Coherent and minimum energy states of a charged particle in a uniform magnetic field

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Conference: ***Coherent states and their applications: A contemporary panorama***

- **Marseille, France, November 18, 2016**

Plan:

1. A brief historical panorama

2. Some new results

Why the problem is interesting

$$\hat{H} = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2$$

$$\mathbf{H} = (0, 0, H_0) = \mathrm{rot}\,\mathbf{A}(\mathbf{r})$$

$\pi = \mathrm{p} - e\mathrm{A}/c = M\mathrm{v}$ is the *kinetic momentum*,

$$[\hat{\pi}_x, \hat{\pi}_y] = i\hbar M\omega_0 \qquad \omega_0 = eH_0/Mc$$

First solutions: eigenstates of the Hamiltonian

Circular gauge: $\mathbf{A}_s = \frac{1}{2}[\mathbf{H} \times \mathbf{r}]$

Fock V
Bemerkung zur quantelung des harmonischen oszillators im magnetfeld
Z. Phys. 47, 446–8 (1928) – **Laguerre polynomials**

Landau gauge $A = (-Hy, 0, 0)$

L. Landau, Diamagnetismus der Metalle Z. Phys. 64, 629-637 (1930)
--- **Hermite polynomials**

Gaussian wave packets:

Darwin C G 1927, Free motion in wave mechanics,
Proc. R. Soc. London A **117**, 258--93

Linear integrals of motion – center of orbit coordinates

$$\widehat{X} = \widehat{x} + \frac{\widehat{\pi}_y}{M\omega_0}, \qquad \widehat{Y} = \widehat{y} - \frac{\widehat{\pi}_x}{M\omega_0},$$

Relative coordinates

$$\widehat{\xi} = \widehat{x} - \widehat{X} = -\frac{1}{M\omega_0}\widehat{\pi}_y, \qquad \widehat{\eta} = \widehat{y} - \widehat{Y} = \frac{1}{M\omega_0}\widehat{\pi}_x$$

$$\left[\widehat{\xi}, \widehat{\eta}\right] = -\left[\widehat{X}, \widehat{Y}\right] = \frac{i\hbar}{M\omega_0},$$

$$\left[\widehat{\xi}, \widehat{X}\right] = \left[\widehat{\xi}, \widehat{Y}\right] = \left[\widehat{\eta}, \widehat{X}\right] = \left[\widehat{\eta}, \widehat{Y}\right] = 0.$$

Angular momentum operator

$$L = x\pi_y - y\pi_x + \frac{1}{2}M\omega_0\left(x^2 + y^2\right)$$

$$\hat{L} = \frac{1}{2}M\omega_0\left(\hat{X}^2 + \hat{Y}^2 - \xi^2 - \eta^2\right)$$

$$\hat{H} = \frac{1}{2}M\omega_0^2\left(\xi^2 + \eta^2\right)$$

$$J = \xi\pi_y - \eta\pi_x = -2H/\omega$$

$$\hat{L} = \frac{1}{2}\left(\hat{J} + M\omega_0\hat{R}^2\right)$$

$$\hat{R}^2 = \hat{X}^2 + \hat{Y}^2$$

circular ("symmetric") gauge

$$\mathbf{A}_s = \frac{1}{2}[\mathbf{H} \times \mathbf{r}]$$

$$L = xp_y - yp_x$$

$$\widehat{a} = \sqrt{\frac{M\omega_0}{2\hbar}}\left(\hat{\eta} - i\hat{\xi}\right) = \frac{\hat{\pi}_x + i\hat{\pi}_y}{\sqrt{2\hbar M\omega_0}}$$

$$\widehat{b} = \sqrt{\frac{M\omega_0}{2\hbar}}\left(\widehat{X} - i\widehat{Y}\right)$$

(assuming $\omega_0 > 0$)

$$\left[\widehat{a}, \widehat{a}^\dagger\right] = \left[\widehat{b}, \widehat{b}^\dagger\right] = 1$$

M. H. JOHNSON AND B. A. LIPPMANN, Phys. Rev. 76 (1949), 828-832
Motion in a constant magnetic field

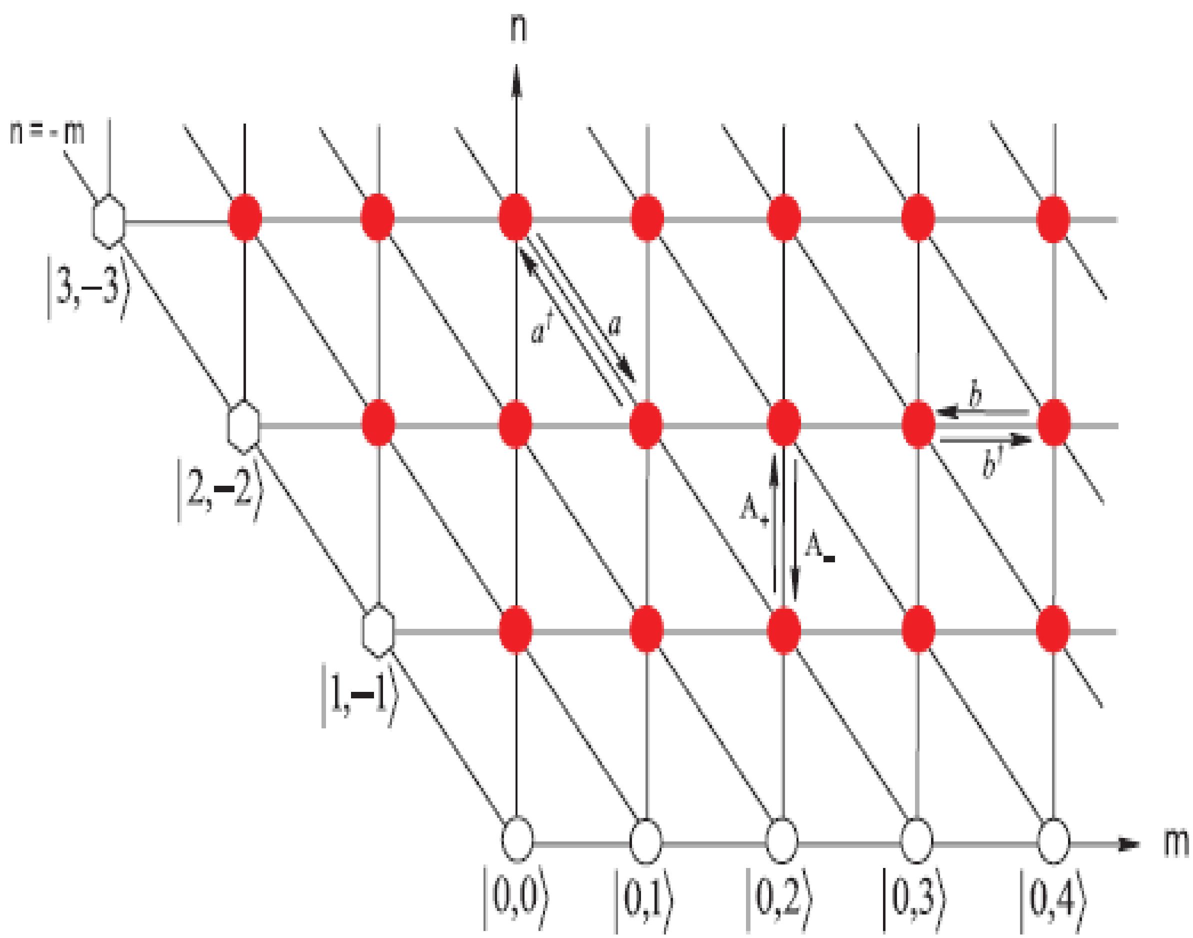
$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

$$\hat{L} = \hbar \left(\hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a} \right)$$

$$\hat{a}^\dagger \hat{a} |n, m\rangle = n |n, m\rangle$$

$$\hat{b}^\dagger \hat{b} |n, m\rangle = m |n, m\rangle$$

l=m-n



Here *m means l*

The common eigenstate of the operators ***a*** and ***b*** can be considered as the simplest coherent state of a charged particle in the uniform magnetic field.

$$\hat{a}|\alpha,\beta\rangle=\alpha|\alpha,\beta\rangle \qquad \hat{b}|\alpha,\beta\rangle=\beta|\alpha,\beta\rangle$$

$$\Phi_{\alpha\beta}^{(MM)}=\sqrt{\frac{m\omega_0}{2\pi\hbar}}\exp\left(-\zeta\zeta^*+\sqrt{2}\beta\zeta+i\sqrt{2}\alpha\zeta^*-i\alpha\beta-\frac{|\alpha|^2+|\beta|^2}{2}\right)$$

$$\zeta=\sqrt{\frac{m\omega_0}{4\hbar}}(x+iy), \qquad \zeta^*=\sqrt{\frac{m\omega_0}{4\hbar}}(x-iy)$$

Malkin I A and Man'ko V I 1969 Coherent states of a charged particle in a magnetic field *Sov. Phys. – JETP* **28** 527-532

Zh. Eksp. Teor. Fiz. **55**, 1014 (1968)

Similar constructions:

Feldman A. and Kahn A.H., Landau diamagnetism from the coherent states of an electron in a uniform magnetic field.
Phys. Rev. B (1970) **1** 4584–4589.
[circular gauge]

Tam W.G., Coherent states and the invariance group of a charged particle in a uniform magnetic field. *Physica* (1971) **54** 557–572.
[circular and Landau gauge]

COHERENT STATES OF AN ELECTRON IN A HOMOGENEOUS CONSTANT MAGNETIC FIELD AND THE ZERO MAGNETIC FIELD LIMIT
VARRO S., JOURNAL OF PHYSICS A, Vol. 17, pp. 1631-1638, (1984)

$$|\alpha, \beta\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\right] \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n!m!}} |n, m\rangle$$

“Partially coherent” (“semi-coherent”) states

Malkin I A and Man’ko V I 1969 Coherent states of a charged particle in a magnetic field *Sov. Phys. – JETP* **28** 527-532

$$a^+ a\Phi_{n_1\beta} = n_1\Phi_{n_1\beta}, \quad b\Phi_{n_1\beta} = \beta\Phi_{n_1\beta}.$$

$$\Phi_{n_1\beta} = |n_1\beta\rangle = D(\beta)\,|n_1 0\rangle = \sqrt{\frac{m\omega}{2\pi}}\,\frac{i^{n_1}}{\sqrt{n_1!}}\,(\sqrt{2}\,\bar{\xi} - \beta)^{n_1} \times \exp\left[-\xi\bar{\xi} + \sqrt{2}\,\beta\xi - \frac{|\beta|^2}{2}\right].$$

$$|\alpha n_2\rangle = D(\alpha)\,|0 n_2\rangle$$

$$|\alpha n_2\rangle = \sqrt{\frac{m\omega}{2\pi}}\,\frac{2^{n_2/2}}{\sqrt{n_2!}}\left(\xi - i\,\frac{\alpha}{\sqrt{2}}\right)^{n_2} \exp\left(-\bar{\xi}\xi + i\sqrt{2}\alpha\bar{\xi} - \frac{|\alpha|^2}{2}\right)$$

Various generalizations

The minimum-uncertainty coherent states for Landau levels
Dehghani A., Fakhri H., Mojaveri B.
JOURNAL OF MATHEMATICAL PHYSICS **53**, 123527 (2012)

su (1, 1) COHERENT STATES BY LEVELS WITH DIFFERENT ENERGY

For a given *m*

$$A_+ = a^\dagger b^\dagger \qquad A_- = ab, \qquad A_3 = \frac{1}{2}(aa^\dagger + b^\dagger b)$$

$$[A_+, A_-] = -2A_3 \qquad [A_3, A_\pm] = \pm A_\pm$$

$$A_+ \left|n-1, m\right\rangle = \sqrt{n(n+m)} \left|n, m\right\rangle$$

$$A_- \left|n, m\right\rangle = \sqrt{n(n+m)} \left|n-1, m\right\rangle$$

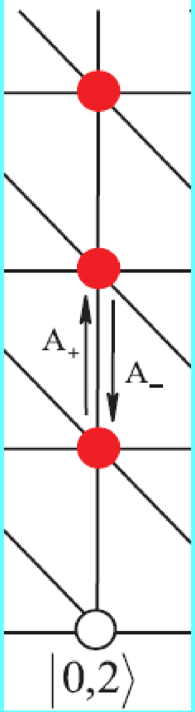
Here *m*=*l*

$$A_3 \left|n, m\right\rangle = \frac{1}{2}(2n+m+1) \left|n, m\right\rangle$$

Klauder-Perelomov coherent states

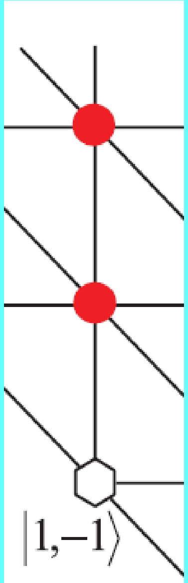
$$\lambda = \frac{\alpha \tanh|\alpha|}{|\alpha|}$$

$$|\lambda\rangle_m^{\mathrm{LLL}} = e^{\alpha A_+ - \bar{\alpha} A_-}\,|0,m\rangle = \frac{(1-|\lambda|^2)^{\frac{m+1}{2}}}{\sqrt{m!}} \sum_{n=0}^{\infty} \lambda^n \sqrt{\frac{(n+m)!}{n!}}\,|n,m\rangle$$



$$\langle r,\varphi|\lambda\rangle_m^{\mathrm{LLL}} = \sqrt{\frac{(\frac{M\omega}{2\hbar})^{m+1}}{\pi m!}} \left(\frac{\sqrt{1-|\lambda|^2}}{1-\lambda}\right)^{m+1} r^m e^{im\varphi} e^{\frac{M\omega r^2}{4\hbar}\frac{\lambda+1}{\lambda-1}}$$

$$|\lambda\rangle_m^{\mathrm{LLLAM}} = e^{\alpha A_+ - \bar{\alpha} A_-}\,|m,-m\rangle = \frac{(1-|\lambda|^2)^{\frac{m+1}{2}}}{\sqrt{m!}} \sum_{n=m}^{\infty} \lambda^{n-m} \sqrt{\frac{n!}{(n-m)!}}\,|n,-m\rangle$$



$$\langle r,\varphi|\lambda\rangle_m^{\mathrm{LLLAM}} = \sqrt{\frac{(\frac{M\omega}{2\hbar})^{m+1}}{\pi m!}} \left(\frac{\sqrt{1-|\lambda|^2}}{1-\lambda}\right)^{m+1} (-r)^m e^{-im\varphi} e^{\frac{M\omega r^2}{4\hbar}\frac{\lambda+1}{\lambda-1}}$$

su (1, 1) COHERENT STATES BY LEVELS WITH THE SAME ENERGY

$$K_+ = \frac{1}{2} b^{\dagger 2}$$

$$K_- = \frac{1}{2} b^2.$$

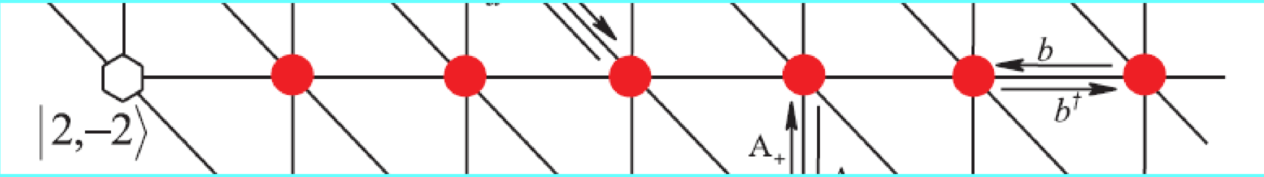
$$K_3 = \frac{1}{2}\left(b^\dagger b + \frac{1}{2}\right)$$

$$[K_+, K_-] = -2K_3$$

$$[K_3, K_\pm] = \pm K_\pm.$$

$$K_- \left| n, m \right\rangle = \frac{1}{2}\sqrt{(n+m-1)(n+m)} \left| n, m-2 \right\rangle$$

$$\left|\gamma\right\rangle_n^{\mathrm{LLLAM}} = e^{\alpha K_+ - \bar{\alpha} K_-} \left| n, -n \right\rangle = \left(\frac{1-|\gamma|^2}{\pi}\right)^{\frac{1}{4}} \sum_{k=0}^{\infty} \gamma^k \sqrt{\frac{\Gamma(k+\frac{1}{2})}{k!}} \left| n, -n+2k \right\rangle$$



su(2) COHERENT STATES

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = \pm J_\pm.$$

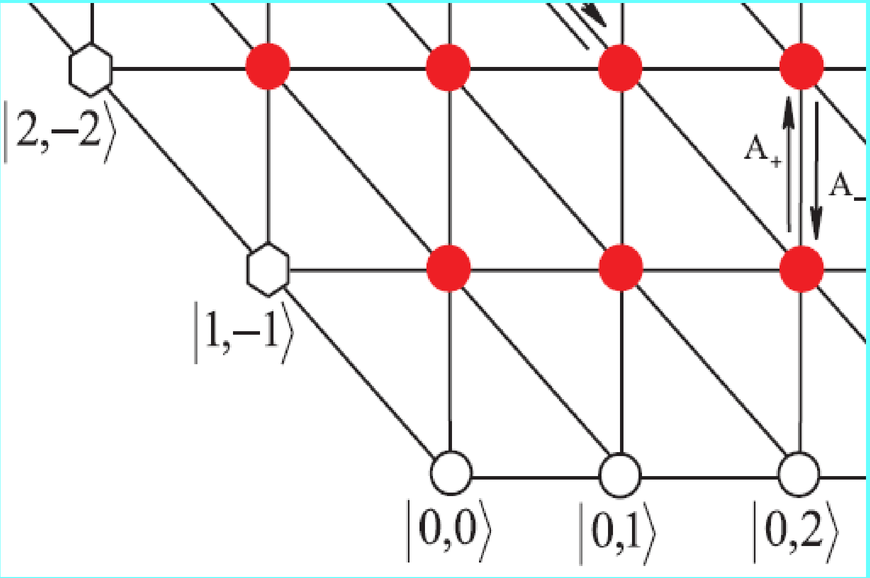
$$J_+ = ab^\dagger \qquad J_- = a^\dagger b \qquad J_3 = \frac{1}{2}(b^\dagger b - a^\dagger a)$$

$$J_+ \,|n+1, m-2\rangle = \sqrt{(n+1)(n+m)}\,|n, m\rangle$$

$$J_- \,|n, m\rangle = \sqrt{(n+1)(n+m)}\,|n+1, m-2\rangle$$

$$J_3 \,|n, m\rangle = \frac{m}{2}\,|n, m\rangle$$

$$|\eta\rangle_n = e^{\alpha J_+ - \bar{\alpha} J_-}\,|n, -n\rangle = (1 + |\eta|^2)^{\frac{-n}{2}} \sum_{k=0}^{n} \eta^k \sqrt{\frac{n!}{k!(n-k)!}}\,|n-k, 2k-n\rangle$$



Properties (squeezing, photon statistics):

Alireza Dehghani and Bashir Mojaveri,

Eur. Phys. J. D (2013), **67**, 264

Generalized su(2) coherent states for the Landau levels and their nonclassical properties

The same states were constructed earlier in

M. Novaes and J. P. Gazeau, J. Phys. A: Math. Gen. (2003) 199–212
Multidimensional generalized coherent states

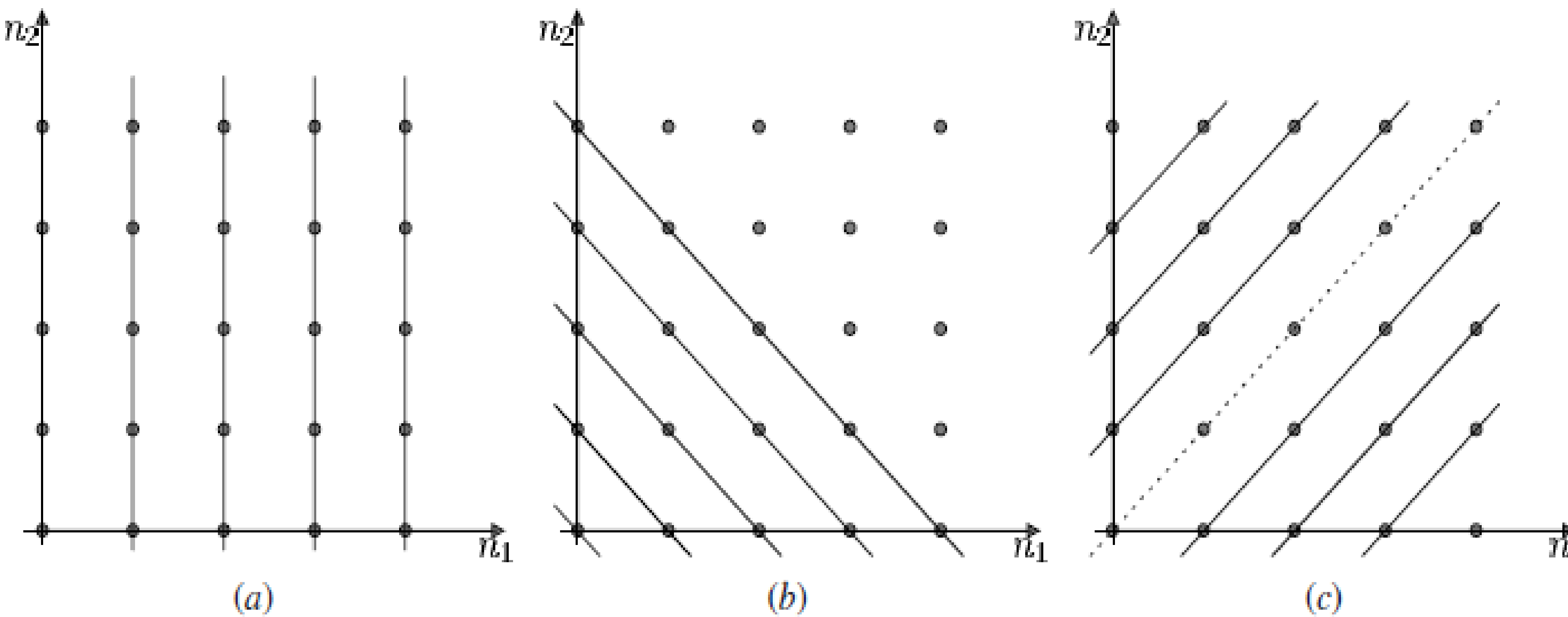


Figure 1. Pictorial representation of the state space $|n_1, n_2\rangle$. (*a*) Representations of 1D harmonic oscillator. (*b*) Irreducible representations of $su(2)$. (*c*) The discrete series of $su(1, 1)$; lines above (below) the dashed line belong to the first (second) sector, as considered in the text.

Dehghani A., Fakhri H., Mojaveri B.
JOURNAL OF MATHEMATICAL PHYSICS **53**, 123527 (2012)

$$B := \frac{b - \lambda a^\dagger}{\sqrt{1-|\lambda|^2}} \qquad B^\dagger := \frac{b^\dagger - \bar{\lambda} a}{\sqrt{1-|\lambda|^2}} \qquad [B, B^\dagger] = 1$$

$$B\,|\lambda\rangle_m^{\mathrm{LLL}} = \sqrt{m}\,|\lambda\rangle_{m-1}^{\mathrm{LLL}} \qquad B^\dagger\,|\lambda\rangle_m^{\mathrm{LLL}} = \sqrt{m+1}\,|\lambda\rangle_{m+1}^{\mathrm{LLL}}$$

$$A := \frac{a - \lambda b^\dagger}{\sqrt{1-|\lambda|^2}} \qquad A^\dagger := \frac{a^\dagger - \bar{\lambda} b}{\sqrt{1-|\lambda|^2}} \qquad [A, A^\dagger] = 1$$

$$A\,|\lambda\rangle_m^{\mathrm{LLLAM}} = \sqrt{m}\,|\lambda\rangle_{m-1}^{\mathrm{LLLAM}} \qquad A^\dagger\,|\lambda\rangle_m^{\mathrm{LLLAM}} = \sqrt{m+1}\,|\lambda\rangle_{m+1}^{\mathrm{LLLAM}}$$

$$|\beta, \lambda\rangle^{\mathrm{LLL}} = e^{\beta B^\dagger - \bar{\beta} B}\,|\lambda\rangle_0^{\mathrm{LLL}} = e^{\frac{-|\beta|^2}{2}} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}}\,|\lambda\rangle_m^{\mathrm{LLL}}$$

$$\langle r, \varphi | \beta, \lambda\rangle^{\mathrm{LLL}} = \sqrt{\frac{M\omega}{2\pi\hbar}}\,\frac{e^{\frac{-|\beta|^2}{2}}\sqrt{1-|\lambda|^2}}{1-\lambda}\,e^{\frac{M\omega}{4\hbar}\frac{\lambda+1}{\lambda-1}r^2 - \frac{\beta\sqrt{\frac{M\omega}{2\hbar}(1-|\lambda|^2)}}{\lambda-1} r e^{i\varphi}}$$

Actually, these are squeezed and correlated states

V.V.Dodonov, E.V.Kurmyshev, V.I.Man'ko: "Correlated coherent states",
in ``Classical and Quantum Effects in Electrodynamics'',
ed. A.A.Komar, Proceedings of Lebedev Physics Institute, v.176
(Nauka, Moscow, 1986, p.128-150;
Nova Science, Commack, N.Y., 1988, p.169-199)

NEW SQUEEZED LANDAU STATES
ARAGONE C.,
PHYSICS LETTERS A, Vol. 175, 377-381 (1993)

Squeezed states of a particle in magnetic field
Ozana M. and Shelankov A.L.
PHYSICS OF THE SOLID STATE **40**, 1276-1282 (1998)

$su(1,1)$-Barut–Girardello coherent states for Landau levels

H. Fakhri, J. Phys. A: Math. Gen. (2004) 5203–5210

$$K_-|z\rangle_m = z|z\rangle_m$$

$$|z\rangle_m = \frac{|z|^{\frac{\alpha+m}{2}}}{\sqrt{I_{\alpha+m}(2|z|)}} \sum_{n=m}^{+\infty} \frac{z^{n-m}|n,m\rangle}{\sqrt{\Gamma(n-m+1)\Gamma(\alpha+n+1)}}$$

$$K_-|m,m\rangle = 0 \qquad A_- = ab.$$

CHIN.PHYS.LETT. Vol. 16, No. 10 (1999) 706

Angular Momentum Conserved Coherent State for an Electron in a Uniform Magnetic Field *

FAN Hong-yi(范洪义)[1,2], ZOU Hui(邹晖)[2], FAN Yue(范悦)[2,3]

COHERENT STATES FOR LANDAU LEVELS:
ALGEBRAIC AND THERMODYNAMICAL PROPERTIES

Aremua I., Hounkonnou M.N., Baloitcha E.,

REPORTS ON MATHEMATICAL PHYSICS **76** 247-269 (2015)

Displaced Landau states

Wen-Long Yang and Jing-Ling Chen,
PHYSICAL REVIEW A 75, 024101(2007)
Berry's phase for **coherent states of Landau levels**

$$|n(\alpha),m\rangle = \exp(\alpha b^{\dagger} - \alpha^{*} b)|n,m\rangle$$

$$|n(\alpha,\beta),m\rangle = D(\alpha)S(\beta)|n,m\rangle$$

Discrete coherent states for higher Landau levels

L.D. Abreu, P. Balazs, M. de Gosson, Z. Mouayn
Annals of Physics 363 (2015) 337–353

$$\langle t \mid (x,y)\,,B,n\rangle = \left(\sqrt{\pi}\,2^n n!\right)^{-\frac{1}{2}} \exp\left(-i\sqrt{B}ty + i\frac{B}{2}xy - \frac{1}{2}\left(t-\sqrt{B}x\right)^2\right) H_n\left(t-\sqrt{B}x\right)$$

$$\mathbf{1}_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2} |(x,y)\,,B,n\rangle\,\langle (x,y)\,,B,n|\, d\mu\,(x,y)$$

Generalization: nonlinear coherent state

K Kowalski and J Rembielinski,
J. Phys. A: Math. Gen. (2005) **38** 8247–8258

J.P. Gazeau, M.C. Baldiotti, D.M. Gitman,
Physics Letters A 373 (2009) 1916–1920

$$J = \xi\pi_y - \eta\pi_x = -2H/\omega$$

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

$$\hat{b}|\zeta, \beta\rangle = \beta|\zeta, \beta\rangle$$

$$\exp\left(\hat{a}^\dagger \hat{a}\right) \hat{a}|\zeta, \beta\rangle = \zeta|\zeta, \beta\rangle$$

$$|\zeta, \beta\rangle = \mathcal{N} \sum_{n,m=0}^{\infty} \frac{\zeta^n \beta^m}{\sqrt{n!m!}} \exp\left[-\frac{1}{2}(n - 1/2)^2\right] |n, m\rangle$$

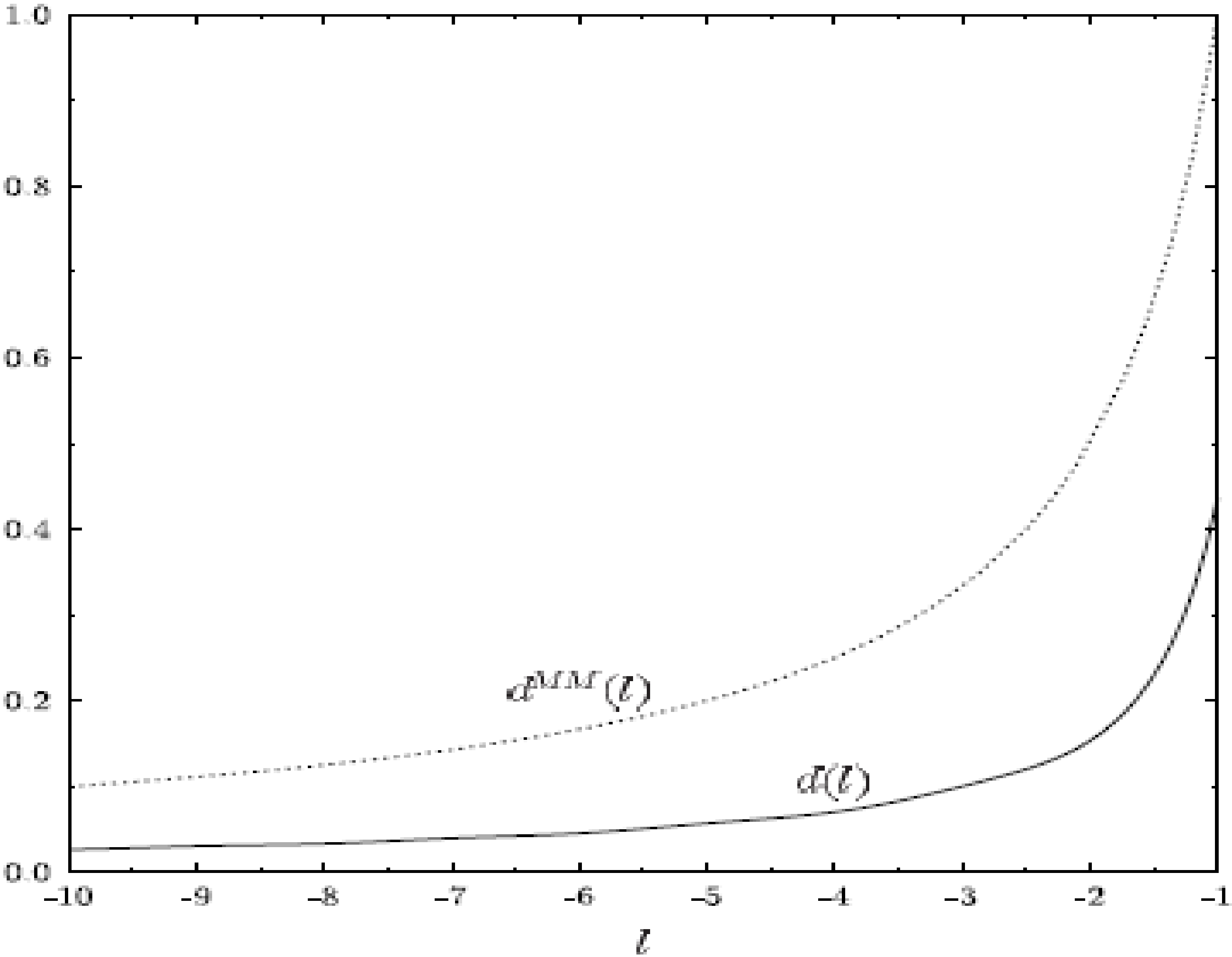


Figure 2. Comparison of the closeness to the phase space of the coherent states introduced in this work (solid line) and the Malkin–Man'ko coherent states (dotted line) by means of the distances $d(l)$ and $d^{\mathrm{MM}}(l)$ given by (5.8) and (5.9), respectively, with $\mu\omega = 1$.

Time-dependent coherent states

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 8 AUGUST 1969

An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field*

H. R. LEWIS, JR., AND W. B. RIESENFELD

$$\left(i\hbar\partial\widehat{I}/\partial t-[\widehat{H}\ ,\ \widehat{I}]\right)\psi(t)=0$$

Volume 30A, number 7 PHYSICS LETTERS 1 December 1969

INVARIANTS AND THE EVOLUTION OF COHERENT STATES FOR A CHARGED PARTICLE IN A TIME-DEPENDENT MAGNETIC FIELD

I. A. MALKIN, V. I. MAN'KO and D. A. TRIFONOV
P. N. Lebedev Physical Institute, Moscow, USSR

Received 22 October 1969

I. A. MALKIN, V. I. MAN'KO, and D. A. TRIFONOV
SOVIET PHYSICS JETP **31**, no. 2 (AUGUST, 1970), p.386-390
[Zh. Eksp. Teor. Fiz. 58, 721-729 (February, 1970)] Submitted August 14, 1969
EVOLUTION OF COHERENT STATES OF A CHARGED PARTICLE IN A VARIABLE MAGNETIC FIELD

Malkin I.A., Man’ko V.I. and Trifonov D.A.,
Coherent states and transition probabilities in a time-dependent electromagnetic field. *Phys. Rev.* D (1970) **2** 1371–1385.

Dodonov V.V., Malkin I.A. and Man’ko V.I.,
Coherent states of a charged particle in a time-dependent uniform electromagnetic field of a plane current. *Physica* (1972) **59** 241–256.

Malkin I.A., Man’ko V.I. and Trifonov D.A.,
Linear adiabatic invariants and coherent states.
J. Math. Phys. (1973) **14** 576–582.

Fiore G. and Gouba L.
Class of invariants for the two-dimensional time-dependent Landau problem and harmonic oscillator in a magnetic field
JOURNAL OF MATHEMATICAL PHYSICS **52** 103509 (2011)

How to calculate the coherent state wave function in the simplest way?

$$\hat{\mathbf{A}} = (\hat{a}_1, ...\hat{a}_N)$$

$$\hat{\mathbf{A}}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}$$

$$[\hat{a}_j, \hat{a}_k] = 0$$

$$f(\mathbf{x};\alpha) = \langle\mathbf{x}|\alpha\rangle \exp\left(|\alpha|^2/2\right)$$

$$\hat{\mathbf{A}} f(\mathbf{x};\alpha) = \alpha f(\mathbf{x};\alpha)$$

$$\hat{\mathbf{A}}^\dagger f(\mathbf{x};\alpha) = \partial f(\mathbf{x};\alpha)/\partial\alpha$$

$$\hat{\mathbf{A}} = \lambda_p \hat{\mathbf{p}} + \lambda_q \hat{\mathbf{x}} + \delta,$$

$$\lambda_p \tilde{\lambda}_q = \lambda_q \tilde{\lambda}_p, \quad \tilde{\lambda}_q \lambda_q^* = \lambda_q^\dagger \lambda_q, \quad \tilde{\lambda}_p \lambda_p^* = \lambda_p^\dagger \lambda_p,$$

$$\lambda_p \lambda_q^\dagger - \lambda_q \lambda_p^\dagger = \lambda_q^\dagger \lambda_p - \tilde{\lambda}_q \lambda_p^* = (i/\hbar) E_N$$

$$\langle \mathbf{x} | \alpha \rangle = \left(2\pi\hbar^2\right)^{-N/4} \left(\det \lambda_p\right)^{-1/2} \exp\left[-\frac{i}{2\hbar} \mathbf{x} \lambda_p^{-1} \lambda_q \mathbf{x} + \frac{i}{\hbar} \mathbf{x} \lambda_p^{-1} (\alpha - \delta) \right.$$
$$\left. + \tfrac{1}{2} (\alpha - \delta) \lambda_p^* \lambda_p^{-1} (\alpha - \delta) + \alpha \delta^* - \tfrac{1}{2} (|\delta|^2 + |\alpha|^2) + i \int_0^t \operatorname{Im}[\dot{\delta}(\tau) \delta^*(\tau)] \mathrm{d}\tau \right].$$

V.V.Dodonov and V.I.Man'ko: Invariants and the evolution of nonstationary quantum systems, Proceedings of Lebedev Physics Institute, v.183 (Nauka, Moscou, 1987;
Nova Science, Commack, N.Y., 1989)

$$\dot{\lambda}_p = \lambda_p b_3 - \lambda_q b_1 \qquad \dot{\lambda}_q = \lambda_p b_4 - \lambda_q b_2$$

$$\dot{\delta} = \lambda_p \mathbf{c}_2 - \lambda_q \mathbf{c}_1$$

$$H = \frac{1}{2} \sum_{j,k=1}^{2N} B_{jk}(t) q_j q_k + \sum_{j=1}^{2N} C_j(t) q_j = \frac{1}{2} \mathbf{q} B(t) \mathbf{q} + \mathbf{C}(t) \mathbf{q}$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{p} \\ \mathbf{x} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}, \qquad B = \begin{Vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Vmatrix}$$

Coherent states of a relativistic charged particle in magnetic field

Malkin I A and Man'ko V I 1969 Coherent states of a charged particle in a magnetic field *Sov. Phys. – JETP* **28** 527-532

Problems with time evolution (temporal stability), since the spectrum is not equidistant.

A possible solution: to use the proper time representation and the null plane variables

V.V.Dodonov, I.A.Malkin, V.I.Man'ko:
"Coherent states and Green functions of relativistic quadratic systems",
Physica, 1976, v.82A, p.113-133

V.G.Bagrov, I.L.Buchbinder, D.M.Gitman, 1976,
J. Phys. A, v.9, no.11, p.1955-1965
Coherent states of a relativistic particle
in an external electromagnetic field

I.M.Ternov and V.G.Bagrov, 1983, Ann. Physik,**40**,p.2-9
On coherent states of relativistic particles

$$\hat{\xi}_3 = \hat{p}_0 - \hat{p}_z \quad \text{and} \quad \hat{\eta}_4 = \tfrac{1}{2}(\hat{p}_0 + \hat{p}_z)$$

the Klein–Gordon equation

$$\left[-\hat{\xi}_3\hat{\eta}_4 + \tfrac{1}{2}(\hat{\pi}_x^2 + \hat{\pi}_y^2) + \tfrac{1}{2}m^2\right]\psi = 0$$

The operator $\hat{\xi}_3 \equiv \hat{I}$ is the integral of the motion.

"new time" $s = (t - z)/I$: $\quad \hat{\xi}_3\hat{\eta}_4 = \mathrm{i}\,\partial/\partial s$

$$\begin{aligned}
&\psi_{\alpha_1,\alpha_2,I}(x, y, z, t) \\
&= (2\pi)^{-2}\,\mathscr{H}^{\frac{1}{2}} \exp\left\{-\frac{\mathrm{i}(\mathscr{H} + m^2)}{2I}(t - z) - \tfrac{1}{2}\mathrm{i}I(t + z)\right. \\
&\quad - \tfrac{1}{4}\mathscr{H}(x^2 + y^2) + (\tfrac{1}{2}\mathscr{H})^{\frac{1}{2}}\left[(x - \mathrm{i}y)\alpha_1 \exp\left(-\frac{\mathrm{i}\mathscr{H}}{I}(t - z)\right)\right. \\
&\quad \left.\left. + \alpha_2(x + \mathrm{i}y)\right] - \alpha_1\alpha_2 \exp\left(-\frac{\mathrm{i}\mathscr{H}}{I}(t - z)\right) - \tfrac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2\right\}
\end{aligned}$$

Recent article: E. Colavita, S. Hacyan, 2014, Ann. Phys. 342, 205-213
Coherent quantum states of a relativistic particle in an electromagnetic plane wave and a parallel magnetic field

Relativistic coherent states and charge structure of the coordinate and momentum operators

Lev BI. Semenov AA, Usenko CV, Klauder JR,
PHYSICAL REVIEW A **66**, 022115 (2002)

They used the **even part** of the annihilation operator, that can be considered as ***a deformed annihilation operator.***

As the result, they arrived at the **nonlinear coherent state**

$$|\alpha,\pm\rangle = \mathcal{N}^{-1/2}(|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}[\varepsilon(n)]!}|n\rangle\otimes|\pm\rangle$$

$$\varepsilon(n) = \frac{E(n-1)+E(n)}{2\sqrt{E(n-1)E(n)}}$$

$$E(n) = mc^2\sqrt{1+2\lambda^2\left(n+\frac{1}{2}\right)},$$

Generalizations: “photon-added states”

Generation of excited coherent states for a charged particle in a uniform magnetic field

B. Mojaveri and A. Dehghani

Journal of Mathematical Physics **56**, 041704 (2015)

$$|\beta,\alpha;\mathbf{n}\rangle := a^{\dagger\mathbf{n}}|\beta,\alpha\rangle$$

Such states for a harmonic oscillator were introduced in:

G. S. Agarwal and K. Tara, Phys. Rev. A 43, 492 (1991).

Coherent states in the combination of homogeneous and Aharonov-Bohm magnetic fields

J. Phys. A: Math. Theor. **43** (2010) 354016
Coherent states of non-relativistic electron in the magnetic–solenoid field
V G Bagrov, S P Gavrilov, D M Gitman and D P Meira Filho

J. Phys. A: Math. Theor. 45 (2012) 244008
Completeness for coherent states in a magnetic-solenoid field
V G Bagrov, S P Gavrilov, D M Gitman and K. Gorska

$$B_z = B + \Phi\delta(x)\delta(y) = B + \frac{\Phi}{\pi r}\delta(r)$$

$$A_x = -y\left(\frac{\Phi}{2\pi r^2} + \frac{B}{2}\right), \qquad A_y = x\left(\frac{\Phi}{2\pi r^2} + \frac{B}{2}\right)$$

Further generalizations: supersymmetries

JOURNAL OF MATHEMATICAL PHYSICS **49**, 032110 (2008)

Supersymmetric associated vector coherent states and generalized Landau levels arising from two-dimensional supersymmetry

S. Twareque Ali[1,a)] and F. Bagarello[2,b)]

SUSY partner Hamiltonians

DYNAMICAL AND KINEMATICAL SUPERSYMMETRIES
OF THE QUANTUM HARMONIC-OSCILLATOR
AND THE MOTION IN A CONSTANT MAGNETIC-FIELD
BECKERS J., DEHIN D., HUSSIN V.
JOURNAL OF PHYSICS A **21** 651-667 (1988)

Spin-orbit coupling (bosons+fermions)

QM on non-commutative plane

Semiclassical and quantum motions on the **non-commutative plane**

M.C. Baldiotti, J.P. Gazeau, D.M. Gitman

Physics Letters A 373 (2009) 3937–3943

Non-Euclidean geometries:

Coherent states for a 2-sphere with a magnetic field
2012 J. Phys. A: Math. Theor. 45 244025
Brian C Hall and Jeffrey J Mitchell

Coherent states attached to Landau levels on the Poincare disc
Mouayn Z
JOURNAL OF PHYSICS A 38 9309-9316 (2005)

Coherent states attached to Landau levels on the Riemann sphere
Mouayn Z
REPORTS ON MATHEMATICAL PHYSICS **55** 269-276 (2005)

Generalizations: semi-coherent states

$$\langle \hat{a} \rangle^2 = \langle \hat{a}^2 \rangle$$

Mathews, P.M., Eswaran, K.: Nuovo Cimento B **17**, 332 (1973)
"Semi-Coherent States of the Quantum Harmonic Oscillator"

$$|\alpha_\perp \beta\rangle = \frac{|\alpha\rangle - |\beta\rangle\langle\beta|\alpha\rangle}{(1 - |\langle\beta|\alpha\rangle|^2)^{\frac{1}{2}}}$$

V V Dodonov and M B Renó,
Nonclassical properties of `semi-coherent' quantum states,
J. Phys. A, v.39, no.23, p. 7411-7422 (2006)

A. Dehghani and B. Mojaveri, Int J Theor Phys (2015) 54:3507–3515,
"New Semi Coherent States: Nonclassical Properties"

$$|(\beta,\alpha)_\perp(\beta',\alpha')\rangle = \frac{|\beta,\alpha\rangle - \mathcal{N}|\beta',\alpha'\rangle}{\sqrt{1-|\mathcal{N}|^2}},$$

Geometrical squeezed states of a charged particle in a time-dependent magnetic field

V.V. Dodonov, V.I. Man'ko, P.G. Polynkin

Physics Letters A 188 (1994) 232-238

$$\sigma_{\alpha\beta} = \tfrac{1}{2}\langle\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\rangle - \langle\hat{\alpha}\rangle\langle\hat{\beta}\rangle$$

$$\alpha, \beta = X, Y, \xi, \eta$$

$$\sigma_{\xi\xi}(t) = \sigma_{\xi\xi}(\tau)\cos^2[\omega(t-\tau)] + \sigma_{\eta\eta}(\tau)\sin^2[\omega(t-\tau)] + \sigma_{\xi\eta}(\tau)\sin[2\omega(t-\tau)]$$

$$\sigma_{\xi\xi}^{(\min)} = \tfrac{1}{2}(T - \sqrt{T^2 - 4d}) = \frac{2d}{T + \sqrt{T^2 - 4d}}$$

$$T = \sigma_{\xi\xi} + \sigma_{\eta\eta}$$

$$d = \sigma_{\xi\xi}\sigma_{\eta\eta} - \sigma_{\xi\eta}^2$$

Lukš, A., Peřinová, V., and Hradil, Z., Principal squeezing. *Acta Phys. Polon.* A (1988) **74** 713–721.

How to create squeezed states from the coherent ones?

$$\omega(t) = \omega_0 \Theta(t)$$

"circular" gauge.

$$\ddot{\varepsilon} + \tfrac{1}{4}\omega^2(t)\varepsilon = 0$$

$$\dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*\varepsilon = 2\mathrm{i}$$

$$\sigma_{\xi\xi} = \sigma_{\eta\eta} = \sigma_{XX} = \sigma_{YY} = \frac{\hbar}{8\omega_0^2 m}\left(\omega_0^2|\varepsilon|^2 + 4|\dot{\varepsilon}|^2\right)$$

$$\geqslant \frac{\hbar|\varepsilon\dot{\varepsilon}|}{2\omega_0 m} \geqslant \frac{\hbar\,\mathrm{Im}(\varepsilon^*\dot{\varepsilon})}{2\omega_0 m} = \frac{\hbar}{2\omega_0 m}$$

the time-dependent Landau gauge,

$$\hat{H}(t) = \tfrac{1}{2} m\omega_0^2 \{\hat{\xi}^2 + \Theta^2(t)\hat{\eta}^2 + [\Theta(t) - 1]^2 \hat{Y}^2 + 2[\Theta(t) - 1]\hat{\eta}\hat{Y}\}$$

$$\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0$$

$$\sigma(t) = \int \omega(\tau)\varepsilon(\tau)\,\mathrm{d}\tau$$

$$s = \tfrac{1}{2}\mathrm{i}(\sigma\varepsilon^* - \varepsilon\sigma^*)$$

$$\kappa(t) = \int [1 - \omega(\tau)s(\tau)]\,\mathrm{d}\tau.$$

$$\tilde{\sigma}_{\alpha\beta} \equiv 2m\omega_0\sigma_{\alpha\beta}/\hbar$$

$$\tilde{\sigma}_{XX}^{(min)} = \frac{1 + \left|\omega_0\sigma + \dot{\varepsilon}\right|^2/\omega_0}{1 + \left|\omega_0\sigma + \dot{\varepsilon}\right|^2/\omega_0 + (\dot{s} - \omega_0\kappa)^2}$$

$$\omega^2(t) = \omega_0^2 + 2\gamma\delta(t), \qquad \gamma > 0$$

$$\varepsilon(t) = \alpha e^{it} + \beta e^{-it}$$

$$\alpha = 1 + i\gamma,\ \beta = -i\gamma.$$

$$\tilde{\sigma}_{\xi\xi}^{(\mathrm{min})} = \frac{\sqrt{1+4\gamma^2}}{\sqrt{1+4\gamma^2}+2\gamma} < 1$$

$$\tilde{\sigma}_{XX}^{(min)} = \frac{1+4\gamma^2}{1+8\gamma^2}$$

Parametric resonance

$$\omega(t) = \omega_0[1 + 2\gamma\cos(2\omega_0 t)]$$

For $|\gamma| \ll 1$ we have an approximate solution, neglecting terms of the order of $\mathcal{O}(\gamma)$

$$\varepsilon(t) = \omega_0^{-1/2}\left[\cosh(\omega_0\gamma t)\, e^{i\omega_0 t} - i\sinh(\omega_0\gamma t)\, e^{-i\omega_0 t}\right]$$

$$\tilde{\sigma}_{\xi\xi}(t) = \cosh(2\omega_0\gamma t) + \sinh(2\omega_0\gamma t)\sin(2\omega_0 t)$$

$$T = 2\cosh(2\omega_0\gamma t)$$

$$d = 1$$

$$\tilde{\sigma}_{XX}^{(min)} = 1$$

$$\tilde{\sigma}_{\xi\xi}^{(min)}(t) = \exp(-2\omega_0\gamma t)$$

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **48** (2015) 435303 (23pp) doi:10.1088/1751-8113/48/43/435303

Rotating quantum Gaussian packets

V V Dodonov

PHYSICAL REVIEW A **93**, 022106 (2016)

Rotating highly mixed Gaussian packets with minimal energy

V. V. Dodonov*

We study two-dimensional Gaussian packets with a fixed value of mean angular momentum.

The main question is:

what is the minimal energy of such packets?

isotropic oscillator: $$E_{n_r m} = \hbar\omega\left(1 + |m| + 2n_r\right)$$

$$E_{min}(m) = \hbar\omega\left(1 + |m|\right)$$

But what is the minimal mean value of energy for superpositions of energy eigenstates with a fixed *mean value* of angular momentum $\hbar\mathcal{L}$

$$\langle E \rangle_{min}(\mathcal{L}) = \hbar\omega(1 + |\mathcal{L}|)$$

This value is achieved in superpositions of states with zero value of radial quantum number and the *same signs* of angular quantum numbers m:

$$\psi_{min} = \sum_m c_m \psi_{0m}$$

$$\sum_m |c_m|^2 = 1$$

$$\sum_m m|c_m|^2 = \hbar\mathcal{L}$$

$$\sum_m |m|\,|c_m|^2 = \hbar|\mathcal{L}|$$

Do such *Gaussian packets* exist?

What can happen for *mixed* quantum states?

Pure quantum states of the 2D harmonic oscillator

$$\psi(x,y) = \tilde{N} \exp\left[-\mu\left(ax^2 + bxy + cy^2\right) + Fx + Gy\right]$$

$$\mu = M\omega/\hbar$$

$$a = \alpha/2 + i\chi_a, \quad b = \beta + i\rho, \quad c = \gamma/2 + i\chi_c,$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$\langle \hat{L}_z \rangle \equiv \hbar \mathcal{L} = \hbar \left(\mathcal{L}_c + \mathcal{L}_i \right)$$

$$\hbar \mathcal{L}_c = x_0 p_{y0} - y_0 p_{x0}$$

$$\mathcal{E} = \mathcal{E}_c + \mathcal{E}_i$$

$$\mathcal{E}_c = \frac{1}{2M} \left(p_{x0}^2 + p_{y0}^2 \right) + \frac{M\omega^2}{2} \left(x_0^2 + y_0^2 \right)$$

$$\mathcal{E}_c^{(min)}(\mathcal{L}_c) = \hbar\omega |\mathcal{L}_c|$$

$$x_0^2 + y_0^2 = R^2 = |\mathcal{L}_c| / \mu$$

$$\mathcal{L}_c = \lambda_c |\mathcal{L}_c|$$

$$\mathcal{L}_i = \left(\overline{xp_y} - \overline{yp_x}\right)/\hbar = \left[2\beta\left(\chi_c - \chi_a\right) + \rho(\alpha - \gamma)\right]/(2\Delta)$$

$$\mathcal{E}_i = \frac{\hbar\omega}{4\Delta}\left[(\alpha+\gamma)\left(\Delta+\rho^2\right) + 4\left(\gamma\chi_a^2 + \alpha\chi_c^2\right) - 4\beta\rho\left(\chi_a + \chi_c\right)\right]$$

$$\Delta = \alpha\gamma - \beta^2$$

$$\overline{AB} \equiv \langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle/2 - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$\mathcal{E}_i^{(min)} = \hbar\omega(1 + |\mathcal{L}_i|)$$

$$\mathcal{E}_{min} = \hbar\omega\left(1 + |\mathcal{L}_i| + |\mathcal{L}_c|\right)$$

$$\psi(x,y) = \tilde{N} \exp\left[-\mu\left(ax^2 + bxy + cy^2\right) + Fx + Gy\right]$$

$$a = \frac{1}{2}\left[1 + \eta\exp(-i\lambda u)\right]$$

$$c = \frac{1}{2}\left[1 - \eta\exp(-i\lambda u)\right]$$

$$b = i\lambda\eta\exp(-i\lambda u)$$

$$\lambda = \pm 1$$

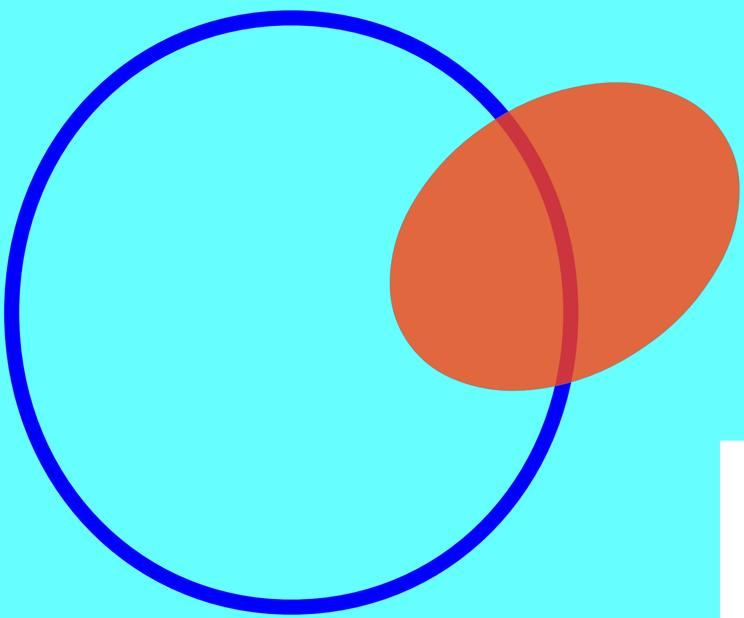
$$\eta^2 = \frac{|\mathcal{L}_i|}{1 + |\mathcal{L}_i|}$$

$$|\psi(x,y)|^2 \;=\; \exp(-\nu)|\psi(x_0, y_0)|^2$$

ellipses

$$S_{circ} = \pi R^2 = \pi |\mathcal{L}_c|$$

$$S_{ellips} = \pi a_+ a_- = 4\nu^2 (1 + |\mathcal{L}_i|)$$



$$\varepsilon = \left[2\eta / \left(1 + \eta\right)\right]^{1/2}$$

$$a_\pm = \frac{2\nu}{1 \mp \eta}$$

The physical meaning of parameters

invariant squeezing

$$S_x = 2\left(\tilde{\mathcal{E}}_x - \sqrt{\tilde{\mathcal{E}}_x^2 - \tilde{U}_x}\right)$$

$$\tilde{\mathcal{E}} = \mathcal{E}/(\hbar\omega)$$

$$\tilde{U} = U/\hbar^2.$$

Robertson–Schrödinger uncertainty product

$$U_x \equiv \overline{x^2}\,\overline{p_x^2} - \left(\overline{xp_x}\right)^2$$

$$U_x = U_y = \frac{\hbar^2}{4}\left(1 + |\mathcal{L}_i|\right)$$

$$S_x = S_y = \frac{1}{1+\eta} < 1$$

$$\eta^2 = \frac{|\mathcal{L}_i|}{1+|\mathcal{L}_i|}$$

the maximal degree of squeezing cannot exceed 50%, since $\eta^2 < 1$

correlation coefficient

$$r_{gf} = \overline{gf}/\sqrt{\overline{g^2}\,\overline{f^2}}.$$

$$r_{p_x p_y} = -r_{xy} = \lambda r_{xp_x} = -\lambda r_{yp_y} = \frac{\eta \sin(u)}{\sqrt{1-\eta^2\cos^2(u)}}$$

$$|r_{max}| = \eta = \sqrt{|\mathcal{L}|/(1+|\mathcal{L}|)}$$

Angular momentum and Energy fluctuations

$$\sigma_L = \langle \hat{L}^2 \rangle - \langle \hat{L} \rangle^2$$

$$\sigma_E = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$$

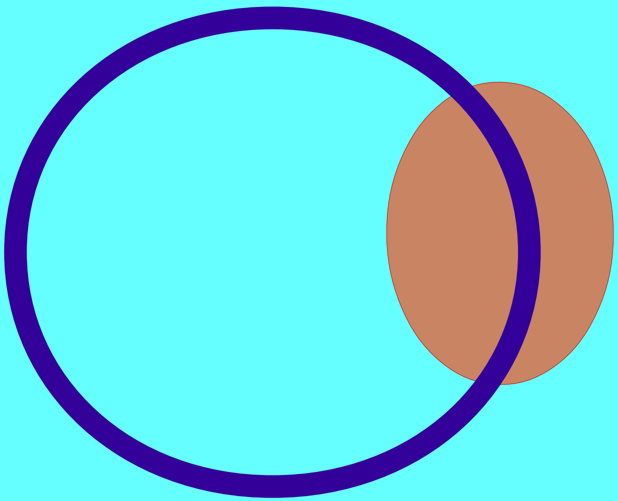
$$\sigma_L / \hbar^2 = |\mathcal{L}_c| + 2\,|\mathcal{L}_i|\,(1 + |\mathcal{L}_i|)\,, \quad \lambda\lambda_c = -1.$$

$$\sigma_L / \hbar^2 = \mathcal{L} + \mathcal{L}_i(1 + 2\mathcal{L}) - 2\mathcal{L}_c\sqrt{\mathcal{L}_i\,(1 + \mathcal{L}_i)}\cos(2w)$$

$$\lambda\lambda_c = +1, \quad \mathcal{L} = \mathcal{L}_i + \mathcal{L}_c$$

co-rotating case

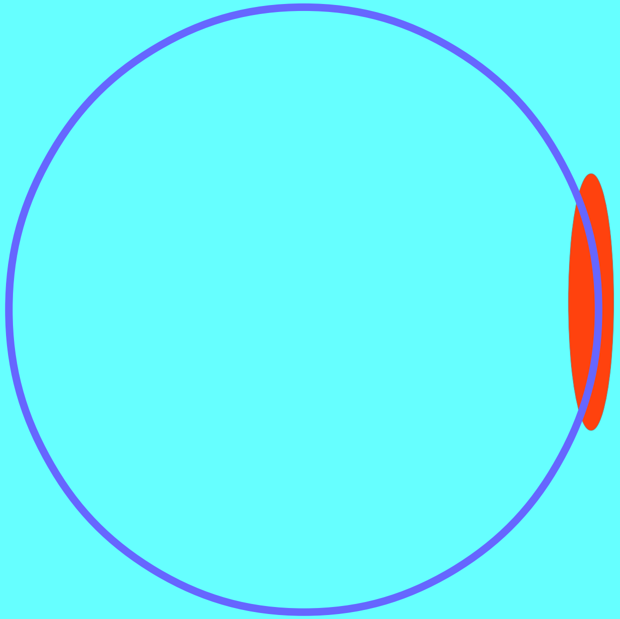
minimum of σ_L



$\mathcal{L}_i \gg 1$ → $\mathcal{L} \approx 16\mathcal{L}_i^3$.

$$\sqrt{\sigma_L^{min}}/(\hbar\mathcal{L}) \approx \sqrt{3/2^{1/3}}(2\mathcal{L})^{-2/3}$$

smaller than the Poissonian width $\mathcal{L}^{-1/2}$



Looking for *mixed* states

$$\rho(\mathbf{r},\mathbf{r}') = \tilde{N}\exp\left[-\frac{1}{2}\left(\mathbf{r}A\mathbf{r} + 2\mathbf{r}B\mathbf{r}' + \mathbf{r}'A^*\mathbf{r}'\right)\right]$$

$$x = x_d(M\omega/\hbar)^{1/2}$$

$$y = y_d(M\omega/\hbar)^{1/2}$$

$$purity\ \mathcal{P} = \mathrm{Tr}\left(\hat{\rho}^2\right) = \frac{1}{4}(\det\mathcal{M})^{-1/2}$$

$$W(\mathbf{q}) = (\det\mathcal{M})^{-1/2}\exp\left[-\frac{1}{2}\left(\mathbf{q}-\langle\mathbf{q}\rangle\right)\mathcal{M}^{-1}\left(\mathbf{q}-\langle\mathbf{q}\rangle\right)\right]$$

$$\mathbf{q} \equiv (\mathbf{p},\mathbf{r})$$

$$\mathcal{P} = \sqrt{\frac{\det[\mathrm{Re}(A+B)]}{\det[\mathrm{Re}(A-B)]}} = \sqrt{\frac{\alpha_{11}^{+}\alpha_{22}^{+} - \left(\alpha_{12}^{+}\right)^2}{\alpha_{11}^{-}\alpha_{22}^{-} - \left(\alpha_{12}^{-}\right)^2}}.$$

$$\alpha_{jk}^{\pm} \equiv a_{jk} \pm b_{jk}$$

We have 10 real parameters

$$A = \left\| \begin{matrix} a_{11} + i\chi_{11} & a_{12} + i\chi_{12} \\ a_{12} + i\chi_{12} & a_{22} + i\chi_{22} \end{matrix} \right\|$$

$$B = \left\| \begin{matrix} b_{11} & b_{12} + i\gamma \\ b_{12} - i\gamma & b_{22} \end{matrix} \right\|$$

$$B = B^{\dagger}$$

$$\mathcal{P} \le 1 \rightarrow b_{11}a_{22} + b_{22}a_{11} - 2b_{12}a_{12} \le 0.$$

$$\hat{H} = \frac{1}{2}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{x}^2 + \hat{y}^2\right)$$

mean energy

$$\mathcal{E} = \frac{1}{4\Delta}\left[\alpha_{11}^{+} + \alpha_{22}^{+} + \left(\alpha_{11}^{-} + \alpha_{22}^{-}\right)\Delta + f\left(\chi_{jk}\right)\right]$$

$$\Delta = \alpha_{11}^{+}\alpha_{22}^{+} - \left(\alpha_{12}^{+}\right)^2$$

$$f\left(\chi_{jk}\right) = \alpha_{11}^{+}\left(\chi_{22}^2 + \chi_{+}^2\right) + \alpha_{22}^{+}\left(\chi_{11}^2 + \chi_{-}^2\right) - 2\alpha_{12}^{+}\left(\chi_{11}\chi_{+} + \chi_{22}\chi_{-}\right).$$

$$\chi_{\pm} = \chi_{12} \pm \gamma.$$

$$\mathcal{L} = \left[\alpha_{12}^{+}\left(\chi_{22} - \chi_{11}\right) + \alpha_{11}^{+}\chi_{+} - \alpha_{22}^{+}\chi_{-}\right]/(2\Delta)$$

The natural first step is to minimize function $f(\chi_{jk})$ with *L=const*

$$2\mathcal{E} = g\left(1-\mathcal{L}^2\right)+\frac{g}{\Delta}+\frac{g}{\eta^2}\left(\mathcal{L}g-\gamma\right)^2+2\gamma\mathcal{L}-b_{11}-b_{22}$$

$$\alpha_{11}^{+}+\alpha_{22}^{+}=2g, \quad \alpha_{11}^{+}-\alpha_{22}^{+}=2\xi$$

$$\alpha_{jj}^{-}=\alpha_{jj}^{+}-2b_{jj}$$

$$\Delta \equiv g^2-\eta^2$$

$$\eta^2=\xi^2+\left(\alpha_{12}^{+}\right)^2$$

7 real parameters remain

The condition of fixed purity:

$$b_{11}b_{22}-b_{12}^2+\alpha_{12}^{+}b_{12}-\left(\alpha_{11}^{+}b_{22}+\alpha_{22}^{+}b_{11}\right)/2=\kappa\Delta$$

$$\kappa \equiv \left(\mathcal{P}^{-2}-1\right)/4 \geq 0$$

Very special case, permitting a simple analytical solution:

$\eta = 0$ $\gamma = g\mathcal{L}$ $b_{12} = 0$ $b_{11} = b_{22}$

$$2\mathcal{E} = \frac{1}{g} + g\left(\mathcal{L}^2 + \sqrt{1+4\kappa}\right) = \frac{1}{g} + g\left(\mathcal{L}^2 + \mathcal{P}^{-1}\right)$$

$$\mathcal{E}_{min}^{mix} = g_*^{-1} = \sqrt{\mathcal{L}^2 + \mathcal{P}^{-1}} = \sqrt{\left(1+\mathcal{L}^2\mathcal{P}\right)/\mathcal{P}}.$$

But this results **does not go to** $\mathcal{E}_{min}^{pure} = 1 + |\mathcal{L}|$ if $\mathcal{P} = 1$

Why ?

Because the statistical operator ρ must be

nonnegatively definite

The consequence of this property is the nonnegative deniteness of matrix $\mathcal{M} - i\Sigma/2$

$$\Sigma = \left\| \begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right\|$$

Some conditions:

$$U_x \equiv \overline{x^2}\,\overline{p_x^2} - (\overline{xp_x})^2 \geqslant 1/4.$$

$$\mathcal{D}_0 \equiv \det \mathcal{M} \geq 1/16.$$

$$\mathcal{D}_0 - \mathcal{D}_2/4 + 1/16 \geq 0.$$

$$\mathcal{D}_2 \equiv U_x + U_y + 2\overline{xy}\,\overline{p_x p_y} - 2\,\overline{xp_y}\,\overline{yp_x}.$$

$$b_{11}b_{22} - b_{12}^2 - \gamma^2 \geq 0.$$

$$\gamma = g\mathcal{L}.$$

→

$$\mathcal{P}^{-1} \geq 1 + 2|\mathcal{L}|, \quad \kappa \geq |\mathcal{L}|(|\mathcal{L}| + 1)$$

Highly mixed "rotating thermal packets"

$$B = (g_*/2)\left[\left(1 - \mathcal{P}^{-1}\right) I_2 + 2i\mathcal{L}J_2\right]$$

$$A = (g_*/2)\left(1 + \mathcal{P}^{-1}\right) I_2$$

$$J_2 = \left\| \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right\|$$

$$\mathcal{M}^{-1} = 2\mathcal{P} \left\| \begin{array}{cc} \mathcal{E}I_2 & \mathcal{L}J_2 \\ -\mathcal{L}J_2 & \mathcal{E}I_2 \end{array} \right\|$$

$$\hat{\rho}_{min} = \mathcal{N}_\rho \exp\left(-\hat{\mathbf{q}}\mathcal{Q}\hat{\mathbf{q}}/2\right)$$

$$\mathcal{Q} = 2\Omega^{-1} \tanh^{-1}\left(\Omega\mathcal{M}^{-1}/2\right)$$

$$\Omega = -i\Sigma$$

$$\hat{\rho}_{min} = \mathcal{N}_\rho \exp\left(-u\hat{H} + v\hat{L}_z\right)$$

$$\mathcal{Q} = \left\| \begin{matrix} uI_2 & vJ_2 \\ -vJ_2 & uI_2 \end{matrix} \right\|$$

$$u = \Lambda_+ + \Lambda_- = \tanh^{-1}\left(2\mathcal{P}\mathcal{E}/[1+\mathcal{P}]\right)$$

$$v = \Lambda_+ - \Lambda_- = \tanh^{-1}\left(2\mathcal{P}\mathcal{L}/[1-\mathcal{P}]\right)$$

$$\Lambda_\pm = \tanh^{-1}\left(X_\pm\right) = \tanh^{-1}\left(\mathcal{P}[\mathcal{E} \pm \mathcal{L}]\right)$$

$$\mathcal{N}_\rho = 4\sinh(\Lambda_+)\sinh(\Lambda_-)$$

Energy and angular momentum fluctuations

$$\sigma_L = \tfrac{1}{2}(\mathcal{E}^2 + \mathcal{L}^2 - 1) = \mathcal{L}^2 + \tfrac{1}{2}(\mathcal{P}^{-1} - 1) \geqslant \mathcal{L}^2 + |\mathcal{L}|,$$

$$\sigma_E = \mathcal{L}^2 + \tfrac{1}{2}(\mathcal{P}^{-1} - 1) = \sigma_L$$

Homogeneous magnetic field

$$\hat{H} = \frac{1}{2M}\hat{\mathbf{p}}^2 + \frac{M}{2}\tilde{\omega}^2\hat{\mathbf{r}}^2 - \omega_L \hat{L}_z,$$

$$\omega_L = \frac{eB}{2Mc},$$

$$\tilde{\omega}^2 = \omega^2 + \omega_L^2$$

$$E_{n_r m} = \hbar\tilde{\omega}\left(1 + |m| + 2n_r\right) - \hbar\omega_L m$$

Pure states:

$$\mathcal{E} = \hbar\tilde{\omega}\left(1 + |\mathcal{L}_i| + |\mathcal{L}_c|\right) - \hbar\omega_L\left(\mathcal{L}_i + \mathcal{L}_c\right)$$

$$\mathcal{E}_{min}(\mathcal{L}) = \hbar\tilde{\omega} + \hbar\left(\tilde{\omega} - |\omega_L|\right)|\mathcal{L}|, \quad \mathcal{L}\omega_L \geq 0$$

$(\omega = 0,\ \tilde{\omega} = |\omega_L|)$ $\quad$ $(\omega_L > 0)$ $\quad$ ➡

$$\mathcal{E} = \hbar\omega_L\left[1 + |\mathcal{L}_i|\left(1 - \lambda\right) + |\mathcal{L}_c|\left(1 - \lambda_c\right)\right]$$

Mixed states:

$$\mathcal{E}_{\min} = \hbar\tilde{\omega}[\mathcal{P}^{-1} + \mathcal{L}^2]^{1/2} - \hbar\omega_L \mathcal{L}$$

In particular, the energy of a free particle

($\omega = 0$) assumes the minimal (ground-state) value $\hbar\omega_L$ for all states with $\mathcal{L} \geqslant 0$ and the critical value of the purity $\mathcal{P}^{-1} = 1 + 2\mathcal{L}$ (here we assume $\omega_L > 0$).

In particular, for a free particle with $\omega_L \geqslant 0$ we have

$$\sigma_E/(\hbar\omega_L)^2 = (\sqrt{\mathcal{L}^2 + \mathcal{P}^{-1}} - \mathcal{L})^2 - 1,$$

so that $\sigma_E = 0$ in critical states with $\mathcal{L} \geqslant 0$ and $\mathcal{P}^{-1} = 1 + 2\mathcal{L}$.

$$\langle n_r = 0, m|\hat{\rho}_{min}|n_r = 0, m\rangle = \frac{\mathcal{L}^m}{(1+\mathcal{L})^{m+1}}$$

$\mathcal{L} \geq 0$

$m \geq 0$

$$\sigma_L = \mathcal{L}^2 + \tfrac{1}{2}(\mathcal{P}^{-1} - 1) \geqslant \mathcal{L}^2 + |\mathcal{L}|$$

$$\overline{\pi_x^2} = \overline{\pi_y^2} = \hbar M \omega_L (1 + |\mathcal{L}| - \mathcal{L})$$

$$\overline{X^2} = \overline{Y^2} = \frac{\hbar}{4M\omega_L}(1 + |\mathcal{L}| + \mathcal{L})$$

$$\hat{L} = \frac{1}{2} M \omega_0 \left(\hat{X}^2 + \hat{Y}^2 - \xi^2 - \eta^2 \right)$$

Open questions

what happens if $\mathcal{P}^{-1} < 1 + 2|\mathcal{L}|$?

It seems that the dependence $\mathcal{E}_{min}(\mathcal{P})$ is not monotonous for a fixed value of $\mathcal{L}$. Indeed, $\mathcal{E}_{min} = 1 + |\mathcal{L}|$ for $\mathcal{P} = 1$, whereas $\mathcal{E}_{min}(\mathcal{P}) > 1+|\mathcal{L}|$ for $0 < 1-\mathcal{P} \ll 1$, according to Eq. (48). But $\mathcal{E}_{min}(\mathcal{P}) = 1+|\mathcal{L}|$ again, if $\mathcal{P}^{-1} = 1+2|\mathcal{L}|$.

A more challenging problem can be to try to find *non-Gaussian* minimal energy states with a fixed value of the mean angular momentum.