Entanglement of quantum circular states of light

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Introduction

Entanglement of bi-partite systems

State space. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$; $\mathcal{H}_A = \mathcal{H}_B = \mathcal{K}$;

Examples. $\mathcal{K} = \mathbb{C}^2$: spin 1/2. $\mathcal{K} = L^2(\mathbb{R}, \mathrm{d}x)$: oscillator/mode of EM field.

Product state: $|\phi_A\rangle \otimes |\phi_B\rangle \in \mathcal{H}$. Entangled state=not a product state.

Question 1. How to measure entanglement?

(a) Given $|\psi\rangle \in \mathcal{H}$, is it entangled: YES or NO?

(b) If YES. How entangled is it? Quantitative measure of entanglement?

Answer 1. Compute $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$.

(a) $\psi \in \mathcal{H}$ is NOT entangled iff $\rho_A^2 = \rho_A$: ρ_A is a projector.

(b) Compute the entanglement of formation $E(|\psi\rangle) = -\mathrm{Tr}\rho_A \ln \rho_A$.

The bigger it is, the more entangled ψ.

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Question 2. Who cares?

Answer 2. Entanglement is a resource for
quantum computation/information/teleportation/cryptography:

The more you can get, the better it is.

Question 3. How to make entangled states? How entangled can they get?

Answer 3. Some elements in the rest of this talk.

How to produce and quantify entanglement: An abstract setup.

Given $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a reference state $|0\rangle_B \in \mathcal{H}_B$ and an evolution operator U on $\mathcal{H}$, do this:

$$\hat{E} : |\phi\rangle \in \mathcal{H}_A \to |\psi\rangle = |\phi\rangle \otimes |0\rangle_B \in \mathcal{H} \to U|\psi\rangle \in \mathcal{H} \to \hat{E}(|\phi\rangle) = E(U|\psi\rangle) \in \mathbb{R}$$

Remark. If $U = U_A \otimes U_B$, then obviously no entanglement is produced. For that, the system components need to interact:

$$H = H_A + H_B + \lambda V_A \otimes V_B,$$

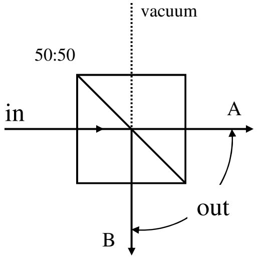
for example, and $U = \exp(iH\tau)$, for some $\tau > 0$.
Some questions:
Question 1. What is $\sup \hat{E}$? Is it attained and if so for which $|\phi\rangle \in \mathcal{H}_A$?
Question 2. In applications, there can be a subset $\mathcal{S}$ of states in $\mathcal{H}_A$ that can actually be constructed experimentally. What is $\sup_{|\phi\rangle \in \mathcal{S}} \hat{E}(|\phi\rangle)$?

How to produce and quantify entanglement: The example of the beamsplitter.

Consider two modes of the EM field in vacuum.
With same frequency ω, but with orthogonal wave vectors $\vec{k}_A, \vec{k}_B$.
$\mathcal{H}_A = L^2(\mathbb{R}) = \mathcal{H}_B$. Choose for $|0\rangle$ the vacuum state of the mode B.
Let

$$U = \exp(i\frac{\pi}{4}(a_A^\dagger a_B - a_A a_B^\dagger)).$$

Then U encodes what happens in a beamsplitter. It couples the two modes.



Question. How much entanglement can you produce this way?
Answer. It depends on what input state you use.

(a) Number state: $U|n\rangle_A|0\rangle_B = \sum_{k=1}^{n} \sqrt{\frac{1}{2^n} C_n^k}|k\rangle_A|n-k\rangle_B$. Entangled.
Entanglement entropy equals that of the binomial distribution:
$\hat{E}(|n\rangle) = \frac{1}{2}\ln(\frac{\pi n}{\mathrm{e}})$.

(b) Coherent state $|\alpha\rangle$ at the point $\alpha = \frac{1}{\sqrt{2}}(q+ip)$ in phase space:
$U|\sqrt{2}\alpha\rangle|0\rangle_B = |\alpha\rangle_A|\alpha\rangle_B$. Not entangled. $\hat{E}(|\alpha\rangle) = 0$. **CLASSICAL**

(c) Superposition of coherent states:
$U\left[\sum_{m=0}^{N-1} c_m|\sqrt{2}\alpha_m\rangle_A|0\rangle_B\right] = \sum_{m=0}^{N-1} c_m|\alpha_m\rangle_A|\alpha_m\rangle_B$. Entangled.

Question. $\hat{E}(N,(\alpha_m),(c_m))$? Maximal entanglement at fixed resources,

$$\hat{E}_{\max}(N,(\alpha_m)) = \max_{(c_m)} \hat{E}(N,(\alpha_m),(c_m))?$$

Answer. Not so obvious.
A partial answer follows for "Rotationally invariant circular states".

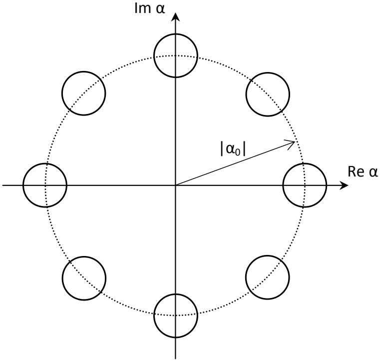
One mode input states: Circular States

A circular one-mode state is a state of the form

$$|\psi_c\rangle = \sum_{m=0}^{N-1} c_m |\sqrt{2}\alpha_m\rangle, \quad \text{where} \quad \alpha_m = \alpha_0 e^{-i2\pi m/N}.$$

Here $\alpha_0 \in \mathbb{C}_*$, and the c_m are arbitrary complex coefficients.
For fixed N and α_0, they form an N-dimensional vector subspace of $\mathcal{H}_A$:

$$\mathcal{S}_N(\sqrt{2}\alpha_0) \subset \mathcal{H}_A = L^2(\mathbb{R}).$$



Two mode output states and their entanglement of formation

Output state after beam splitter:

$$U\left[\sum_{m=0}^{N-1} c_m|\sqrt{2}\alpha_m\rangle_A|0\rangle_B\right] = \sum_{m=0}^{N-1} c_m|\alpha_m\rangle_A|\alpha_m\rangle_B \in \mathcal{H}_A\otimes\mathcal{H}_B.$$

Lemma.

(a) $\forall\alpha_0, \forall N$, $\hat{E}_{\max}(\alpha_0, N) = \max_{(c_m)} \hat{E}(N, \alpha_0, (c_m)) \leq \ln(N)$.

(b) $\forall N$, $\lim_{\alpha_0\to+\infty} \hat{E}_{\max}(\alpha_0, N) = \ln(N)$.

Conjectures. (i) $\forall\alpha_0, \exists c_{\alpha_0} > 0$ so that, for all N, $c_{\alpha_0}\ln(N) \leq \hat{E}_{\max}(\alpha_0, N)$.
(ii) $\forall\alpha_0$, $c_{\alpha_0} \geq \frac{1}{2}$.

We will study these questions by using Multi-component Cat States as trial states.

Rotationally Invariant Circular States: RICS

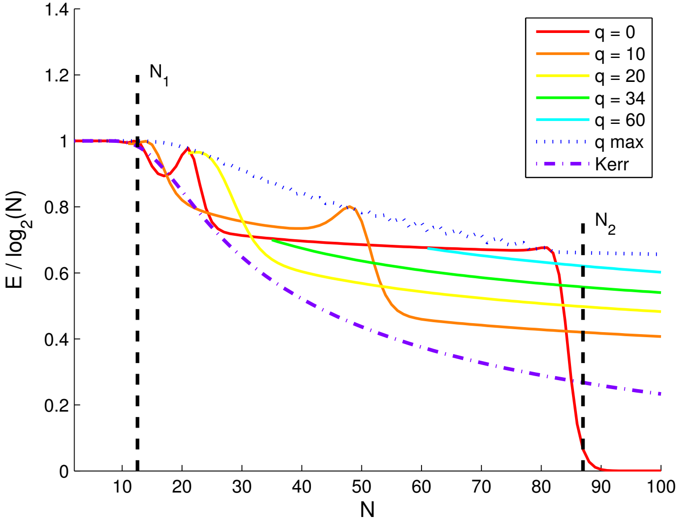
For all $0 \leq q < N$, define

$$|q\rangle = \frac{1}{N\sqrt{\tilde{g}(q)}} \sum_{m=0}^{N-1} e^{i2\pi mq/N} |\alpha_0 e^{-i2\pi m/N}\rangle, \quad \langle q|q'\rangle = \delta_{qq'},$$

These are circular states with constant amplitudes $|c_m|$ and linear phase: phase space rotation invariant.
$N = 2$: Odd and even "Schrödinger Cat" state.
$N > 2$: "Multi-component cats" (Raimond-Haroche (2006));
Can be experimentally produced in optical and microwave domain.
Then

$$\max_q \hat{E}_{\max}(\alpha_0, N, q) \leq \hat{E}_{\max}(\alpha_0, N).$$

Numerical computation of the entropy of formation of RICS: $\alpha_0 = 4$.



Explaining the figure

• Small N. $N \leq N_1 = \pi|\alpha_0|$. Separated coherent states. Basically orthogonal (semiclassical)

$$\hat{E}(\alpha_0, N, q) \simeq \ln N.$$

• Limiting value as $N \to +\infty$

$$\lim_{N\to\infty} |q\rangle = |n = q\rangle \Rightarrow \lim_{N\to\infty} \hat{E}(\alpha_0, N, q) = \frac{1}{2}\ln(\frac{\pi \mathrm{e}}{2} q).$$

Note that the limit does not depend on α_0, but the rate of convergence does.

• For large enough N (depending on α_0), we have

$$\max_{0\leq q<(N-1)} \hat{E}(\alpha_0, N, q) = \hat{E}(\alpha_0, N, N-1) \simeq \frac{1}{2}\ln(\frac{\pi \mathrm{e}}{2}(N-1)).$$

- Behaviour $\hat{E}(\alpha_0, N, q=0)$ as a function of N (red curve):

$$|q=0\rangle \approx \frac{1}{\sqrt{1+X}}\left(|n=0\rangle + \sqrt{X}|n=N\rangle\right),$$

where $X = (2|\alpha_0|^2)^N/N!$ decays fast for $N > N_2 = 2|\alpha_0|^2$. Then

$$\hat{E}(\alpha_0, N, q=0) \simeq B(N) + S(N),$$

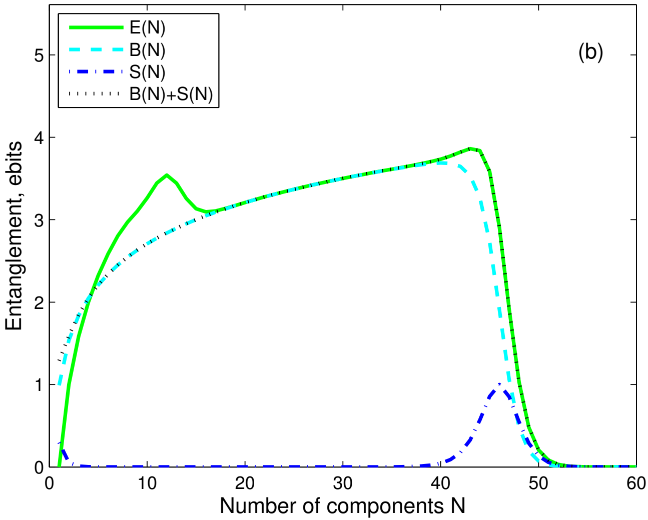
where

$$B(N) = \frac{X}{1+X}\frac{1}{2}\log_2\left(\frac{\pi e N}{2}\right),$$

is the entropy of the binomial distribution times its weight, while

$$S(N) = -\frac{1}{1+X}\log_2\frac{1}{1+X} - \frac{X}{1+X}\log_2\frac{X}{1+X},$$

is the Shannon entropy of the distribution of weights.



$$|\alpha_0| = 3, \quad q = 4.$$

Conclusions

• A beamsplitter allows one to create two mode entangled states by coupling an "in" state of one mode to the vacuum of the other mode.
• The entanglement entropy that can be generated from a superposition of N coherent states at distinct sites coupled to the vacuum cannot exceed $\ln N$.
• Using constant amplitude superpositions of coherent states placed on a regular polygon, one can get at least $\frac{1}{2}\ln N$ in entanglement entropy.
• Entanglement entropy can be used as a measure of non-classicality.

For more information: talk to me or read Phys. Rev. A 93, 062323, 2016

Coherent states are like a dirty mind

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THANK YOU!