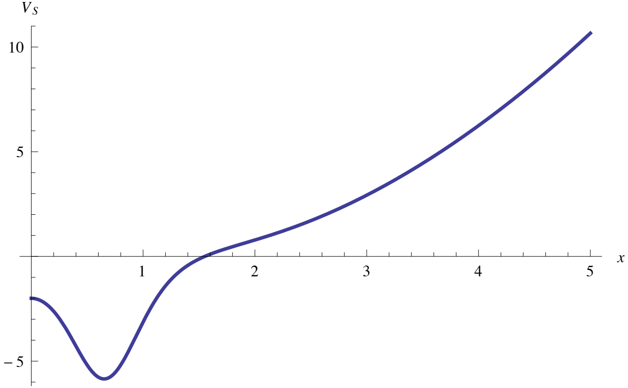

Figure 3 - SUSY potential $V_{\mathrm{S}}(x)$.

The intertwining operators are given by

$$Q_x = \frac{1}{2}\left(\frac{d^2}{dx^2} - \eta\frac{d}{dx} + \gamma\right)$$

and

$$Q_x^\dagger = \frac{1}{2}\left(\frac{d^2}{dx^2} - \eta\frac{d}{dx} + \eta' + \gamma\right),$$

with

$$\eta = \frac{2x\left(4x^4 + 8x^2 + 3\right)}{4x^4 + 3}, \quad \gamma = \frac{4x^6 + 12x^4 + 27x^2 - 15}{4x^4 + 3}.$$

The level which has been added, below the original ground state $E_0 = 3/2$, is given by $\varepsilon = -5/2$ and the associated eigenfunction is

$$\psi_{\varepsilon_0} = -\frac{2\sqrt{2}\mathrm{e}^{-\frac{x^2}{2}}x\left(2x^2+3\right)}{\sqrt[4]{\pi}\left(4x^4+3\right)}.$$

This is the ground state of H_{S}. The eigenfunctions of the excited states are given by

$$\phi_n = \frac{Q\psi_n}{\sqrt{(2n+4)(2n+5)}},$$

3. Truncated oscillator, ladder operators and coherent states

3.2.Supersymmetric partner

The annihilation operator of the new system may be written as $L^- = Q_x l^- Q_x^\dagger$. It is a differential operator in x of order 5.
Let us note that we have

$$L^- \phi_n = Q l^- Q^\dagger \phi_n = k(n)\phi_{n-1}$$

with $k(n) = \sqrt{2n(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}$.
It shows that this annihilation operator cancel both the ground state and the first excited state.
For the construction of the CS, we use again the linearized annihilation operator L_S^- for which

$$L_S^- \phi_n = \sqrt{n}\phi_{n-1}.$$

The linearized coherent states of H_{S} in the isospectral subspace are given by

$$|z\rangle_{\mathrm{iso}} = D_{\mathcal{L}}(z)\phi_0 = \exp\left(-|z|^2\right) \sum_{n=0}^{\infty} \frac{(\sqrt{2}z)^n}{\sqrt{n!}} \phi_n,$$

where the factor $\sqrt{2}$ multiplying the complex parameter z comes from the spacing of 2 in the energy levels.
We show some behaviour of those states.

3. Truncated oscillator, ladder operators and coherent states

3.2.Supersymmetric partner

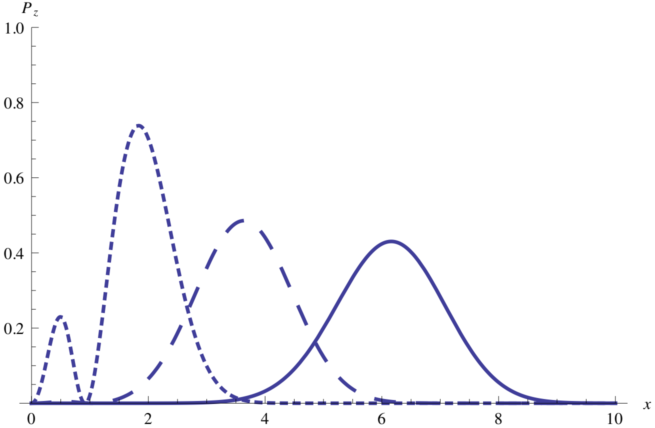

Figure 4 - Probability density of $|z\rangle_{\text{iso}}$ for $|z| = 0.1$ (- - -), $|z| = 1$ ($-\ -\ -$) and $|z| = 2$ (——).

Now let us study the uncertainty relation.
The expectation value of an operator O when the system is in a coherent state $|z\rangle_{\rm iso}$ is given by

$$\langle z|_{\rm iso} O |z\rangle_{\rm iso} = \sum_{m,n=0}^{\infty} \Lambda_{mn}(z) \, \langle O_{nm} \rangle \, ,$$

where $\Lambda_{m,n}(z) = \exp(-|z|^2) \, \frac{(\sqrt{2}z)^m(\sqrt{2}z^*)^n}{m!n!}$.
It is used to compute the standard deviations of the position and momentum operators $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, and their product $\sigma_x \sigma_p$ as functions of the complex parameter z.

3. Truncated oscillator, ladder operators and coherent states

3.2.Supersymmetric partner

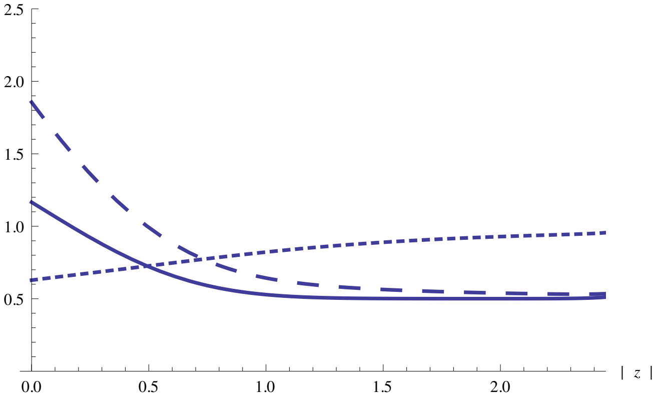

Figure 5- Uncertainty relation for $D_{\mathcal{L}}(z)$-CS where σ_x (- - - -), σ_p (— — —), $\sigma_x\sigma_p$ (———).

4. Conclusion

- Susy partners hamiltonian have been studied in two cases:
1) infinite well
2) truncated oscillator.
- Linearized ladder operators $Q/Q^\dagger$ have been realized as differential operators of order 1 in x with a dependence in N.
- CS have been constructed using the displacement operator action.

4. Conclusion

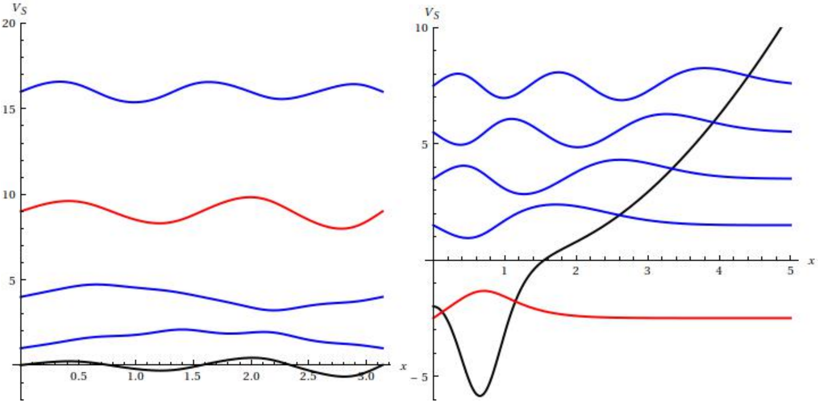

Figure 6 - Susy infinite well (left) and truncated oscillator (right).

For the infinite well, we see that

$$\tilde{\psi}(x; k, \omega; z) = \sum_{n=2}^{k-1} c_n(z)\tilde{\psi}_{n-1} + c_k\tilde{\psi}_k + \sum_{n=k+1}^{\infty} c_n(z)\tilde{\psi}_{n-1}.$$

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