Wavelet approximation theory in higher dimensions

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Lehrstuhl A für Mathematik, RWTH

Outline

1. Introduction: Nice wavelets and sparse signals in dimension one

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2. Continuous wavelet transforms over general dilation groups

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5. Constructing compactly supported nice wavelets

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Overview

Wavelet orthonormal bases

Definition

A wavelet ONB $(\psi_{j,k})_{j,k\in\mathbb{Z}} \subset \mathrm{L}^2(\mathbb{R})$ is an ONB of the form

$$(\psi_{j,k})_{j,k\in\mathbb{Z}} \subset \mathrm{L}^2(\mathbb{R}) \ , \psi_{j,k} = 2^{j/2}\psi(2^j x - k) \ , \psi \text{ fixed}$$

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- For sufficiently nice wavelets ψ, the wavelet expansion

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$$

converges in the norm of a homogeneous Besov space $\dot{B}^{\alpha}_{p,q}$, as soon as $f \in \dot{B}^{\alpha}_{p,q}$.

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- There exist arbitrarily nice compactly supported wavelets. (Daubechies)

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- Important byproduct: Consistency.
  All sufficiently nice wavelets have the same spaces of sparse signals!
- Yields mathematically rigourous justification for many wavelet-based methods and algorithms, such as denoising, compression etc.

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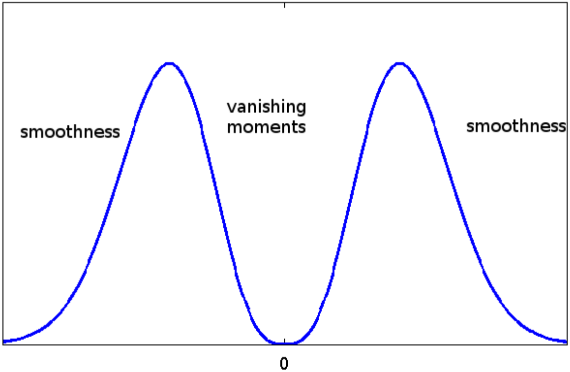
(Note: Frequency-side localization is understood away from zero.)

Cartoon: Fourier side decay of wavelets



Plot of $|\widehat{\psi}|$.

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- Better yet: Identify easily accessible classes of nice wavelets ($\rightsquigarrow$ bandlimited Schwartz functions, vanishing moment criteria)

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$$(\pi(x,h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y-x)) \ .$$
- Continuous wavelet transform: Given suitable $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ and $f \in \mathrm{L}^2(\mathbb{R}^d)$, let
$$\mathcal{W}_\psi f : G \to \mathbb{C} \ , \ \mathcal{W}_\psi f(x,h) = \langle f, \pi(x,h)\psi\rangle$$

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Wavelet inversion

If ψ is admissible, we obtain the wavelet inversion formula

$$f = \int_G \mathcal{W}_\psi f(x,h)\ \pi(x,h)\psi\ d(x,h)\ .$$

with weak-sense convergence.

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- Informal interpretation of $\mathcal{W}_\psi$: The transform acts as a filter bank labelled by elements of h, the frequencies associated to the "channel" ψ_h are contained in $h^{-T}\mathrm{supp}(\widehat{\psi})$.

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Theorem (Bernier/Taylor, 1996; HF, 2010)

H is irreducibly admissible iff there exists a single open orbit

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under the dual action, with the additional property that the associated stabilizer

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Standing assumption

From now on: H is assumed irreducibly admissible.

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Complete list in dimension two, up to conjugacy.

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- Main issues addressed by coorbit theory:
  - Consistency: When is the norm independent of ψ? (I.e., is there a notion of nice wavelets?)
  - Discretization: When can the norm be expressed in terms of a discrete set of sampled wavelet coefficients?

Nice wavelets according to coorbit theory

Definition

Let v denote the control weight for $\mathrm{L}^p(G)$, given by

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Desirable: Explicit criteria for nice wavelets.

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Also: $Co(L^p)$*-norm is equivalent to norm on discrete coefficients!*
I.e., $Co(L^p)$ *is a consistently defined space of sparse signals.*

Overview

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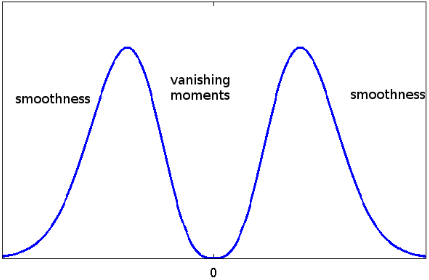
Overview

Reminder: Nice wavelets in dimension one

Desirable properties of wavelets

A nice wavelet $\psi \in \mathrm{L}^2(\mathbb{R})$ typically has three properties: Fast decay, smoothness, vanishing moments.
Concisely: Nice wavelets have good time-frequency localization.
(Note: Frequency-side localization is understood away from zero.)



Vanishing moments and wavelet coefficient decay

Assumptions on nice wavelet ψ guarantee fast decay of $\mathcal{W}_\psi\psi$:

$$|\mathcal{W}_\psi\psi(x,s)| \le \sum_{j<\ell} \left\| \partial^j \left(\widehat{\psi} \cdot \overline{\widehat{\psi}(s\cdot)} \right) \right\|_1 |s|^{-1/2}(1+|x|)^{-\ell}$$

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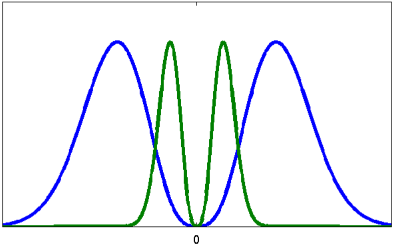
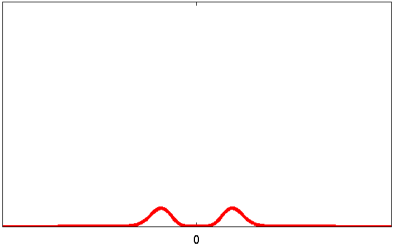


Plot of $\widehat{\psi}$ and $\widehat{\psi}(3\cdot)$



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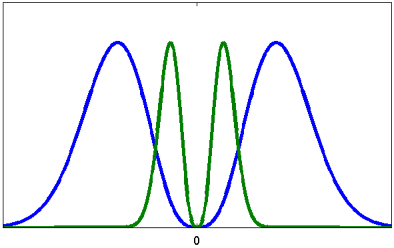

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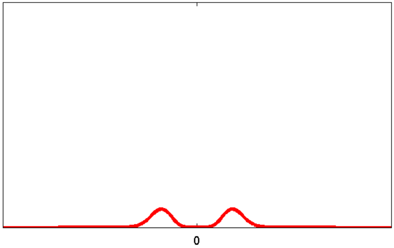

Overlap $\widehat{\psi} \cdot \widehat{\psi}(3\cdot)$

$\Rightarrow$ vanishing moments, smoothness govern decay of overlap, as $|s| \to 0, \infty$

Generalizing this to higher dimensions

Central idea

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- Characterize nice wavelets in terms of smoothness,

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Definition

Let $r \in \mathbb{N}$ be given. $f \in \mathrm{L}^1(\mathbb{R}^d)$ has vanishing moments in $\mathcal{O}^c$ of order r if all distributional derivatives $\partial^{\alpha} \widehat{f}$ with $|\alpha| < r$ are continuous functions, identically vanishing on $\mathcal{O}^c$.

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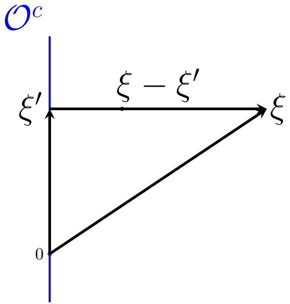

$$A(\xi) = \min\left(\frac{|\xi - \xi'|}{1+|\xi'|}, \frac{1}{1+|\xi|} \right)$$

with ξ' = point in $\mathcal{O}^c$ closest to ξ

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Lemma (HF, '98)

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- Then $\psi \in C_c^\infty(\mathbb{R}^d)$ has vanishing moments of order r in $\mathcal{O}^c$.
- Clearly, picking $f \in C_c^k(\mathbb{R}^d)$, for k sufficiently large, is enough.

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Making sense of wavelet coorbit spaces

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Answer

Decomposition space description!

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Let $\mathcal{Q} = (Q_i)_{i \in I}$ denote a covering of an open set $\mathcal{O} \subset \mathbb{R}^d$. Let $(\varphi_i)_{i \in I} \subset C_c^\infty(\mathcal{O})$ denote a partition of unity subordinate to Q_i.

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and the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_v)$ as the space of all u for which this norm is finite.

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Theorem (HF, F. Voigtlaender, 2015)

For any admissible matrix group H and weight u on H there exists an admissible covering $\mathcal{Q} = (Q_j)_{j \in J}$ and a weight v on J such that

$$Co(L^{p,q}_u) = \mathcal{D}(\mathcal{Q}, L^p, \ell^q_v)$$

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- As a rule, the criteria for embeddings between or equality of decomposition spaces are based on explicit computations involving the induced coverings.

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- Decomposition space approach also covers other types of smoothness spaces that are not associated to dilation groups, such as (α-)modulation spaces, anisotropic Besov spaces, etc.
- The scheme extends to quasi-Banach setting (e.g. $p < 1$).

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