Coherent state transforms for compact groups and their large-N limits

Brian C. Hall   University of Notre Dame

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Going *hog wild* for coherent states!

Motivation: Klauder, *Annals of Physics*, 1960

KLAUDER

Transforming to polar coordinates one readily determines that (5a) reduces to

$$\int_0^\infty \exp(-|a|^2) \sum_{N'=0}^{\infty} (N'!)^{-1} |a|^{2N'} |N'\rangle\langle N'| \; d|a|^2 \tag{5b}$$

$$= \sum_{N'=0}^{\infty} |N'\rangle\langle N'| = 1.$$

This "resolution of the identity" is a very useful tool ...

Credits and references

- Large-N limit is joint work with **Bruce Driver** and **Todd Kemp**, Univ. Calif., San Diego
- Web site: www.nd.edu/~bhall/
- **Expository paper** on large-N limit: arXiv:1308.0615 [math.RT] (printed copies available)
- Shameless self-promotion: Textbook

Brian Hall

Quantum Theory for Mathematicians

Springer, Graduate Texts in Mathematics, 2013

Compact-type Lie groups

- Lie group K of "compact type," i.e., compact groups, $\mathbb{R}^n$, and products
- Examples: $SO(3)$ for rigid body motion, or $SU(2) = S^3$
- View K as **configuration space** (position)
- **Phase space** is cotangent bundle $T^*(K)$ (position and momentum)
- Quantum Hilbert space is $L^2(K)$

Complexification

- **Complexified group** $K_{\mathbb{C}} \supset K$
- **Examples**:

$$\begin{array}{ll} K = \mathbb{R}^n & K_{\mathbb{C}} = \mathbb{C}^n \\ K = SU(N) & K_{\mathbb{C}} = SL(n;\mathbb{C}) \end{array}$$

- Identify $K_{\mathbb{C}}$ as **phase space**, as follows.

$$T^*(K) \underset{\text{metric}}{\cong} T(K) \underset{\substack{\text{polar} \\ \text{decomp.}}}{\cong} K_{\mathbb{C}}$$

Heat kernel (or "complexifier") coherent states on K

- **Heat kernel** ρ_t on K, based at identity:

$$\begin{aligned} \frac{d\rho_t}{dt} &= \frac{1}{2}\Delta_K \rho_t \\ \lim_{t\to 0} \rho_t &= \delta_e. \end{aligned}$$

- **Holomorphic extension** to $K_{\mathbb{C}}$
- **Coherent states**: For $g \in K_{\mathbb{C}}$, define

$$\chi_g(x) = \overline{\rho_\hbar(gx^{-1})}, \quad g \in K_{\mathbb{C}},\ x \in K.$$

- **Not** of Perelomov type
- $\hbar$ plays role of "time" in heat equation

Euclidean case

- If $K = \mathbb{R}$, heat kernel is **Gaussian**
- Coherent states are usual **Gaussian wave packets**:

$$\begin{aligned}\chi_z(x) &= \overline{C\exp\left\{-\frac{1}{2\hbar}(z-x)^2\right\}} \\ &= C'\exp\left\{-\frac{1}{2\hbar}(x-a)^2\right\}\exp\left\{-\frac{ibx}{\hbar}\right\}, \quad z = a+ib\end{aligned}$$

- Packet centered at $x = a = \operatorname{Re} z$, with expected momentum $b = \operatorname{Im} z$
- For general K, heat kernel is "most Gaussian" function

Results: Resolution of identity

- Let ν_t be K-**invariant heat kernel** on $K_{\mathbb{C}}$:

$$\begin{aligned} \frac{d\nu_t}{dt} &= \frac{1}{4}\Delta_{K_{\mathbb{C}}}\nu_t \\ \lim_{t\to 0}\nu_t &= \delta_K \end{aligned}$$

- If $K = \mathbb{R}$ and $K_{\mathbb{C}} = \mathbb{C}$ then $\nu_t(x+iy) = (\pi t)^{-1/2}e^{-y^2/2t}$

Theorem (H 1994)

We have a resolution of the identity as follows:

$$I = \int_{K_{\mathbb{C}}} |\chi_g\rangle\langle\chi_g| \ \nu_\hbar(g)\ dg$$

- If $K = \mathbb{R}$, gives the resolution of identity of **John Klauder** (1960)

Results: Segal–Bargmann representation

- For $\psi \in L^2(K)$, define **Segal–Bargmann transform**:

$$\begin{aligned}(C_\hbar \psi)(g) &= \langle \chi_g | \psi \rangle \\ &= \int_K \rho_\hbar(gx^{-1})\psi(x)\ dx, \quad g \in K_{\mathbb{C}}\end{aligned}$$

- **Segal–Bargmann space**: $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_\hbar)$ (square-integrable **holomorphic** functions)

Theorem (H 1994)

For each $\hbar > 0$, the map $C_\hbar$ is a **unitary map** *of $L^2(K)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_\hbar)$*

Results: Geometric quantization

- Do **geometric quantization with half-forms** of $T^*(K) \cong K_{\mathbb{C}}$ using complex polarization
- Hilbert space turns out to be (isomorphic to) $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_{\hbar})$
- Geometric quantization somehow reproduces heat kernel!
- Segal–Bargmann transform $=$ **BKS pairing map**
- References: H, 2002, C. Florentino, P. Matias, J. Mourão, J Nunes, 2006

Results: (1+1)-dimensional Yang–Mills theory

- Spacetime cylinder $S^1 \times \mathbb{R}$, structure group K
- **Configuration space** is $\mathcal{A} = \mathfrak{k}$-valued connections over spatial circle
- **Based gauge group**: $\mathcal{G}_0 =$ gauge transformations equal to e at basepoint
- $\mathcal{A}/\mathcal{G}_0 = K$ (holonomy around spatial circle)
- **Goal**: Project coherent states for $\mathcal{A}$ into gauge-invariant subspace

Results: (1+1)-dimensional Yang–Mills theory

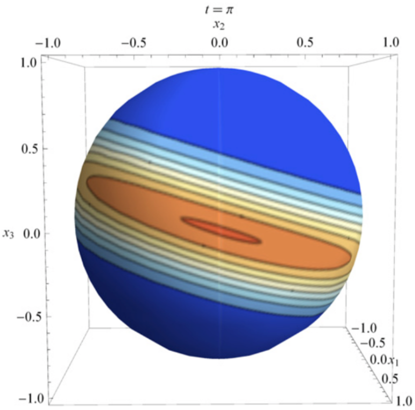
- **First approach**: Wren, using group-integration method of Landsman
- **Second approach**: Driver–H, using Segal–Bargmann transform for $\mathcal{A}$
- **Gaussian** coherent states for $\mathcal{A}$ project to **heat kernel** coherent states for $\mathcal{A}/\mathcal{G}_0 \cong K$
- "Quantization commutes *unitarily* with reduction"
- References: K. K. Wren 1998; Driver–Hall 1999

Results: Coherent states on spheres

- **Configuration space** S^n, phase space $T^*(S^n)$
- **Project** coherent states from $L^2(SO(n+1))$ to $L^2(S^n)$
- Coherent states, resolution of identity, Segal–Bargmann representation
- **Kowalski–Rembieliński** polar decomposition method
- **Thiemann complexifier** method
- References: T. Thiemann 1996, M. Stenzel 1999, Kowalski–Rembieliński 2000 & 2001, Hall–Mitchell 2002

Results: Coherent states on spheres

- **Coherent states** given in terms of heat kernel on S^n
- Resemble Gaussian wave packets:



Ref: K Kowalski, J Rembieliński and J Zawadzki, *J. Phys. A* 2015

Results: Coherent states on spheres

- **Large-radius** limit gives back Gaussian
- Results for particle on S^2 in **magnetic field**
- Results for general compact symmetric spaces
- Refs: Hall–Mitchell 2002 & 2012, Stenzel 1999

Applications: quantum gravity

- A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann: "Coherent state transform for spaces of connections" (1996)
- H. Sahlmann, T. Thiemann, O. Winkler, "Gauge field theory coherent states" (four papers in 2001)
- Much additional work since then ...

New direction: Large-N limit

- **Popular idea**: Gauge theory for $U(N)$, let $N \to \infty$
- **Master field**: Expect path-integral for $U(N)$ Yang–Mills to concentrate in limit to a **single** connection called "master field" ['t Hooft, 1974]
- **Gross and Taylor**: "Two-dimensional QCD is a String Theory" (1993)
- **J. Maldacena**: "The large-N Limit of superconformal field theories and supergravity" (1999): 3,000 citations
- **Here**: one specific aspect of large-N limit!

Geometry of the unitary groups

- $U(N)$ = group of $N \times N$ unitary matrices ($U^*U = I$)
- Lie algebra = $u(N)$ = skew matrices ($X^* = -X$)
- Use on $u(N)$ **scaled Hilbert–Schmidt** inner product:

$$\langle X, Y\rangle = N\,\mathrm{Re}[\mathrm{Trace}(X^*Y)].$$

- This inner product determines a bi-invariant Riemannian metric on $U(N)$
- Metric determines Laplacian Δ_N (with $\Delta_N \leq 0$)

B-version Segal–Bargmann transform

- $(U(N))_{\mathbb{C}} = GL(N;\mathbb{C}) =$ group of all $N \times N$ invertible matrices
- Transform **as before**

$$
\begin{aligned}
(B_t^N f)(g) &= \int_{U(N)} \rho_t(gx^{-1}) f(x)\ dx \\
&= \left(e^{t\Delta_N/2} f\right)_{\mathbb{C}} (g)
\end{aligned}
$$

- **Full heat kernel** μ_t on $GL(N;\mathbb{C})$

Theorem (H 1994)

The map B_t^N is a unitary map of $L^2(U(N),\rho_t)$ onto $\mathcal{H}L^2(GL(N;\mathbb{C}),\mu_t)$

Large-N behavior of Laplacian

- Consider normalized trace,

$$\mathrm{tr}(U) = \frac{1}{N}\mathrm{Trace}(U) = \frac{1}{N}\sum_{j=1}^{N} U_{jj}.$$

- Now consider **trace polynomials**, i.e., polynomials in traces of powers of U. E.g.

$$f(U) = 7\mathrm{tr}(U^2)\mathrm{tr}(U^3) - (\mathrm{tr}(U^2))^3.$$

- The action of Δ_N on trace polynomials decomposes as:

$$\Delta_N = \Delta_\infty + \frac{1}{N^2}L$$

for operators "Δ_∞" and "L" whose actions are **independent of** N.

Large-N behavior of Laplacian, cont'd

- **Two basic properties** determine Δ_∞
- First,

$$\Delta_\infty[\mathrm{tr}(U^k)] = -k\,\mathrm{tr}(U^k) - 2\sum_{j=1}^{k-1} j\,\mathrm{tr}(U^j)\mathrm{tr}(U^{k-j}).$$

- Second, Δ_∞ satisfies the **first-order product rule**:

$$\Delta_\infty(fg) = \Delta_\infty(f)g + f(\Delta_\infty g).$$

- **Cross terms are small** in product rule for Δ_N

Concentration of heat kernels

- Look at **heat kernel measure**

$$d\rho_t^N(U) := \rho_t(U)\ d\mathrm{vol}(U) \quad \text{on } U(N)$$

- Measure is concentrating to set where $\mathrm{tr}(U)$ **has definite value**:

$$\left\| \mathrm{tr}(U) - e^{-t/2} \right\|_{L^2(U(N),\rho_t^N)} \to 0$$

- Trace polynomials effectively become **constants** (as elements of $L^2(U(N), \rho_t^N)$)!

Concentration of heat kernels, cont'd

- Concentration related to the **first-order product rule** for Δ_∞.
- If Δ_∞ behaves like a first-order operator, then **heat doesn't diffuse**.
- Similar concentration results on $GL(N;\mathbb{C})$ w.r.t.

$$d\mu_t^N(Z) := \mu_t(Z)\ d\mathrm{vol}(Z)$$

on $GL(N;\mathbb{C})$

Back to Segal–Bargmann transform

- Limiting SBT on trace polynomials **makes sense but is trivial** (constants map to constants)
- Consider **matrix-valued trace polynomials**, e.g.,

$$f(U) = 2U^2\mathrm{tr}(U^3) - 9U\mathrm{tr}(U^4).$$

- Product rule extends **only** if one of the polynomials is scalar:

$$\Delta_\infty(U^2U^3) \neq \Delta_\infty(U^2)U^3 + U^2\Delta_\infty(U^3).$$

Example

- **Function**:
$$f(U) = U^2, \quad U \in U(N).$$
- **Transform**:
$$\begin{aligned} B_t^N(f)(Z) &= e^{-t}\left[\cosh(t/N)\ Z^2 - t\frac{\sinh(t/N)}{t/N}\ Z\mathrm{tr}(Z)\right] \\ &\approx e^{-t}\left[Z^2 - tZ\mathrm{tr}(Z)\right], \quad Z \in GL(N;\mathbb{C}) \end{aligned}$$
- **Concentration**: on $GL(N;\mathbb{C})$ we have $\mathrm{tr}(Z) \approx 1$ (w.r.t. μ_t^N) in the large-N limit
- **Limit**:
$$\lim_{N\to\infty} B_t^N(f)(Z) = e^{-t}(Z^2 + tZ)$$

Main result

- Traces disappear: **only powers of U survive**

Theorem (Driver-H-Kemp, 2013)

Let p be a polynomial in a single variable and let

$$f(U) = p(U), \quad U \in U(N).$$

Then for each $t > 0$, there exists a unique polynomial q_t in a single variable such that

$$\left\| B_t^N(f)(Z) - q_t(Z) \right\|_{L^2(GL(N;\mathbb{C}),\mu_t^N)} \to 0$$

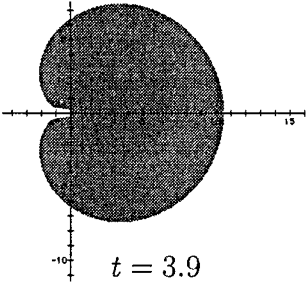
as $N \to \infty$.

- E.g., if $p(u) = u^2$, then

$$q_t(z) = e^{-t}(z^2 + tz).$$

Comparison with Biane

- Map $p \mapsto q_t$ coincides with the "free Hall transform" of Biane 1997
- Biane uses "free unitary Brownian motion" for large-N limit
- Map $\mathcal{G}_t : L^2(S^1, \gamma_t) \to \mathcal{H}(\Sigma_t)$ for domain $\Sigma_t \subset \mathbb{C}$. Here γ_t is limiting eigenvalue distribution of ρ_t^N.



$t = 3.9$

Comparison with Biane

- Driver–H–Kemp shows that $\mathcal{G}_t$ is limit of B_t^N on trace polynomials
- New recursive method of computation
- Two-parameter version

Computing large-N limit

- Step 1: Start with U^k and **apply heat operator** $e^{t\Delta_\infty/2}$.
- Step 2: **Evaluate the traces**. Actually, $\mathrm{tr}(Z^k) \approx 1$ for every k.
- Example: $f(U) = U^3$. Applying $e^{t\Delta_\infty/2}$ gives

$$e^{-3t/2}\left\{Z^3 + t[2Z^2\mathrm{tr}(Z) + Z\mathrm{tr}(Z^2)] + \frac{3t^2}{2}Z\mathrm{tr}(Z)^2\right\}$$

- Evaluating all traces to 1 gives

$$B_t^\infty(U^3) = q_t(Z) = e^{-3t/2}\left\{Z^3 + t(2Z^2 + Z) + \frac{3t^2}{2}Z\right\}$$

Computing large-N limit

- **Recursive** method of computing on U^k
- **Generating function** for transform and inverse
- References: Biane 1999, Driver–Hall–Kemp 2015, G. Cébron 2015
- Expository paper: arXiv:1308.0615 [math.RT]

Conclusion

- Thank you for your attention!