Phase retrieval in Infinite Dimensions

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Phase retrieval:

e.g. • measure $|\psi(x)|$, $|\hat{\psi}(\xi)|$ but want ψ

or

• measure $|\hat{f}(\xi)|$, know that $f(x) \geq 0$; want f

or

• measure Wigner distribution for a wave function ψ (corresponding to projection operator on $\mathbb{C}\psi$); want ψ

↳ $\langle W(v)\, \Pi\, \psi, \psi \rangle$

In most cases: measurements also noisy.

Algorithms proposed: typically iterative.

$$\hat{f}_0 = |\hat{f}| \Rightarrow f_0 \Rightarrow f_1(x) = |f(x)|\, e^{i \operatorname{Arg} f_0(x)}$$

$$\Downarrow$$

$$\cdots \Leftarrow \hat{f}_2 = |\hat{f}|\, e^{i \operatorname{Arg} \hat{f}_1} \Leftarrow \hat{f}_1$$

or

$$\hat{f}_0 = |\hat{f}| \Rightarrow f_0 \Rightarrow f_1 = \max(f_0, 0)$$

$$\Downarrow$$

$$f_3 = \max(f_2, 0) \Leftarrow f_2 \Leftarrow \hat{f}_2 = |\hat{f}|\, e^{i \operatorname{Arg} \hat{f}_1} \Leftarrow \hat{f}_1$$

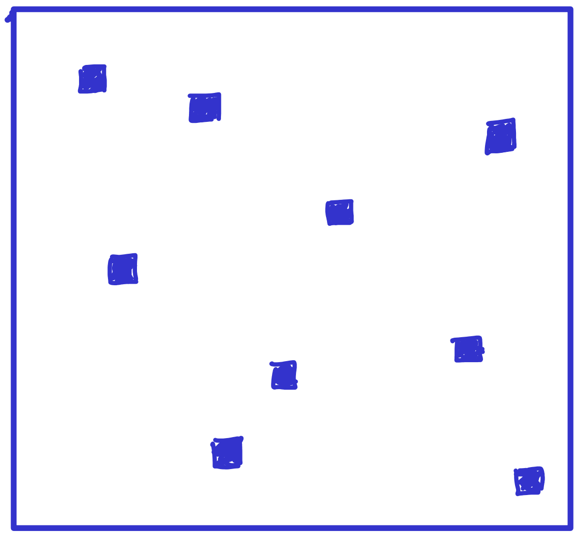
$$\Downarrow$$

$$\vdots$$

often: no proof of convergence.

Candès - Strohmer:

. reconstruction of low-rank matrices from incomplete data.



+ low-rank $\longrightarrow$ M.

+ algorithms : $\underset{T}{\operatorname{argmin}} \left[\sum_{(m,n) \in S} |T_{mn} - M_{mn}|^2 + \lambda \operatorname{Tr} |T| \right]$

. Wigner distribution $\longrightarrow$ Ψ.

Phase retrieval for frames:

- Suppose you have $(\varphi_\lambda)_{\lambda\in\Lambda}$ s.t. $\forall\, \psi \in \mathcal{H}$:

$$A \|\psi\|^2 \leq \sum_{\lambda\in\Lambda} |\langle \psi, \varphi_\lambda\rangle|^2 \leq B\|\psi\|^2$$

$A>0, B<\infty$ indep. of ψ.

then: $\psi = \sum_{\lambda\in\Lambda} \langle \psi, \varphi_\lambda\rangle \tilde{\varphi}_\lambda$

stable reconstr. of ψ from $(\langle\psi,\varphi_\lambda\rangle)_{\lambda\in\Lambda}$

What if your data are not $(\langle\psi,\varphi_\lambda\rangle)_{\lambda\in\Lambda}$

but $(|\langle\psi,\varphi_\lambda\rangle|)_{\lambda\in\Lambda}$?

Phase retrieval for frames of coherent states.

- canonical coherent states:

$\langle f, g_{p,q} \rangle$ $\qquad\qquad$ $g_{p,q}(x) = C\, e^{-\frac{i}{2}pq}\, e^{ipx}\, e^{-\frac{1}{2}(x-q)^2}$

$\hookrightarrow e^{-\frac{1}{4}(p^2+q^2)}\, F(p+iq)$

$\hookrightarrow$ entire, of growth limited by condition that $\iint |\langle f, g_{p,q} \rangle|^2 \, dp\, dq < \infty$.

$$f = C \iint \langle f, g_{p,q} \rangle\, g_{p,q} \, dp\, dq.$$

$|\langle f, g_{pq} \rangle|$ completely determines f

even just knowing zero-set of $|\langle f, g_{p,q} \rangle|$ determines f! (but very unstably)

- $(p,q) = (m,n)$ with $n,m \in \mathbb{Z}$

$\Rightarrow$ $\langle f, g_{m,n} \rangle$ do not determine f in <u>stable</u> manner

(in fact: map $\ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$

$(\langle f, g_{mn} \rangle)_{m,n \in \mathbb{Z}} \longmapsto f$

is unbounded)

- $(p,q) = \left(\frac{m}{2}, \frac{n}{2}\right)$ with $n,m \in \mathbb{Z}$

$\Rightarrow$ very nice reconstruction formula

$$f = \sum_{m,n} \langle f, g_{\frac{m}{2},\frac{n}{2}} \rangle \, \tilde{g}_{\frac{m}{2},\frac{n}{2}}$$

what if only $|\langle f, g_{\frac{m}{2},\frac{n}{2}} \rangle|$ are given?

Phase retrieval in infinite-dimensional Hilbert spaces

J. Cahill, P. Casazza, I. Daubechies, '16

(Discrete) frames of Hilbert spaces

We fix:

- a separable Hilbert space $\mathcal{H}$ with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$
- a discrete index set I
- a family of functions $\Phi = (\varphi_n)_{n \in I} \subset \mathcal{H}$.

If for $0 < A \leq B < \infty$

$$A \|f\|^2 \leq \sum_{n \in I} |\langle f, \varphi_n \rangle|^2 \leq B \|f\|^2 \text{ for all } f \in \mathcal{H},$$

then Φ is a frame of $\mathcal{H}$.

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Phase retrieval: Inversion of

$$\mathcal{A}_{\Phi} : \mathbb{P}_{\mathbb{K}}\mathcal{H} \to \ell^2(I), \quad x \mapsto (|\langle f, \varphi_n\rangle|)_{n\in I}.$$

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"Φ does phase retrieval" $\iff \mathcal{A}_\Phi$ is injective.

Injectivity and the complement property (CP)[1]

[1] Balan, Casazza and Edidin '06

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Definition (Complement property)

A frame $\Phi = (\varphi_n)_{n \in I}$ for $\mathcal{H}$ satisfies the *complement property* (CP) if for every subset $S \subseteq I$, either

$$\overline{\text{span}}\{\varphi_n\}_{n \in S} = \mathcal{H},$$

or

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Theorem

- Φ *does phase retrieval* $\Longrightarrow$ Φ *has the CP*
- *For* $\mathbb{K} = \mathbb{R}$*:* Φ *does phase retrieval* $\Longleftrightarrow$ Φ *has the CP*

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We need: $c_1, c_2 > 0$ s.t.

$$c_1 d_{\mathcal{H}}(f,g) \leq \|\mathcal{A}_\Phi(f) - \mathcal{A}_\Phi(g)\| \leq c_2 d_{\mathcal{H}}(f,g) \text{ for all } f,g \in \mathcal{H}.$$

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Lemma

$$c_2 = B^{1/2}.$$

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Phase retrieval is always stable when $\mathcal{H}$ is finite dimensional.

$\mathcal{H}$ infinite dimensional: such a $c_1 > 0$ can never exist!

Theorem

Phase retrieval is never uniformly stable when $\mathcal{H}$ is infinite dimensional.

$(\varphi_n)_{n\in\mathbb{N}}$ frame.

$$A\|\psi\|^2 \leq \sum_n |\langle \psi, \varphi_n\rangle|^2 \leq B\|\psi\|^2$$

Thm. $\forall \varepsilon>0, \forall N : \exists k, \exists m>k$ so that

$$\sum_{n=1}^{N} |\langle \varphi_k, \varphi_n\rangle|^2 + \sum_{n=m}^{\infty} |\langle \varphi_k, \varphi_n\rangle|^2 < \varepsilon.$$

Pf. $V = \mathrm{Span}(\varphi_1, \dots \varphi_N)$; orthon. basis $e_1, \dots e_L$.

- $$\sum_{n\in\mathbb{N}} \|P_V \varphi_n\|^2 = \sum_{n\in\mathbb{N}} \sum_{\ell=1}^{L} |\langle \varphi_n, e_\ell\rangle|^2$$
$$\leq \sum_{\ell=1}^{L} B\|e_\ell\|^2 = BL < \infty$$

- $\exists k$ so that $\|P_V \varphi_k\|^2 < \varepsilon/2B$

$$\Rightarrow \sum_{n=1}^{N} |\langle \varphi_k, \varphi_n\rangle|^2 = \sum_{n=1}^{N} |\langle P_V\varphi_k, \varphi_n\rangle|^2 \leq B\|P_V\varphi_k\|^2 < \frac{\varepsilon}{2}$$

$$\sum_n |\langle \varphi_k, \varphi_n \rangle|^2 < \infty$$

$$\Rightarrow \exists\, m \text{ so that } \sum_{n=m}^{\infty} |\langle \varphi_k, \varphi_n \rangle|^2 < \varepsilon/2$$

$\square$

Now, pick $\psi \perp \operatorname{Span}(\varphi_{N+1}, \dots, \varphi_{m-1})$

set $f = \varphi_k + \psi$, $g = \varphi_k - \psi$

for $n : N+1 \to m$: $|\langle f, \varphi_n \rangle| = |\langle g, \varphi_n \rangle|$

$$\Rightarrow \sum_n \left(|\langle f, \varphi_n \rangle| - |\langle g, \varphi_n \rangle| \right)^2$$

$$= \left(\sum_{n=1}^{N} + \sum_{n=m}^{\infty} \right) \left(|\langle f, \varphi_n \rangle| - |\langle g, \varphi_n \rangle| \right)^2$$

use $|z_1 + z_2| - |z_1 - z_2| \leq 2|z_1|$

$$\leq \left(\sum_{n=1}^{N} + \sum_{n=m}^{\infty} \right) \left(4 |\langle \varphi_k, \varphi_n \rangle|^2 \right)$$

$$\sum_n \left(|\langle f, \varphi_n \rangle| - |\langle g, \varphi_n \rangle| \right)^2 \leq 4\varepsilon$$

but $\| f - \alpha g \|^2 = \| (1-\alpha)\varphi_k + (1+\alpha)\psi \|^2$

$$= |1-\alpha|^2 + |1+\alpha|^2 = 4$$

$(|\alpha| = 1)$

Finite dimensional subspaces

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Good news: Stable phase retrieval is possible for elements in $\mathcal{H}$ that can be approximated sufficiently well by finite-dimensional expansions.

Bad news: Stability can deteriorate very fast with increasing dimension.

Example: Unsigned samples of real-valued band-limited signals

$$\mathcal{H} = \{f \in L^2(\mathbb{R}, \mathbb{R}) : \text{supp } \widehat{f} \subseteq [-\pi, \pi]\}$$

and

$$\varphi_n(x) = \text{sinc}(x - \frac{n}{4})$$

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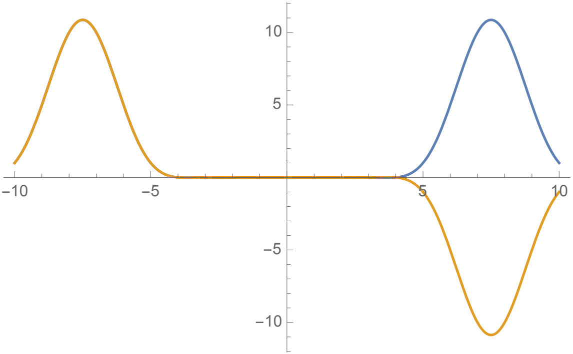
There exist $f_m, g_m \in V_{2m}$ such that with $C > 0$ (m-independent):

$$d_{\mathcal{H}}(f_m, g_m) > C(m+1)^{-1} 2^{3m} \|\mathcal{A}_\Phi(f_m) - \mathcal{A}_\Phi(g_m)\| \text{ for all } m \in \mathbb{N}.$$

The proof is constructive.



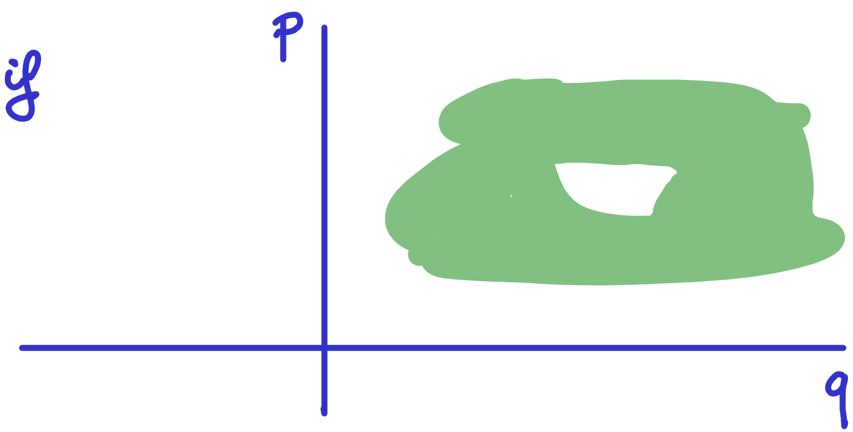
f_m, g_m for $m = 5$.

In practice: when attempting to reconstruct f from $|\langle f, g_{m\alpha, n\beta} \rangle|$

$$e^{2\pi i m \alpha t} \, e^{-\frac{1}{2}(t-n\beta)^2}$$

$$\alpha\beta < 1/4$$

↳ OK if



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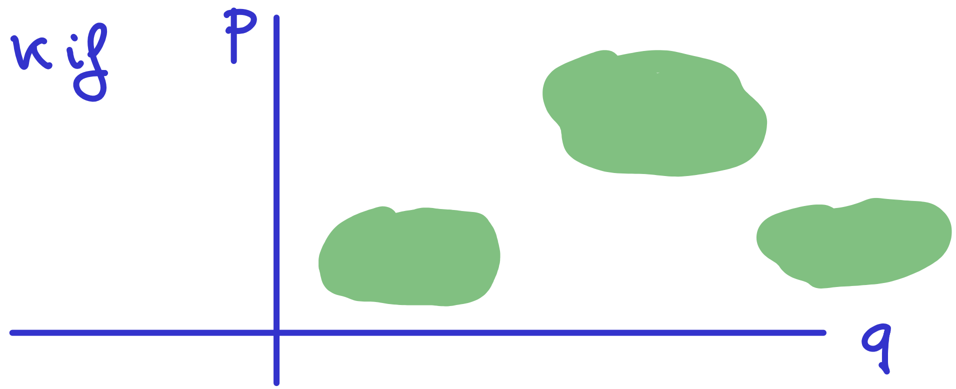

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