construction of linear and nonlinear coherent states using GHA

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outline

- Generalized Heisenberg Algebra - GHA
- examples
- coherent states using GHA
- infinite square-well potential case (iswp)
- nonlinear cs - iswp
- linear cs - iswp
- canonically conjugate variables
- "analogous" quantum Hamiltonian

Generalized Heisenberg Algebra - GHA

$(\mathcal{H}, A, A^\dagger)$ $\mathcal{H}^\dagger = \mathcal{H}$ (adimensional Hamiltonian)

$$\begin{aligned} \mathcal{H}A^\dagger &= A^\dagger f(\mathcal{H}) \\ A\mathcal{H} &= f(\mathcal{H})A \\ [A, A^\dagger] &= f(\mathcal{H}) - \mathcal{H} \end{aligned} \longrightarrow \begin{aligned} [\mathcal{H}, A^\dagger] &= A^\dagger (f(\mathcal{H}) - \mathcal{H}) \\ [\mathcal{H}, A] &= -(f(\mathcal{H}) - \mathcal{H})A \\ [A, A^\dagger] &= f(\mathcal{H}) - \mathcal{H} \end{aligned}$$

$C = A^\dagger A - \mathcal{H} = AA^\dagger - f(\mathcal{H})$ $\mathcal{H} = A^\dagger A + \epsilon_0$

f: a monotonically increasing function (differentiable)

$f(\mathcal{H}) = \mathcal{H} + 1$ -> Heisenberg algebra $\mathcal{H} \to N$

Generalized Heisenberg Algebra - GHA

representation theory

$|0\rangle$: state with the lowest eigenvalue of H

$$\mathcal{H}A^\dagger = A^\dagger f(\mathcal{H})$$
$$A\mathcal{H} = f(\mathcal{H})A$$
$$[A, A^\dagger] = f(\mathcal{H}) - \mathcal{H}$$

$$\mathcal{H}|0\rangle = \epsilon_0|0\rangle$$

1) $$\mathcal{H}\underline{A^\dagger|0\rangle} = A^\dagger f(\mathcal{H})|0\rangle = f(\epsilon_0)\underline{A^\dagger|0\rangle}$$

$A^\dagger|0\rangle$: eigenstate of $\mathcal{H}$ with eigenvalue $f(\epsilon_0)$ $\qquad \epsilon_1 = f(\epsilon_0)$

$$\mathcal{H}(\underline{A^\dagger|n\rangle}) = A^\dagger f(\mathcal{H})|n\rangle = f(\epsilon_n)(A^\dagger|n\rangle) = \epsilon_{n+1}\underline{A^\dagger|n\rangle}$$

$$\epsilon_{n+1} = f(\epsilon_n) = f^{(n+1)}(\epsilon_0) \qquad A^\dagger|n\rangle \propto |n+1\rangle$$

$\Rightarrow$ $A^\dagger|n\rangle$ is an eigenvector of the Hamiltonian with eigenvalue ϵ_{n+1}

Generalized Heisenberg Algebra - GHA

$$\mathcal{H}|n\rangle = \epsilon_n|n\rangle \qquad \mathcal{H}|0\rangle = \epsilon_0|0\rangle \qquad A|0\rangle = 0$$

$$\begin{aligned} \mathcal{H}A^\dagger &= A^\dagger f(\mathcal{H}) \\ A\mathcal{H} &= f(\mathcal{H})A \\ [A, A^\dagger] &= f(\mathcal{H}) - \mathcal{H} \end{aligned}$$

2) $$f(\mathcal{H})A|n\rangle = A\mathcal{H}|n\rangle = \epsilon_n A|n\rangle = f(\epsilon_{n-1})A|n\rangle$$

$$A|n\rangle \propto |n-1\rangle \quad \rightarrow \quad \epsilon_{n-1}$$

$$N|n\rangle = n|n\rangle$$

representation theory

$$\mathcal{H}|n\rangle = \epsilon_n |n\rangle$$

$$A^\dagger|n\rangle = N_n|n+1\rangle \; = \sqrt{\epsilon_{n+1} - \epsilon_0}\,|n+1\rangle$$

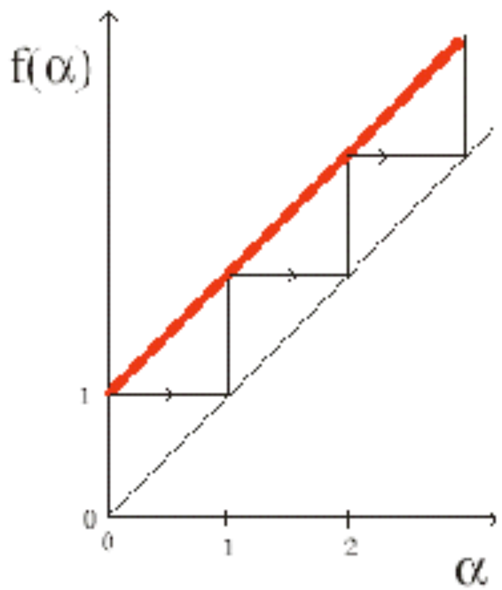
$$A|n\rangle = N_{n-1}|n-1\rangle = \sqrt{\epsilon_n - \epsilon_0}\,|n-1\rangle$$

$$N|n\rangle = n|n\rangle$$

$$N_n^2 = \epsilon_{n+1} - \epsilon_0$$

$$\Rightarrow \; A|0\rangle = 0$$

Heisenberg algebra

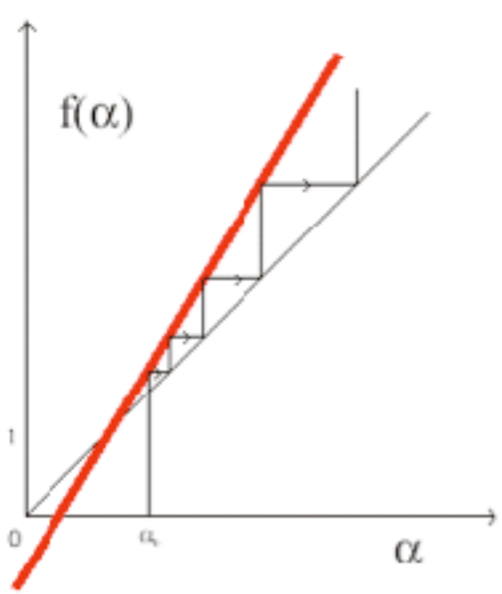


$$f(\alpha) = \alpha + 1$$

$$\mathcal{H} = N + 1/2$$

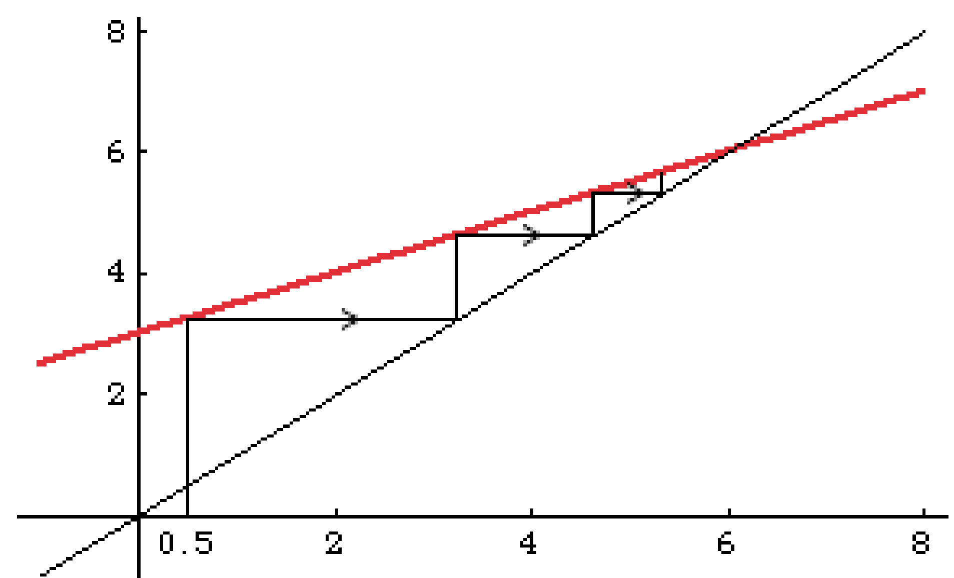
$$[N, A^\dagger] = A^\dagger$$
$$[N, A] = -A$$
$$[A, A^\dagger] = 1$$

deformed Heisenberg algebra - $f(\mathcal{H}) = r\mathcal{H} + s$



$r > 1$

q-oscillators



$r < 1$

$$N_{n-1}^2 = \frac{r^n - 1}{r - 1}$$

coherent state

$$A|z\rangle = z|z\rangle \qquad |z\rangle = \sum_{n\geq 0} c_n|n\rangle$$

$$A|z\rangle = \sum_{n\geq 0} c_n A|n\rangle = \sum_{n\geq 1} c_n\sqrt{\epsilon_n - \epsilon_0}\,|n-1\rangle \qquad n \to m+1$$

$$= \sum_{m\geq 0} \underline{c_{m+1}\sqrt{\epsilon_{m+1} - \epsilon_0}}\,|m\rangle$$

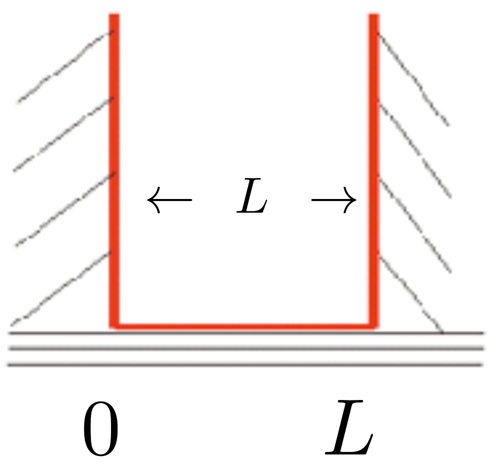
$$z|z\rangle = \sum_{m\geq 0} \underline{c_m z}|m\rangle$$

$$c_{m+1}\sqrt{\epsilon_{m+1} - \epsilon_0} = zc_m \;\to c_{m+1} = \frac{z}{\sqrt{\epsilon_{m+1} - \epsilon_0}}c_m$$

$$|z\rangle \propto \sum_{n\geq 0} \frac{z^n}{\sqrt{\prod_{k=1}^{n}(\epsilon_k - \epsilon_0)}}|n\rangle$$

continuity, normalization and resolution of identity

infinite square-well potential



$$H = \frac{p^2}{2m} \rightarrow \; e_n = bn^2 \;\; (n \geq 1)$$

$$b = \pi^2/2mL^2; \quad e_1 = b$$

$$\epsilon_{n+1} = (n+1)^2 = (\sqrt{\epsilon_n} + 1)^2$$

$$f(x) = \left(\sqrt{x} + 1\right)^2$$

$$\Psi_n(x) = \sqrt{2/L}\,\sin(n\pi x/L)$$

$$\mathcal{H} = N^2$$

$$[\mathcal{H}, A^\dagger] = 2A^\dagger\sqrt{\mathcal{H}} + A^\dagger$$

$$[\mathcal{H}, A] = -2\sqrt{\mathcal{H}}A - A$$

$$[A, A^\dagger] = 2\sqrt{\mathcal{H}} + 1$$

$$\mathcal{H}|n\rangle = n^2\,|n\rangle \quad (n \geq 1)$$

$$A^\dagger|n\rangle = \sqrt{(n+1)^2 - 1}\,|n+1\rangle$$

$$A|n\rangle = \sqrt{n^2 - 1}\,|n-1\rangle \qquad (A|1\rangle = 0)$$

$$N_{n-1} = \sqrt{n^2 - 1}$$

physical realization of the operators

$$\mathcal{H} = -\frac{L^2}{\pi^2}\frac{d^2}{dx^2}$$

$$A = \sqrt{1+\frac{2}{N}}\left(-\frac{L}{\pi}\frac{d}{dx}\sin\frac{\pi x}{L} + N\cos\frac{\pi x}{L}\right)$$

$$A^\dagger = \left(\frac{L}{\pi}\sin\frac{\pi x}{L}\frac{d}{dx} + \cos\frac{\pi x}{L}N\right)\sqrt{1+\frac{2}{N}}$$

ladder operators

$$\Psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad N\Psi_n(x) = n\Psi_n(x)$$

$$A\Psi_n(x) = \sqrt{\frac{2}{L}} \sqrt{1 + \frac{2}{N}} \left(-\frac{L}{\pi} \frac{d}{dx} \left(\sin \frac{\pi x}{L} \sin \frac{n\pi x}{L} \right) + N \left(\cos \frac{\pi x}{L} \sin \frac{n\pi x}{L} \right) \right)$$

$$A\Psi_n(x) = (n-1)\sqrt{1 + \frac{2}{N}}\, \Psi_{n-1}(x) = \sqrt{n^2 - 1}\, \Psi_{n-1}(x)$$

$$\boxed{A\Psi_1(x) = 0}$$

$$A^\dagger \Psi_n(x) = \sqrt{1 + \frac{2}{n}} \left(\frac{L}{\pi} \sin \frac{\pi x}{L} \frac{d}{dx} + N \cos \frac{\pi x}{L} \right) \Psi_n(x)$$

$$A^\dagger \Psi_n(x) = \sqrt{n^2 + 2n}\, \Psi_{n+1}(x) = \sqrt{(n+1)^2 - 1}\, \Psi_{n+1}(x)$$

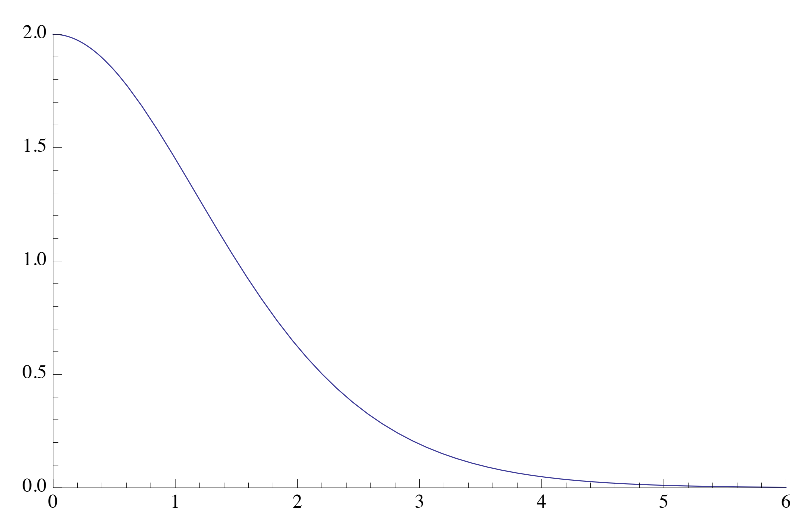
nonlinear coherent states

$$A|z\rangle_{NL} = z|z\rangle_{NL} \qquad (z \in \mathbb{C})$$

$$|z\rangle_{NL} = \sum_{n\geq 1} c_n |n\rangle \qquad A|n\rangle = \sqrt{n^2-1}\,|n-1\rangle$$

$$|z\rangle_{NL} = \mathcal{N}_{NL}(|z|) \sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n^2-1)!}} |n\rangle$$

$$\mathcal{N}^2_{NL}(|z|) = \left[\sum_{n=1}^{\infty} \frac{|z|^{2(n-1)}}{(n^2-1)!}\right]^{-1} = \frac{|z|^2}{I_2(2|z|)} \qquad 0 \leq |z| < \infty$$



D operators

$$D = \frac{1}{\sqrt{N+2}} A$$
$$D^\dagger = A^\dagger \frac{1}{\sqrt{N+2}}$$

$$D|n\rangle = \frac{1}{\sqrt{N+2}} A\,|n\rangle = \sqrt{n-1}\,|n-1\rangle$$

$$D|1\rangle = 0$$

$$D^\dagger|n\rangle = A^\dagger \frac{1}{\sqrt{N+2}}\,|n\rangle = \sqrt{n}\,|n+1\rangle$$

$$[D, D^\dagger] = 1$$
$$[N, D^\dagger] = D^\dagger$$
$$[N, D] = -D$$

$$\mathcal{H} = (D^\dagger D + 1)^2$$

linear coherent states

$$D|n\rangle = \sqrt{n-1}\,|n-1\rangle \qquad\qquad D|1\rangle = 0$$

$$D|z\rangle_L = z|z\rangle_L \qquad\qquad (z \in \mathbb{C})$$

$$|z\rangle_L = \sum_{n\geq 1} c_n|n\rangle$$

$$|z\rangle_L = e^{\frac{-|z|^2}{2}} \sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n-1)!}}\,|n\rangle$$

$$|n\rangle \to \sin\frac{n\pi x}{L}$$

canonical conjugate operators

$$\begin{aligned} [D, D^\dagger] &= 1 \\ [N, D^\dagger] &= D^\dagger \\ [N, D] &= -D \end{aligned}$$

$$\begin{aligned} \xi &= \frac{L}{\sqrt{2}}(D + D^\dagger) \\ \rho &= i\frac{\hbar}{\sqrt{2}L}(D^\dagger - D) \end{aligned}$$

$$[\xi, \rho] = i\hbar$$

cs uncertainty relation

$$\langle \xi \rangle_z = \langle z|\xi|z\rangle_L = \frac{L}{\sqrt{2}}\langle z|D + D^\dagger|z\rangle_L = \sqrt{2}\,L\,\mathbb{R}(z)\,,$$

$$\langle \rho \rangle_z = \langle z|\rho|z\rangle_L = i\frac{\hbar}{L\sqrt{2}}\langle z|D^\dagger - D|z\rangle_L = \frac{\sqrt{2}\,\hbar}{L}\mathbb{I}(z)$$

$$\begin{aligned}\langle \xi^2 \rangle_z &= \langle z|\xi^2|z\rangle_L = \frac{L^2}{2}\langle z|D^2 + (D^\dagger)^2 + DD^\dagger + D^\dagger D|z\rangle_L \\ &= \frac{L^2}{2}(z^2 + \bar{z}^2 + 2|z|^2 + 1)\,, \\ \langle \rho^2 \rangle_z &= \langle z|\rho^2|z\rangle_L = -\frac{\hbar^2}{2\,L^2}\langle z|D^2 + (D^\dagger)^2 - DD^\dagger - D^\dagger D|z\rangle_L \\ &= -\frac{\hbar^2}{2\,L^2}(z^2 + \bar{z}^2 - 2|z|^2 - 1)\end{aligned}$$

uncertainty relation

$$\Delta\xi = \sqrt{\langle \xi^2 \rangle_z - \langle \xi \rangle_z^2} = \frac{L}{\sqrt{2}}\,,$$

$$\Delta\rho = \sqrt{\langle \rho^2 \rangle_z - \langle \rho \rangle_z^2} = \frac{\hbar}{L\sqrt{2}}$$

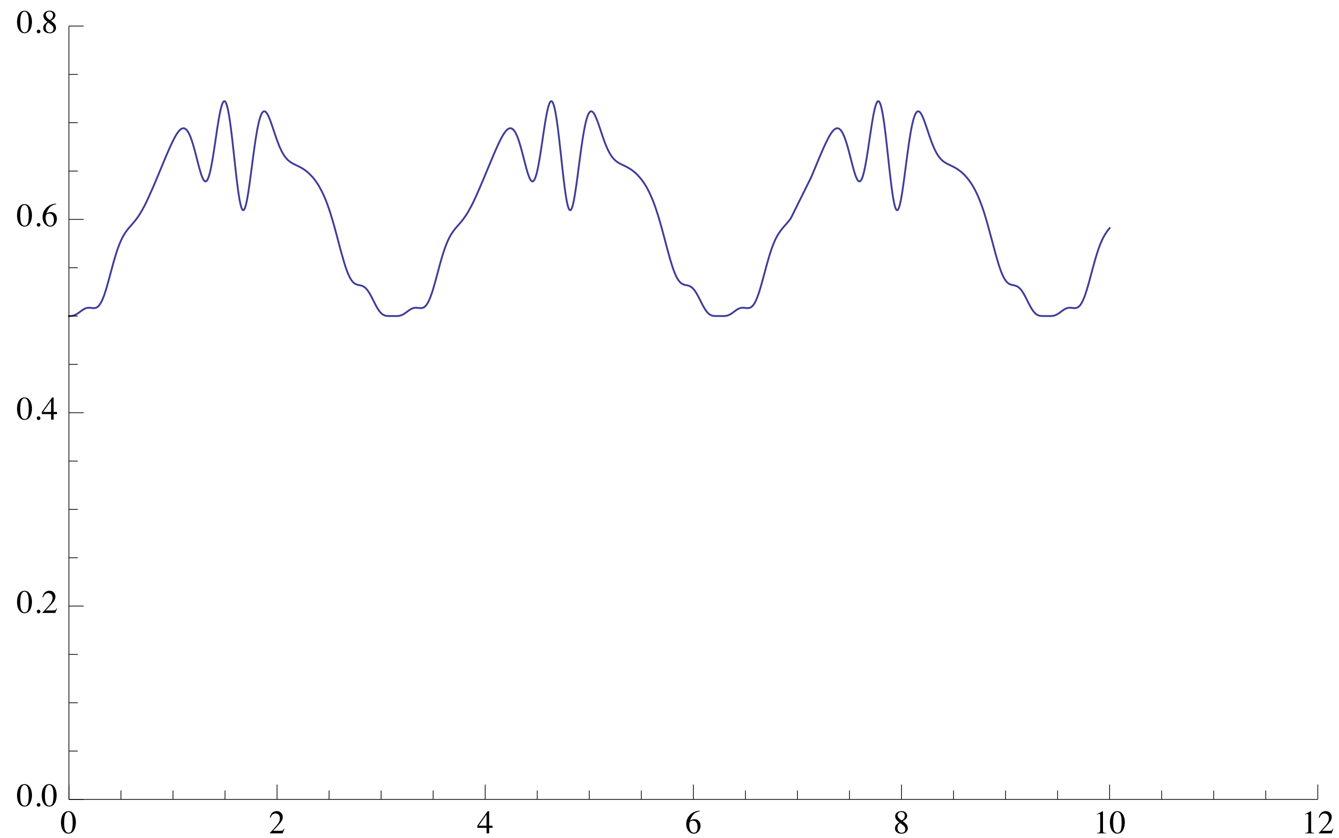
$$\Delta\xi\,\Delta\rho = \frac{\hbar}{2}$$

cs time evolution

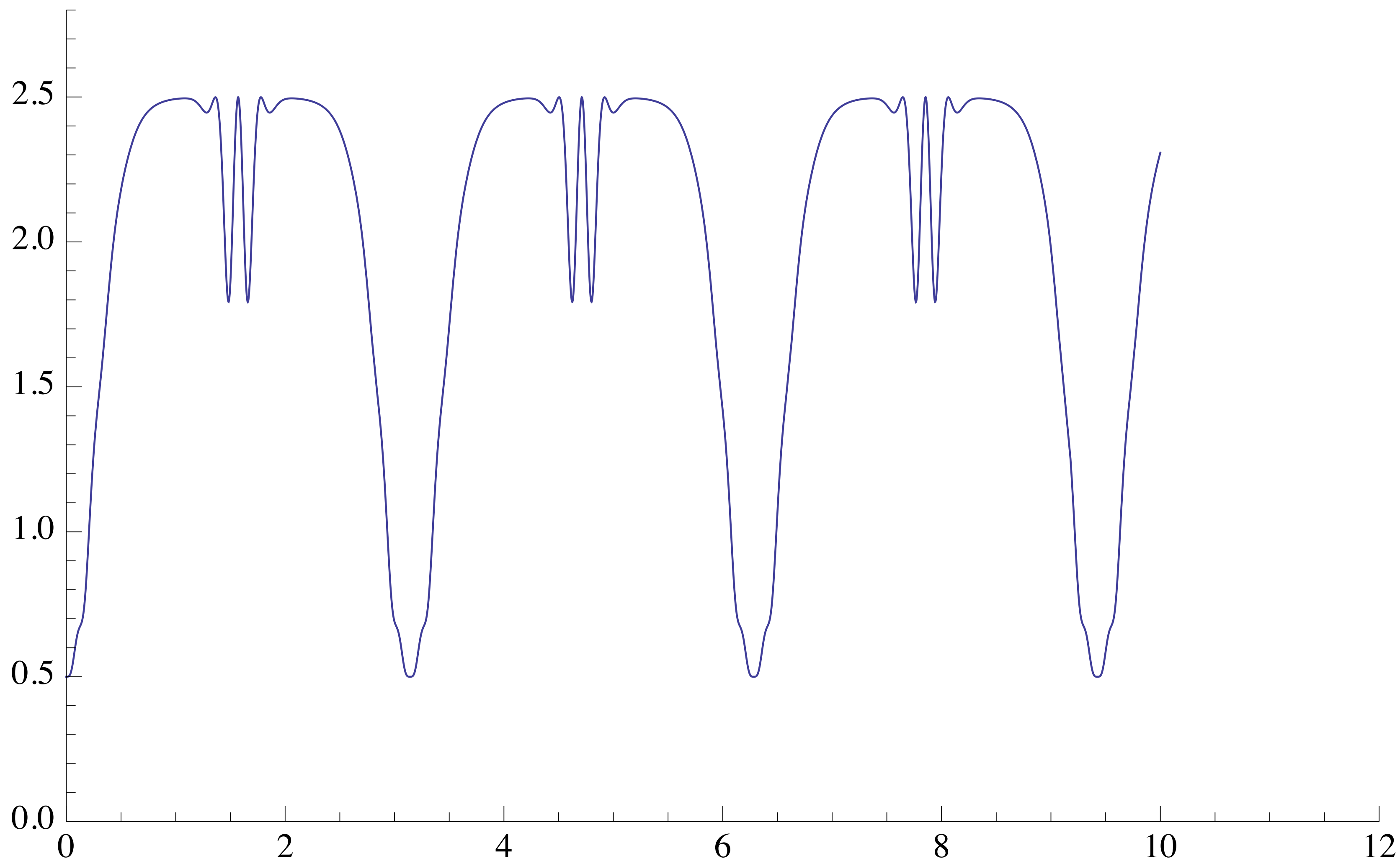
$$U(t) = \exp(-i\mathcal{H}t/\hbar)$$

$$\begin{aligned} U(t)\,|z\rangle_L &= e^{-i\mathcal{H}t/\hbar}\,|z\rangle_L \\ &= e^{-|z|^2/2} \sum_{n\geq 1} \frac{z^{n-1}}{\sqrt{(n-1)!}}\, e^{-i\mathcal{H}t/\hbar}\,|n\rangle \\ &= e^{-|z|^2/2} \sum_{n\geq 1} \frac{z^{n-1}}{\sqrt{(n-1)!}}\, e^{-ibn^2t/\hbar}\,|n\rangle \end{aligned}$$

$$\langle\xi(t)\rangle;\ \langle\rho(t)\rangle;\ \langle\xi(t)^2\rangle;\ \langle\rho(t)^2\rangle \longrightarrow \Delta\xi(t)\,\Delta\rho(t)$$



$$z = 0.5 + 0.25i$$



$$z = 1 + i$$

analogous Hamiltonian - Dr Jekyll and Mr Hyde

$$\xi = \frac{L}{\sqrt{2}}(D + D^\dagger)$$
$$\rho = i\frac{\hbar}{\sqrt{2}L}(D^\dagger - D)$$

$$[\xi, \rho] = i\hbar$$

$$\xi$$
$$\rho = -i\hbar\frac{d}{d\xi}$$

$$D = \frac{1}{\sqrt{2}L}\xi + \frac{L}{\sqrt{2}}\frac{d}{d\xi}$$
$$D^\dagger = \frac{1}{\sqrt{2}L}\xi - \frac{L}{\sqrt{2}}\frac{d}{d\xi}$$

$$\mathcal{H} = (D^\dagger D + 1)^2 \longrightarrow H = \frac{\pi^2\hbar^2}{2mL^2}\left[-\frac{L^2}{2}\frac{d^2}{d\xi^2} + \frac{1}{2L^2}\xi^2 + \frac{1}{2}\right]^2$$
$$= \frac{\pi^2\hbar^2}{2mL^2}\left[\mathcal{H}_{HO} + \frac{1}{2}\right]^2$$

$$H_{HO} = \left[-\frac{\hbar^2}{2m}\frac{d^2}{d\xi^2} + \frac{m\omega^2}{2}\xi^2 \right]$$

$$\omega = \frac{\hbar}{mL^2}$$

$$H = \frac{\pi^2\hbar^2}{2mL^2}\left[-\frac{L^2}{2}\frac{d^2}{d\xi^2} + \frac{1}{2L^2}\xi^2 + \frac{1}{2} \right]^2 \qquad H\Phi_n(x) = E_n\Phi_n(x)$$

$$\Phi_n(y) = (\pi L^2)^{-1/4}\frac{1}{2^n n!}H_n(y)e^{-y^2/2}$$

$$E_n = \left(\frac{\pi^2\hbar^2}{2mL^2} \right)(n+1)^2 \qquad n \geq 0$$

same spectrum!

summary

- GHA -> CS
- ISWP -> A, A^+ and D, D^+
- nonlinear and linear cs
- conjugate variables
- analogous Hamiltonian

basic papers

- EMFC, MA Rego-Monteiro, JPA 34 (2001) 3253
- Y Hassouni, EMFC, MA Rego-Monteiro, PRA 71 (2005) 022104
- EMFC, Y. Hassouni, MA Rego-Monteiro, LMCS Rodrigues, PLA 372 (2008) 3350
- ST Ali, L Balkova, EMFC, JP Gazeau, MA Rego-Monteiro, LMCS Rodrigues, JMP 50 (2009) 043517
- EMFC, LMCS Rodrigues, MA Rego-Monteiro - in preparation