Integrable part of the regularized Mixmaster

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Introduction

- It seems that the isotropy of space is dynamically unstable towards the big-bang singularity[1].
- If the present Universe originated from an inflationary phase, then the pre-inflationary universe is supposed to have been both inhomogeneous and anisotropic.
- Numerical evidence[2] suggests that the dynamics of such universe backwards in time becomes ultralocal: approximately identical with the homogeneous but anisotropic one at each spatial point.
- Therefore an anisotropic model, comprising the Friedmann model as a particular case, is expected to be better suited for describing the earliest Universe.
- Mixmaster universe, Bianchi IX model, has sufficient generality.

[1] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).
[2] D. Garfinkle, Phys. Rev. Lett. **93**, 161101 (2004).

Introduction (cont)

- The Mixmaster describes the space-time metric:

$$ds^2 = -dt^2 + a^2(e^{2\beta})_{ij}\sigma^i\sigma^j \tag{1}$$

σ^i are differential forms on a three-sphere, satisfying $d\sigma^i = \frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k$.

- The diagonal form of the metric is assumed in the absence of matter or for simple fluids:

$$(e^{2\beta})_{ij} := \mathrm{diag}\,(e^{2(\beta_+ + \sqrt{3}\beta_-)}, e^{2(\beta_+ - \sqrt{3}\beta_-)}, e^{-4\beta_+}),$$

where

- $\beta_\pm$ are distortion parameters, a is the averaged scale factor:

$$\beta_+ = \ln\frac{a_3}{\sqrt{a_1 a_2}}, \quad \beta_- = \frac{1}{2\sqrt{3}}\ln\frac{a_1}{a_2}, \quad a = \sqrt[3]{a_1 a_2 a_3}$$

a_1, a_2, a_3 being the anisotropic scale factors.

Introduction (cont): Mixmaster universe

- The canonical description of diagonal Bianchi IX model is given in terms of Misner's variables[3].
- The dynamics resembles motion of a particle in a three-dimensional Minkowskian space-time and in a space-and-time-dependent confining potential.
- The spatial coordinates $\beta_\pm$ of this particle describe the distortion to the spherical shape.
- The particle is moving in a potential representing the curvature of spatial geometry, undergoing infinitely many oscillations.

[3] C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969); Phys. Rev. **186**, 1319 (1969).

Classical Bianchi IX potential

- The potential of the Bianchi IX model has the form

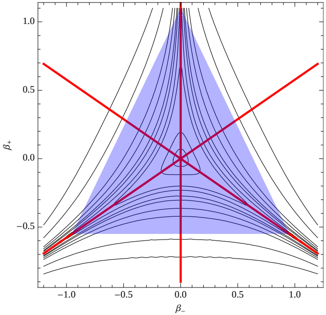
$$V_{\mathfrak{n}}(\beta) = \mathfrak{n}^2 \frac{e^{4\beta_+}}{3} \left[\left(2\cosh(2\sqrt{3}\beta_-) - e^{-6\beta_+} \right)^2 - 4 \right] + \mathfrak{n}^2 ,$$

  where $\mathfrak{n}$ is the structure constant and may be put $\mathfrak{n} = 1$ in the subsequent considerations

- Equivalent form more suitable to the subsequent considerations:

$$V(\beta) = \frac{1}{3} \Big[2e^{4\beta_+} \left(e^{4\sqrt{3}\beta_-} + e^{-4\sqrt{3}\beta_-} \right) - 2e^{4\beta_+} \left(e^{2\sqrt{3}\beta_-} + e^{-2\sqrt{3}\beta_-} \right) + e^{-8\beta_+} - 2e^{4\beta_+} \Big] + 1.$$

Classical Bianchi IX potential



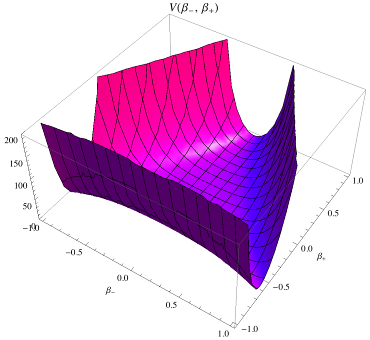


Figure: The plot of Bianchi IX potential near its minimum.

- This potential has three “open” C_{3v} symmetry directions.
- They can be viewed as three deep “canyons”, increasingly narrow until their respective wall edges close up at the infinity whereas their respective bottoms tend to zero.

Comment on Bianchi IX potential

- $V(\beta)$ is *bounded* from below and reaches its minimum value, $V(\beta) = -\mathfrak{n}^2$, at $\beta_\pm = 0$.
- $V(\beta)$ is expanded around its minimum as follows (harmonic approximation)

$$V(\beta) = -\mathfrak{n}^2 + 8\mathfrak{n}^2(\beta_+^2 + \beta_-^2) + o(\beta_\pm^2)\,.$$

- $V(\beta)$ is asymptotically *confining* except for the following three directions , in which $V(\beta) \to 0$:

$$(i)\ \beta_- = 0,\ \beta_+ \to +\infty,\ (ii)\ \beta_+ = -\frac{\beta_-}{\sqrt{3}},\ \beta_- \to +\infty,$$
$$(iii)\ \beta_+ = \frac{\beta_-}{\sqrt{3}},\ \beta_- \to -\infty$$

Comment on Bianchi IX potential

- The motion of the Misner particle in this potential is chaotic: though the curvature, which is proportional to the potential, flattens with time, the confined particle undergoes infinitely many oscillations.
- In the so-called steep wall approximation, the particle is locked in the triangular potential with its infinitely steep walls moving apart in time. At the quantum level, the confining shape originates a discrete spectrum.
- On the other hand, it is unclear (but probably not) whether or not the Bianchi-IX potential also originates a continuum spectrum.

- The idea is to attempt to regularize a potential itself, by applying the Weyl-Heisenberg quantization scheme.
- We expect this procedure should smooth out the potential, specially problematic escape canyons, which can give contribution to non-discrete spectrum of the quantum model.

Weyl-Heisenberg integral quantization

- From the resolution of the identity obeyed by the operator-valued function $\mathfrak{Q}(\hat{r})$ on phase space $\mathbb{R}^2 = \{(q,p) \equiv \hat{r}\}$

$$\int_{\mathbb{R}^2} \mathfrak{Q}(\hat{r}) \frac{\mathrm{d}^2\hat{r}}{2\pi c_{\mathfrak{Q}_0}} = I\,, \quad \mathfrak{Q}(\hat{r}) = U(\hat{r})\mathfrak{Q}_0 U(\hat{r})^\dagger$$

  where $U(\hat{r}) = e^{\mathrm{i}(pQ-qP)}$, $[Q,P] = \mathrm{i}\,\hbar I \equiv \mathrm{i}\,I$ is the unitary displacement operator and $\mathfrak{Q}_0$ an operator, the choice of it is left to us provided that $0 < c_{\mathfrak{Q}_0} < \infty$

- Equipped with one choice of $\mathfrak{Q}_0$, the corresponding WH covariant integral quantization reads

$$f(\hat{r}) \mapsto A_f = \int_{\mathbb{R}^2} f(\hat{r})\mathfrak{Q}(\hat{r}) \frac{\mathrm{d}^2\hat{r}}{2\pi c_{\mathfrak{Q}_0}}$$

- Quantization based on $\mathfrak{Q}_0$ is only possible **IF** $\mathfrak{Q}_0$ is trace class, i.e. $\mathrm{Tr}(\mathfrak{Q}_0)$ is finite

Weight or "apodization" function, WH transform, and constant $c_{\mathfrak{Q}_0}$

- Introduce the "WH-transform" of operator $\mathfrak{Q}_0$ and its inverse

$$\Pi(\hat{r}) = \mathrm{Tr}\left(U(-\hat{r})\mathfrak{Q}_0\right) \;\Leftrightarrow\; \mathfrak{Q}_0 = \int_{\mathbb{R}^2} U(\hat{r})\,\Pi(\hat{r})\,\frac{\mathrm{d}^2\hat{r}}{2\pi}$$

  where $\mathsf{P} = \mathsf{P}^{-1}$ is the parity operator defined as $\mathsf{P}U(\hat{r})\mathsf{P} = U(-\hat{r})$

- The function $\Pi(\hat{r})$ is like a weight, or better, an apodization, on the plane, which determines the extent of our coarse graining of the phase space

- The value of constant $c_{\mathfrak{Q}_0}$ derives from the above

$$c_{\mathfrak{Q}_0} = \mathrm{Tr}\left(\mathfrak{Q}_0\right) = \Pi(\vec{\mathbf{0}})$$

Alternative quantization formula through symplectic Fourier transform

- Symplectic Fourier transform

$$\mathfrak{F}_{\mathfrak{s}}[f](\hat{r}) = \int_{\mathbb{R}^2} e^{-\mathrm{i}\hat{r}\wedge\vec{\mathbf{r}'}} f(\vec{\mathbf{r}'}) \frac{\mathrm{d}^2\vec{\mathbf{r}'}}{2\pi}$$

  It is involutive, $\mathfrak{F}_{\mathfrak{s}}[\mathfrak{F}_{\mathfrak{s}}[f]] = f$ like its "dual" defined as $\overline{\mathfrak{F}_{\mathfrak{s}}}[f](\hat{r}) = \mathfrak{F}_{\mathfrak{s}}[f](-\hat{r})$

- Equivalent form of WH integral quantization

$$A_f = \int_{\mathbb{R}^2} U(\hat{r})\, \overline{\mathfrak{F}_{\mathfrak{s}}}[f](\hat{r}) \frac{\Pi(\hat{r})}{\Pi(\vec{\mathbf{0}})} \frac{\mathrm{d}^2\hat{r}}{2\pi}$$

Permanent issues of WH covariant integral quantizations

- Canonical commutation rule is preserved

$$A_q = Q + c_0\,, \quad A_p = P + d_0\,, \quad c_0, d_0 \in \mathbb{R}\,, \Rightarrow [A_q, A_p] = \mathrm{i}I\,,$$

- Kinetic energy

$$A_{p^2} = P^2 + e_1\,P + e_0\,, \quad e_0, e_1 \in \mathbb{R}$$

- Dilation

$$A_{qp} = A_q\,A_p + \mathrm{i}f_0\,, \quad f_0 \in \mathbb{R}$$

- Potential energy is multiplication operator in position representation

$$A_{V(q)} = \mathfrak{V}(Q)\,, \quad \mathfrak{V}(Q) = \frac{1}{\sqrt{2\pi}}\,V * \overline{\mathcal{F}}[\Pi(0,\cdot)](Q)$$

  where $\overline{\mathcal{F}}$ is the inverse 1-D Fourier transform

- If $F(\hat{r}) \equiv h(p)$ is a function of p only, then A_h depends on P only

$$A_h = \frac{1}{\sqrt{2\pi}}\,h * \overline{\mathcal{F}}[\Pi(\cdot,0)](P)\,.$$

WH Integral quantization of the anisotropic part

- For each canonical pair $(\beta_\pm, p_\pm)$ we choose separable Gaussian weights

$$\Pi(\beta_\pm, p_\pm) = e^{-\frac{\beta_\pm^2}{2\sigma_\pm^2}} e^{-\frac{p_\pm^2}{2\tau_\pm^2}}$$

  which yield manageable formulae with familiar probabilistic content
- The “limit” Weyl-Wigner case holds as the widths $\sigma_\pm$ and $\tau_\pm$ are infinite (Weyl-Wigner is singular in this respect!)

WH Integral quantization of the anisotropic part (cont.)

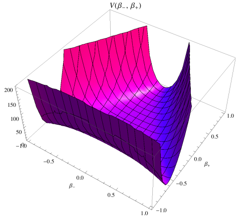
- It results in the quantized form of the Bianchi IX potential (as a multiplication operator)

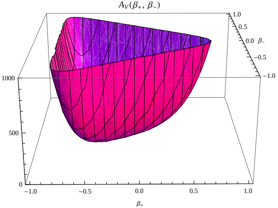
$$A_{V(\beta)} = \frac{1}{3}\Big(2D_+^4 D_-^{12} e^{4\beta_+} \cosh 4\sqrt{3}\beta_- - 4D_+ D_-^3 e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + D_+^{16} e^{-8\beta_+} - 2D_+^4 e^{4\beta_+}\Big) + 1,$$

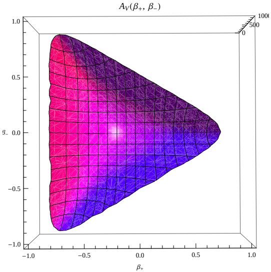
  where $D_\pm := e^{\frac{2}{\sigma_\pm^2}}$

- The original Bianchi IX potential $V(\beta) \equiv V(\beta_+, \beta_-)$ is recovered for $D_+ = 1 = D_-$, thus for weights $\sigma_+,\ \sigma_- \to \infty$.

Regularized BIX potentials after quantization







- Plot of the original Bianchi IX potential $V(\beta)$ (top) and its regularized version after quantization, near its minimum, for sample values $D_+ = 1.1$, $D_- = 1.4$.
- The original escape canyons became regularized and the whole potential is now fully confining.

Regularized BIX potentials after quantization (cont.)

- However the potential has become anisotropic in the variables β_+ and β_- and its minimum is shifted from the $(0, 0)$ position, namely it is at the $(\beta_0, 0)$ point, where the value β_0 is obeys

$$-D_+^{16} e^{-8\beta_0} + D_+ D_-^3 e^{-2\beta_0} - D_+^4 e^{4\beta_0} + D_+^4 D_-^{12} e^{4\beta_0} = 0$$

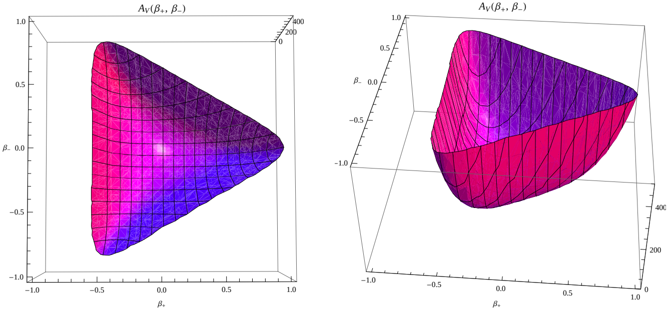
arriving from the condition $\partial A_{V(\beta_+,\beta_-)}/\partial\beta_+ = 0$ for $(\beta_0, 0)$. Condition $\partial A_{V(\beta_+,\beta_-)}/\partial\beta_- = 0$ is fulfilled automatically at this point.

After suppressing shift of the minimum

- Imposing anisotropy or no shift condition yields the same result $D_+ = D_-$, which also preserves $\mathbb{C}_{3v}$ symmetry.
- The resulting potential reads as

$$A_{V(\beta_+,\beta_-)} = \frac{1}{3}\left(D_+^{16}\left(2e^{4\beta_+}\cosh 4\sqrt{3}\beta_- + e^{-8\beta_+}\right) - D_+^4\left(4e^{-2\beta_+}\cosh 2\sqrt{3}\beta_- - 2e^{4\beta_+}\right)\right) + 1$$

- The form of this potential is shown on the picture below. Direct verification shows it is invariant with respect to rotations by $2\pi/3$ and $4\pi/3$.

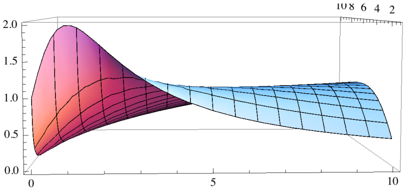


Proximity to an integrable system

- The regularized potential may be viewed as perturbation of the following integrable one:

$$A_0 = \frac{1}{3}\left(2D_+^{16}e^{4\beta_+}\cosh 4\sqrt{3}\beta_- + D_+^{16}e^{-8\beta_+}\right) + 1$$

with $A_1 = D_+^4\left(4e^{-2\beta_+}\cosh 2\sqrt{3}\beta_- - 2e^{4\beta_+}\right)$. Indeed direct verification shows that $\left|\frac{A_1}{A_0-1}\right| \leq 2D_+^{-12} \ll 1$.



Proximity to an integrable system

- Thus in the first order of approximation we deal with Hamiltonian of the following form:

$$H_0 = \frac{1}{2}(p_+^2 + p_-^2) + \frac{D_+^{16}}{3}\left(2e^{4\beta_+}\cosh 4\sqrt{3}\beta_- + e^{-8\beta_+}\right) + 1$$
$$= \frac{1}{2}(p_+^2 + p_-^2) + \frac{D_+^{16}}{3}\left(e^{4(\beta_+ + \sqrt{3}\beta_-)} + e^{4(\beta_+ - \sqrt{3}\beta_-)} + e^{-8\beta_+}\right) + 1$$

- Let us introduce new non-intuitive coordinates [4] as follows:

$$q_3 - q_2 := 4(\beta_+ + \sqrt{3}\,\beta_-),\ q_1 - q_3 := 4(\beta_+ - \sqrt{3}\,\beta_-),$$
$$q_2 - q_1 := -8\beta_+,$$

and corresponding momenta p_i.

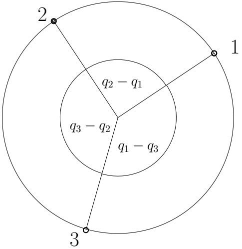
[4] M. Berry, Topics in Nonlinear Mechanics, ed. S Jorna, Am. Inst. Ph. Conf. Proc No. **46** 1978, 16-120

Liouville integrable Hamiltonian

- First approximation Hamiltonian H_0 may be rewritten in terms of those coordinates as follows:

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{D_+^{16}}{3}\left(e^{q_3-q_2} + e^{q_1-q_3} + e^{q_2-q_1}\right) + 1.$$

- The system described by the above Hamiltonian is a well known the three particle periodic Toda lattice, up to multiplication coefficient.
  It is the simplest non trivial crystal consisting of three particles moving on a ring and interacting via exponential forces.



Liouville integrable Hamiltonian

- This system has three independent conserved quantities: total momentum, energy and a third invariant:

$$K = -p_1 p_2 p_3 + a D_+^{16} \left(p_1 e^{q_3 - q_2} + p_2 e^{q_1 - q_3} + p_3 e^{q_2 - q_1} \right) ,$$

  where a is an arbitrary coefficient.
- We know that 2D system is Liouville-integrable if we can find a first integral K different of the energy, that is a function $K \neq f(H)$ on phase space such as $\{H, K\} = 0$.
- Thus the above system is completely integrable, with complete solution given by *e.g.* M. Kac M and P van Moerbeke, *A complete solution of the periodic Toda problem*. Proceedings of the National Academy of Sciences of the United States of America, 1975; 72(8), 2879-2880.

Future prospects

- Classical solutions of the periodic Toda lattice give rise to solving dynamic of the Bianchi IX model in the first order of approximation.
- The quantization of a three particle Toda system should provide the spectrum of the main, integrable part of the quantum Bianchi IX. There exist numerical simulations [5] for canonical quantization and Taylor expansion of the Toda potential.
- The full quantum Mixmaster might be obtained by adding the second order part of the potential as a perturbation to the existing solutions.
- The work is in progress.

[5] S. Isola, H. Kantz and R. Livi, *On the quantization of the three-particle Toda lattice*, Journal of Physics A, **24** 24 (1991), 3061