NEW APPLICATIONS OF COHERENT STATES IN QUANTUM INFORMATION

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Coherent States and their applications: a contemporary panorama
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Which applications?

1) State transformation games:

what is the best way to turn a given input into a desired output?

2) Quantum benchmarks:

how to certify a quantum advantage?

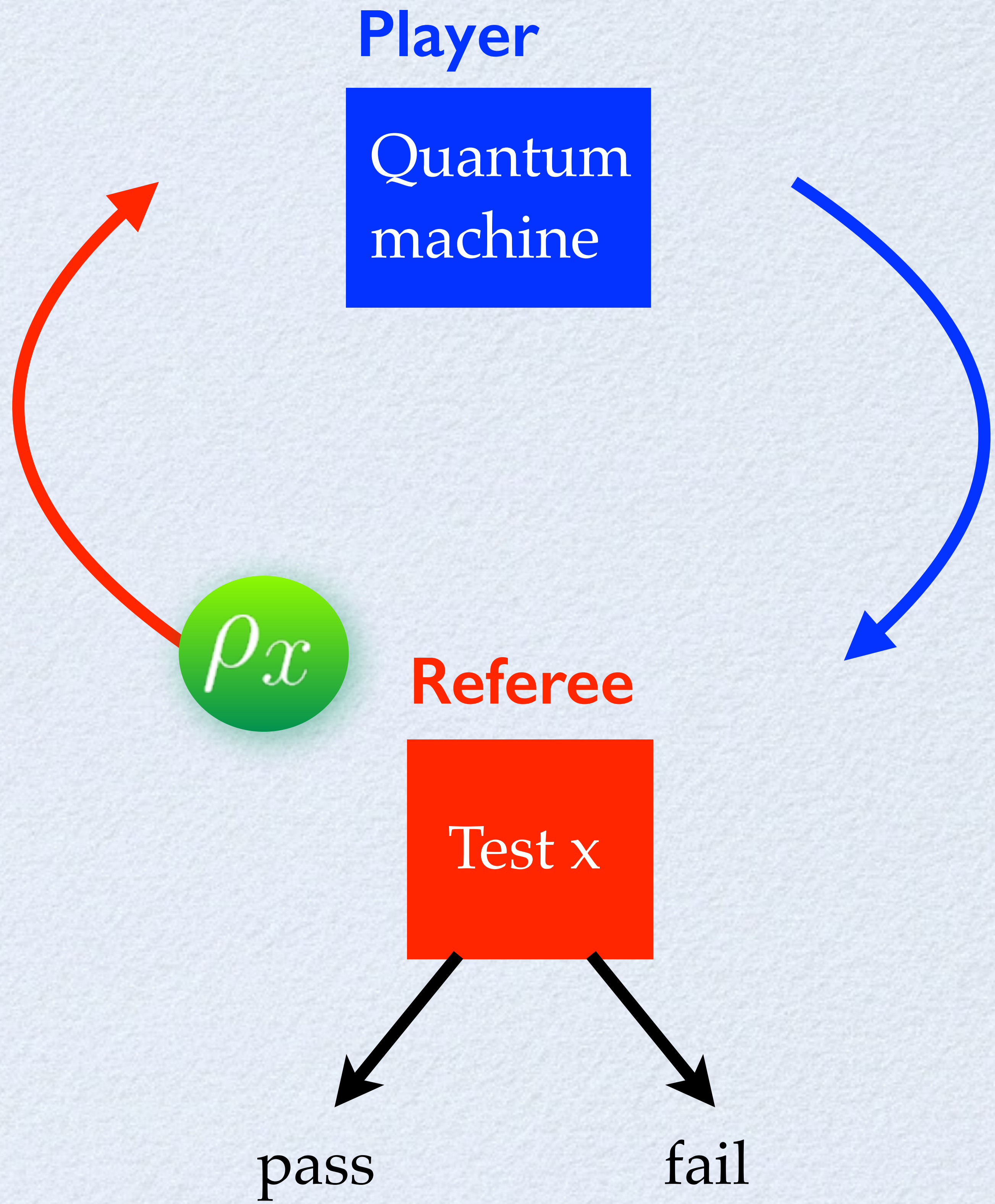
APPLICATION 1:

STATE TRANSFORMATION GAMES

State transformation games

A Referee prepares a quantum system in a random state and sends it to a Player,
who has to return an output state.

The Referee then tests the output state and assigns score
1 if the test is passed and 0 otherwise.



Mathematical description

- Random input state: ensemble $\{\rho_x \,, p_x\}_{x \in \mathsf{X}}$

- Test x: binary POVM $\{T_x \,, I - T_x\}$

- Average payoff: $F = \sum_{x \in \mathsf{X}} p_x \, \mathrm{Tr}[T_x \, \widetilde{\rho}_x]$

Examples

- Preserving pure states:
$$\rho_x = |\psi_x\rangle\langle\psi_x|$$
$$T_x = |\psi_x\rangle\langle\psi_x|$$

- Making copies:
$$\rho_x = (|\psi_x\rangle\langle\psi_x|)^{\otimes N}$$
$$T_x = (|\psi_x\rangle\langle\psi_x|)^{\otimes M}$$

- Amplifying coherent states
$$\rho_\alpha = |\alpha\rangle\langle\alpha|$$
$$T_\alpha = |g\,\alpha\rangle\langle g\,\alpha|\,, \qquad g > 1$$

Strategies

The player wants to maximize her payoff.
What is the best strategy?

- Strategy: physical transformation
  Mathematically: completely positive, trace-preserving map

$$\mathcal{C} : \mathrm{St}(\mathcal{H}_{\mathrm{in}}) \to \mathrm{St}(\mathcal{H}_{\mathrm{out}})$$

- Output state: $\widetilde{\rho}_x := \mathcal{C}(\rho_x)$

- Maximum payoff: $F_Q := \max_{\mathcal{C}} \sum_x p_x \left[T_x \, \mathcal{C}(\rho_x) \right]$

The maximum payoff

Optimizing over all physical transformations,
the Player can reach the payoff

$$F_Q = \min_{\sigma > 0\,, \mathrm{Tr}[\sigma]=1} \left\| (I_{out} \otimes \sigma^{-\frac{1}{2}})\, \Omega\, (I_{out} \otimes \sigma^{-\frac{1}{2}}) \right\|_\infty$$

$$\Omega := \sum_x p_x\, T_x \otimes \rho_x^T$$ "game operator"

$$\text{where } \|A\|_\infty := \max_{\||\Psi\rangle\|=1} \langle \Psi | A | \Psi \rangle\,, \quad A \geq 0$$

Koenig, Renner, Schaffner, IEEE Trans. Inf.Th. 55, 4337 (2009)
Chiribella and Xie, PRL 110, 213602 (2013)

Variant: games with abstention

Now the Referee gracefully grants the Player the right to pass, as many times as she wants.

- Strategy: ***probabilistic*** transformation

$|\psi\rangle$

Probabilistic Machine

Mathematically: completely positive, trace non-increasing map $\mathcal{P}$

- Maximum payoff (conditional)

$$F_Q^{\rm prob} := \sup_{\mathcal{P}} \frac{\sum_x p_x \, {\rm Tr}[T_x \, \mathcal{P}(\rho_x)]}{\sum_x p_x {\rm Tr}[\mathcal{P}(\rho_x)]}$$

The maximum payoff

Optimizing over all probabilistic transformations, the Player can reach the payoff

$$F_Q^{prob} = \left\| (I_{out} \otimes \rho^{-\frac{1}{2}})^T \; \Omega \; (I_{out} \otimes \rho^{-\frac{1}{2}})^T \right\|_\infty$$

$$\Omega := \sum_x p_x \, T_x \otimes \rho_x^T$$

$$\rho := \sum_x p_x \, \rho_x$$

Chiribella and Xie, PRL 110, 213602 (2013)

EXAMPLE:

AMPLIFYING COHERENT STATES OF THE HARMONIC OSCILLATOR

Amplifying coherent states of light

Coherent state:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$\alpha \in \mathbb{C}$

Ideally we wish to transform $|\alpha\rangle$ into $|g\alpha\rangle,\ g > 1$ ("amplifier gain")

$\longrightarrow$

Modelling the source

To model the source of coherent states we assume a Gaussian distribution:

$$p_\lambda(\alpha) = \lambda e^{-\lambda|\alpha|^2}$$

with this choice the expected photon number is $\langle n \rangle = 1/\lambda$

λ represents our prior information about the input:

$$\lambda = 0 \quad \Rightarrow \quad \text{no information}$$

$$\lambda = \infty \quad \Rightarrow \quad \text{complete information}$$

No perfect amplification

The transformation $|\alpha\rangle \to |g\alpha\rangle \qquad \forall \alpha \in \mathbb{C}$
is not physically realizable.

For good reasons:

- it would violate the uncertainty principle
- it would lead to faster-than-light communication
- it would violate the no-cloning theorem
- ...

How can we approximate amplification with a physical process allowed by quantum mechanics?

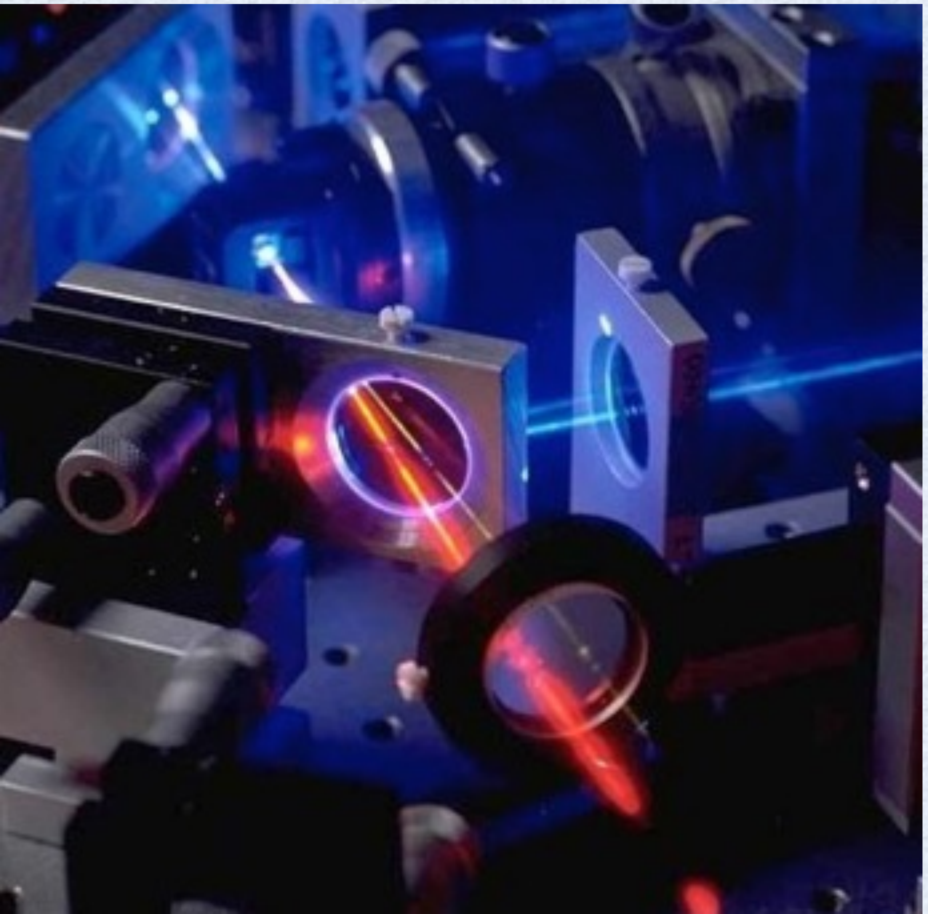
Approximate amplification

Most popular example: parametric amplifier

$$\mathcal{C}_r(\rho) = \mathrm{Tr}_B[\underbrace{e^{r(a^\dagger b^\dagger - ab)}}_{\text{two-mode squeezing operator}}(\rho \otimes \underbrace{|0\rangle\langle 0|}_{\text{ancillary mode in the vacuum state}})e^{-r(a^\dagger b^\dagger - ab)}]$$

two-mode squeezing operator

ancillary mode in the vacuum state



For the input $|\alpha\rangle$

the output is a thermal state displaced by

$$g\alpha, \qquad g = \cosh r$$

The best deterministic amplifiers

Theorem (Namiki 2008,GC-Xie 2013): the best deterministic amplifiers are two-mode squeezing processes with squeezing parameter depending on the amount of information about the input.

The maximum fidelity that can be achieved using two-mode squeezing is given by

$$F^{opt}_{g,\lambda} = \begin{cases} \dfrac{\lambda+1}{g^2}, & \lambda \leq g-1 \\ \dfrac{\lambda}{\lambda+(g-1)^2}, & \lambda > g-1. \end{cases}$$

amplification of $\frac{g}{\lambda+1}$ (for $\lambda \leq g-1$)

no amplification at all (!) (for $\lambda > g-1$)

critical behaviour at the value $\lambda_c^{det} = g-1$

The best probabilistic amplifiers

Dramatic effect of the critical value:

When the prior information is larger than the critical value, nearly perfect amplification becomes possible!

$$F^{prob}_{g,\lambda} = \begin{cases} \dfrac{\lambda+1}{g^2}, & \lambda \leq g^2 - 1 \\ 1 & \lambda > g^2 - 1. \end{cases}$$

critical behaviour at the value $\lambda_c^{prob} = g^2 - 1$

Chiribella and Xie, PRL 110, 213602 (2013)

The importance of the prior information

The dramatic difference between deterministic and probabilistic amplifiers discovered here is a genuine effect of the finite photon number.

In the idealized scenario of "no prior information" ($\lambda = 0$) there is no difference!

The best amplifier is just two-mode squeezing and has fidelity

$$F^{opt}_{g,\lambda=0} = \frac{1}{g^2}$$

How to achieve unit fidelity?

Ralph and Lund (2008) proposed a probabilistic scheme that achieves almost perfect amplification.

$$\mathcal{Q}_N(\rho) = Q_N\,\rho\,Q_N^\dagger \qquad Q_N := \sum_{n=0}^{N} \frac{g^n}{g^N}\,|n\rangle\langle n|$$

For large N: $\mathcal{Q}_N(|\alpha\rangle\langle\alpha|) \approx |g\alpha\rangle\langle g\alpha|$

(*caveat:* the probability of success drops exponentially)

Probabilistic amplifiers in the lab

PRL 104, 123603 (2010) PHYSICAL REVIEW LETTERS week ending 26 MARCH 2010

Implementation of a Nondeterministic Optical Noiseless Amplifier

Franck Ferreyrol, Marco Barbieri, Rémi Blandino, Simon Fossier, Rosa Tualle-Brouri, and Philippe Grangier

LETTERS
PUBLISHED ONLINE: 28 MARCH 2010 | DOI: 10.1038/NPHOTON.2010.35
nature photonics

Heralded noiseless linear amplification and distillation of entanglement

G. Y. Xiang[1], T. C. Ralph[2], A. P. Lund[1,2], N. Walk[2] and G. J. Pryde[1]*

nature physics
LETTERS
PUBLISHED ONLINE: 11 NOVEMBER 2012 | DOI: 10.1038/NPHYS2469

Heralded noiseless amplification of a photon polarization qubit

S. Kocsis[1,2], G. Y. Xiang[2,3], T. C. Ralph[1,4] and G. J. Pryde[1,2]*

nature physics
LETTERS
PUBLISHED ONLINE: 15 AUGUST 2010 | DOI: 10.1038/NPHYS1743

Noise-powered probabilistic concentration of phase information

Mario A. Usuga[1,2†], Christian R. Müller[1,3†], Christoffer Wittmann[1,3], Petr Marek[4], Radim Filip[4], Christoph Marquardt[1,3], Gerd Leuchs[1,3] and Ulrik L. Andersen[2]*

IOPscience
New Journal of Physics > Volume 15 > September 2013

A complete characterization of the heralded noiseless amplification of photons

OPEN ACCESS

N Bruno, V Pini, A Martin and R T Thew[1]

ARTICLES
PUBLISHED ONLINE: 21 NOVEMBER 2010 | DOI: 10.1038/NPHOTON.2010.260
nature photonics

A high-fidelity noiseless amplifier for quantum light states

A. Zavatta[1,2], J. Fiurášek[3] and M. Bellini[1,2]*

QUANTUM BENCHMARKS

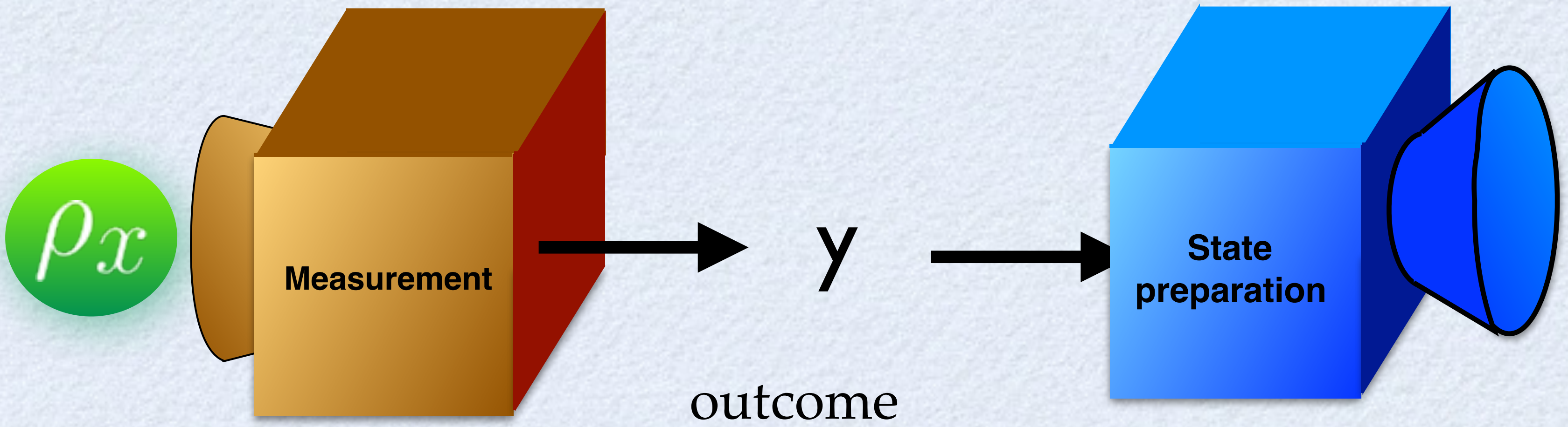
How to certify genuine quantum information processing?

Suppose that you have an experimentalist friend who claims that she built a device that implements a *quantum* strategy for certain state transformation game.

How can she prove her claim?

Unfortunately, there are experimental imperfections
and the actually fidelity is smaller
than the optimal fidelity...

Classical strategies: measure and prepare (m&p)



"Classical way": processing via measurement

information about the input state is extracted by a measurement,

the output state copies is prepared based on this information

Mathematical formulation

Classical strategies: measure-and-prepare channels

- Measurement: POVM

$$\{P_y\}_{y\in \mathsf{Y}}\,, \quad P_y \geq 0\,\forall y \in \mathsf{Y}\,, \quad \sum_{y\in\mathsf{Y}} P_y = I$$

- Measure-and-prepare channel $\mathcal{C}(\rho) := \sum_{y\in\mathsf{Y}} \widetilde{\rho}_y \,\mathrm{Tr}[P_y\,\rho]$

- Quantum benchmark:

$$F_{MP} = \sup_{\{P_y\}\,,\{\widetilde{\rho}_y\}} \sum_{x,y} p_x \,\mathrm{Tr}[T_x\widetilde{\rho}_y]\,\mathrm{Tr}[P_y\,\rho_x]$$

Quantum benchmark

$$F_{MP} = \min_{\sigma>0\,,\mathrm{Tr}[\sigma]=1} \left\| (I_{out} \otimes \sigma^{-\frac{1}{2}})\,\Omega\,(I_{out} \otimes \sigma^{-\frac{1}{2}}) \right\|_{\times}$$

$$\Omega := \sum_x p_x\, T_x \otimes \rho_x^T$$

where $\quad \|A\|_\times := \max_{\||\psi\rangle\|=\||\psi\rangle\|=1} \langle\varphi|\langle\psi|A|\varphi\rangle|\psi\rangle \qquad A \geq 0$

Chiribella and Xie, PRL 110, 213602 (2013)

Probabilistic benchmarks

Suppose that the Player is allowed to pass.

Mathematically: POVM $\{P_y\}_{y\in\mathsf{Y}} \cup \{P_{\text{pass}}\}$

$$\sum_{y\in\mathsf{Y}} P_y + P_{\text{pass}} = I$$

What is the maximum payoff?

$$F_{MP}^{prob} = \left\| (I_{out} \otimes \rho^{-\frac{1}{2}})^T \, \Omega \, (I_{out} \otimes \rho^{-\frac{1}{2}})^T \right\|_{\times}$$

$$\Omega := \sum_x p_x \, T_x \otimes \rho_x^T \qquad \rho := \sum_x p_x \, \rho_x$$

Chiribella and Xie, PRL 110, 213602 (2013)

QUANTUM BENCHMARKS FOR GILMORE-PERELOMOV COHERENT STATES

Yang, Chiribella, and Adesso, PRA 90, 042319 (2014)

Chiribella and Adesso, Phys. Rev. Lett. 112, 010501 (2014)

Gilmore-Perelomov CS

- **Gilmore-Perelomov coherent states (GPCS):**

$$|\psi_g\rangle := U_g|\psi\rangle \qquad g \in \mathsf{G}\,,\ U = \text{irrep}$$

(quotient w.r.t. stabilizer implicit)

- **Mutually coherent GPCS:** two families of GPCS

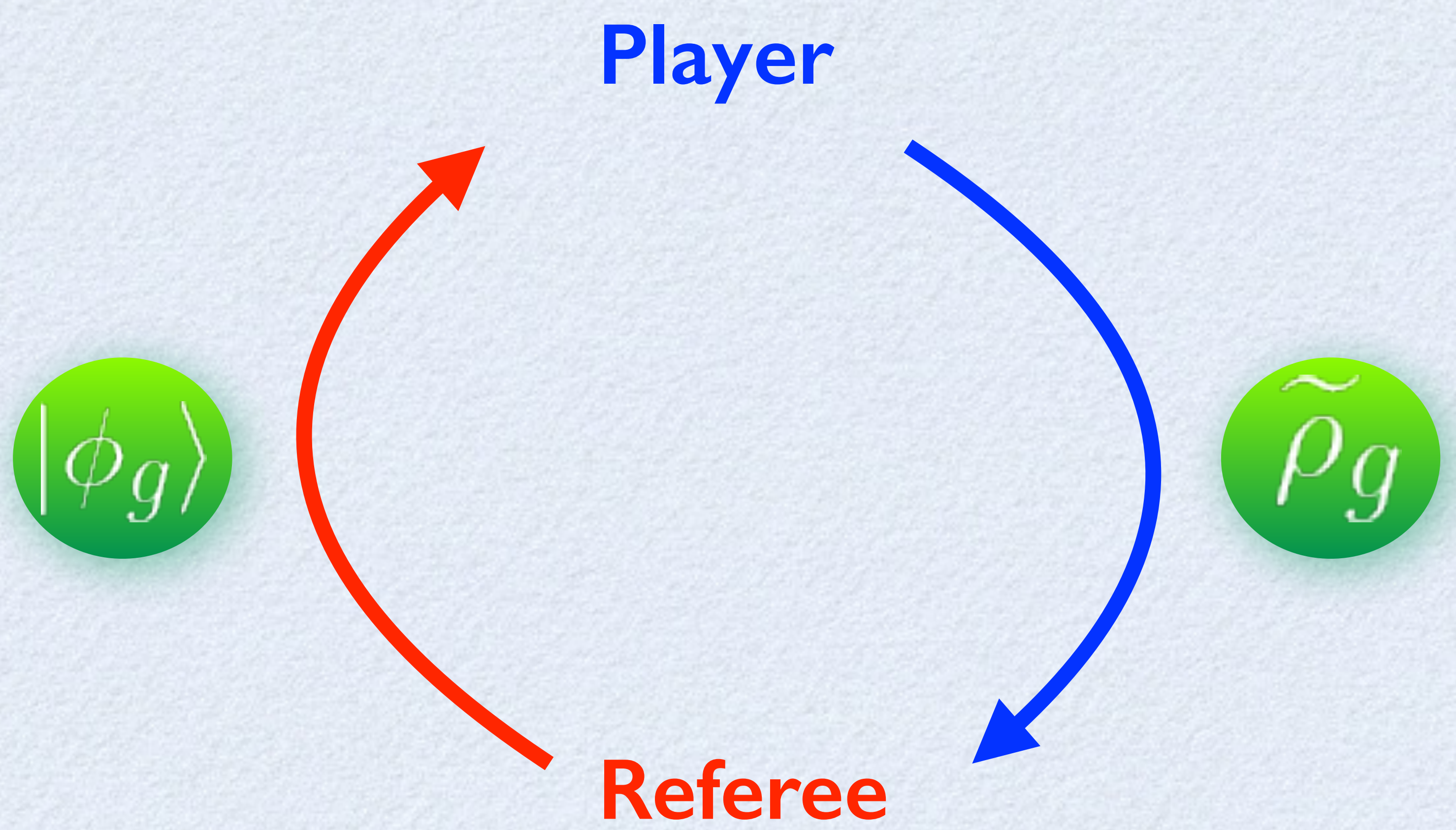
$$\{|\varphi_g\rangle\} \quad \text{and} \quad \{|\psi_g\rangle\}$$

are **mutually coherent** if

$$\{|\varphi_g\rangle \otimes |\psi_g\rangle\}$$

is a family of GPCS

GPCS state transformation games



$$\{|\phi_g\rangle, dg\}$$

input ensemble

(limit of "uniform prior")

$$\{|\psi_g\rangle\langle\psi_g|, I - |\psi_g\rangle\langle\psi_g|\}$$

test g

The benchmark

If $\{|\varphi_g\rangle\}$ and $\{|\psi_g\rangle\}$

are mutually coherent GPCS

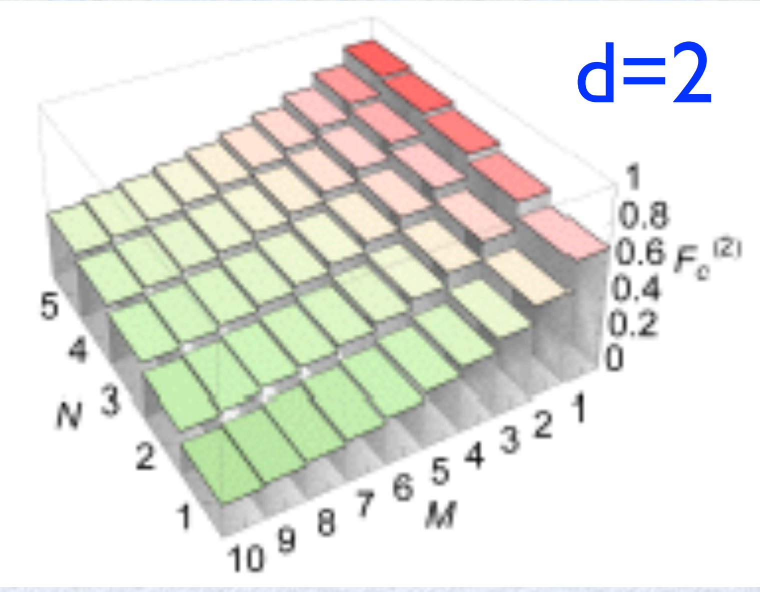
then

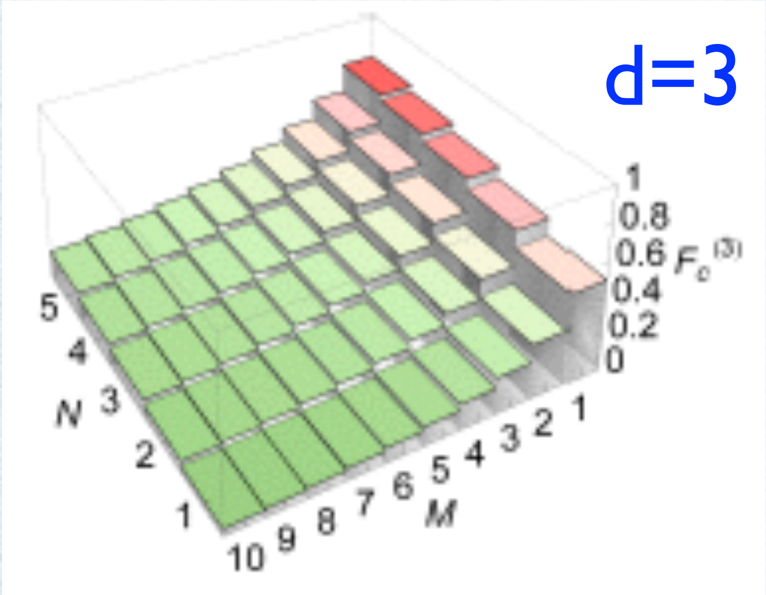
- the quantum benchmark is $F_{MP} = \frac{\int dg\, |\langle\phi|\phi_g\rangle|^2 \, |\langle\psi|\psi_g\rangle|^2}{\int dg\, |\langle\phi|\phi_g\rangle|^2}$
- the optimal measurement has POVM
  $P_g = |\varphi_g\rangle\langle\varphi_g|$
- the optimal state preparation is $|\psi_g\rangle$

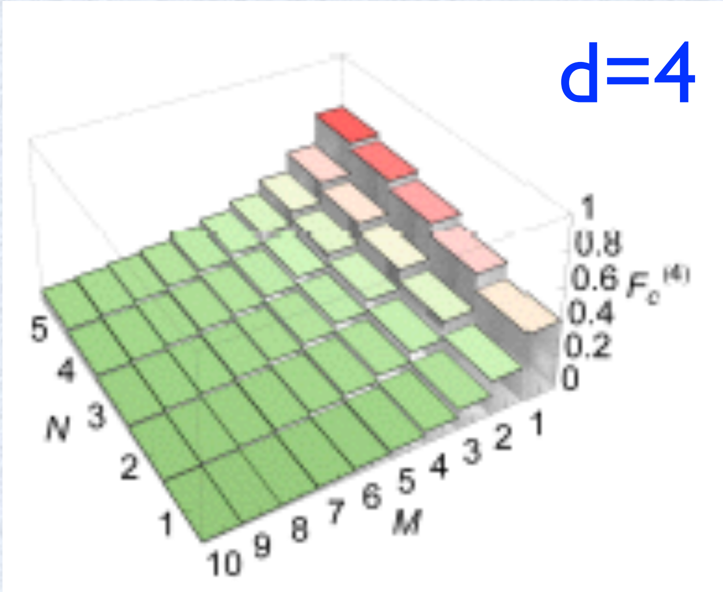
Example 1: benchmark for quantum copy machines

State transformation game: given N copies
of a completely unknown pure state
produce M≥N copies
of the same quantum state

Quantum benchmark: $$F_{MP}(N \to M) = \prod_{k=1}^{d-1} \frac{N+k}{N+M+k}$$







Example 2: quantum benchmark for coherent state modulation

State transformation game: given a coherent state $|\alpha\rangle$
transform it into the state $|g\alpha\rangle\,, \quad g \geq 0$

- g>1 ⟶ amplification
- g=1 ⟶ storage/transmission of the coherent state $|\alpha\rangle$
- g<1 ⟶ attenuation

Quantum benchmark: $$F_{MP}(g) = \frac{1}{1+g^2}$$

Recovers
Hammerer, M. M. Wolf, E. S. Polzik, and J. I. Cirac, PRL 94, 150503 (2005)
Namiki, Koashi, Imoto, PRL 101, 100502 (2008).

Including prior information

So far, we considered the limit of "uniform prior distribution".

What about the realistic case where the input GPCS have a non-uniform prior?

Good priors:

$$p_\lambda(g) = d_\lambda \, |\langle \lambda | \lambda_g \rangle|^2 \qquad d_\lambda = \left(\int dg \, |\langle \lambda | \lambda_g \rangle|^2 \right)^{-1}$$

Example: gaussian prior $\quad p_\lambda(\alpha) = \lambda \, e^{-\lambda |\alpha|^2} = d_\lambda \, |\langle 0 | \lambda \alpha \rangle|^2$

The probabilistic benchmark

If $\{|\phi_g\rangle\}\,, \quad \{|\psi_g\rangle\}\,, \quad \text{and} \quad \{|\lambda_g\rangle\}$

are mutually coherent GPCS

then

the probabilistic quantum benchmark is

$$F_{MP} = \frac{\int dg\, p_\lambda(g)\, |\langle\phi|\phi_g\rangle|^2\, |\langle\psi|\psi_g\rangle|^2}{\int dg\, p_\lambda(g)\, |\langle\phi|\phi_g\rangle|^2}$$

Example 1: coherent state modulation

State transformation game: given a coherent state $|\alpha\rangle$
transform it into the state $|g\alpha\rangle \,, \quad g \geq 0$

Probabilistic benchmark: $$F_{MP}(g, \lambda) = \frac{1 + \lambda}{1 + g^2 + \lambda}$$

(accidentally,
this benchmark can be achieved with
deterministic operations)

Comparison with experiment

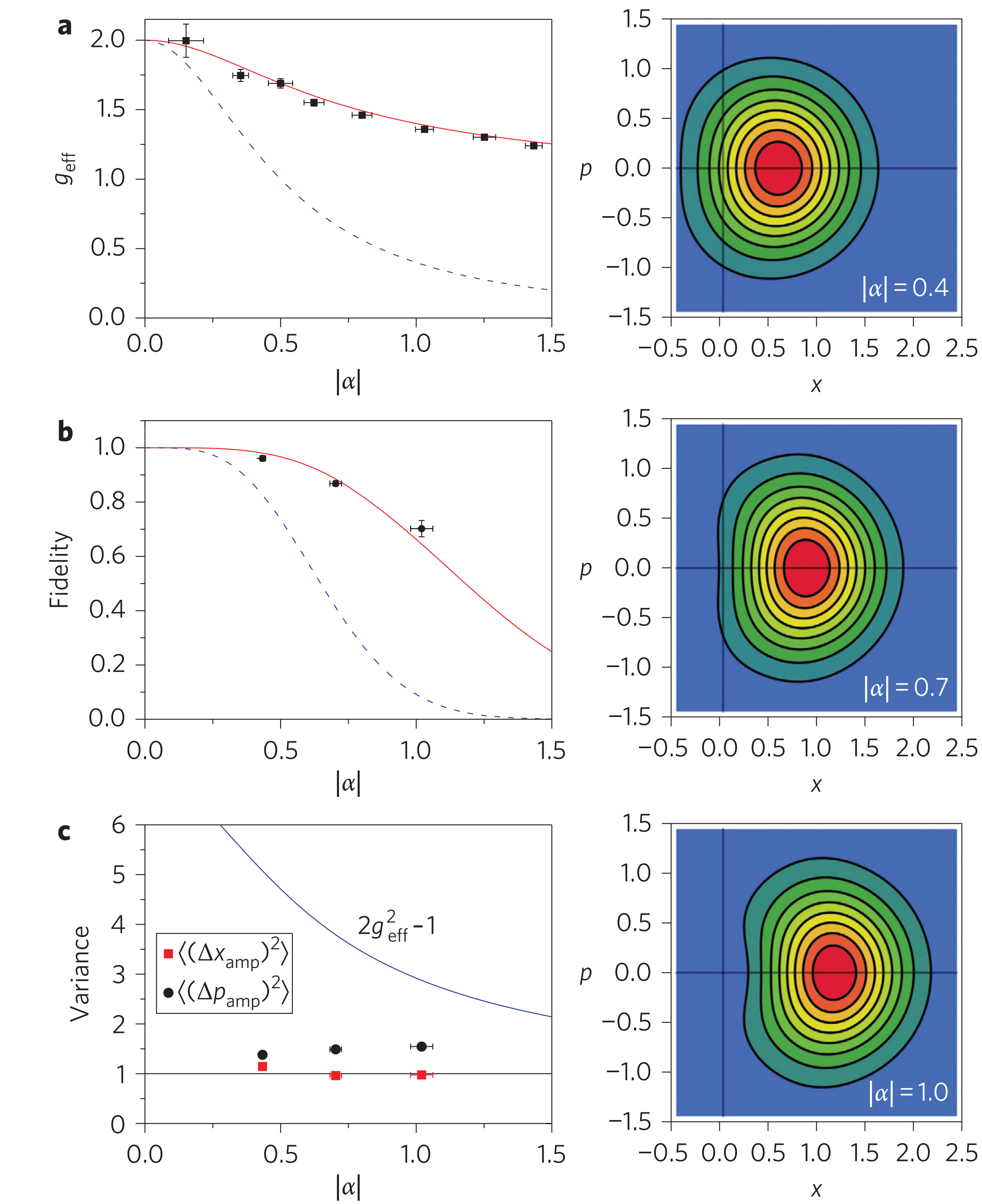
Experiment designed to demonstrate high-fidelity probabilistic amplification with gain g = 2.

Values tested in the experiment:
$|\alpha| \approx 0.4/0.7/1.0$.
Experimental fidelities:
$F_{exp} \approx 0.99/0.91/0.67$.

Reasonable choice of λ: $\lambda = 3$
gives the quantum benchmark

$$\widetilde{F}_{g=2,\lambda=3} = 50\%$$

passed by the experiment
(although more data would be needed…)



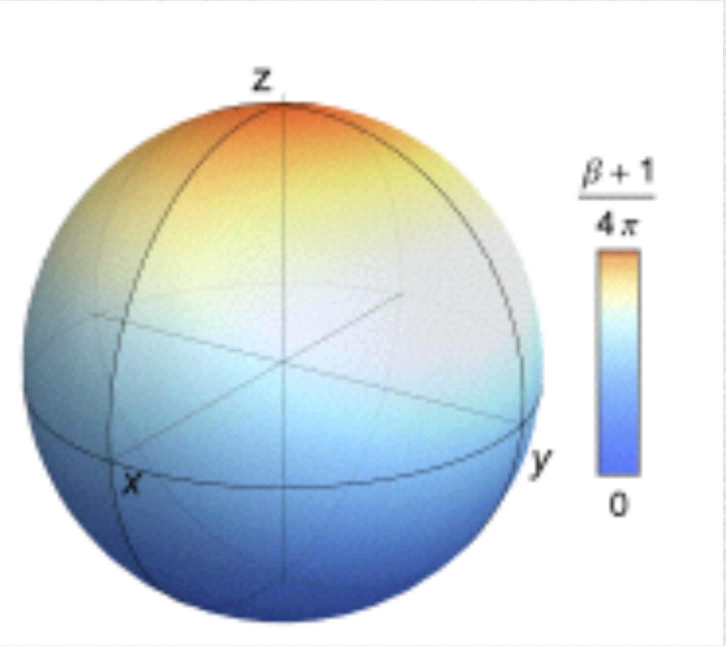
from Zavatta, Bellini, Fiurasek, Nature Photonics 5, 5260 (2011)

Example 2: storage/transmission/cloning of qubit states

State transformation game:
given N copies a qubit state $|\theta, \varphi\rangle := \cos\frac{\theta}{2}|0\rangle + e^{-i\varphi}\sin\frac{\theta}{2}|1\rangle$
produce M copies of the same state.

Probability distribution:

$$p_\beta(d\theta, d\varphi) = (\beta+1)\left(\cos\frac{\theta}{2}\right)^{2\beta+1}\sin\frac{\theta}{2}\,d\theta\frac{d\varphi}{2\pi}$$



Probabilistic benchmark: $F_c^{(2)}(\beta) = \frac{N+\beta+1}{M+N+\beta+1}$

Example 2: d-dimensional states

State transformation game:
given N copies an unknown state $|\psi\rangle \in \mathbb{C}^d$
produce M copies of the same state.

Probability distribution: $p_\beta(\psi) = d_\beta \, |\langle 0|\psi\rangle|^\beta$

Probabilistic benchmark: $F^{(d)}_{MP}(\beta) = \prod_{k=1}^{d-1} \frac{N+\beta+k}{N+M+\beta+k}$

Example 5: squeezed vacuum states

State transformation game:

given N copies an unknown squeezed vacuum state

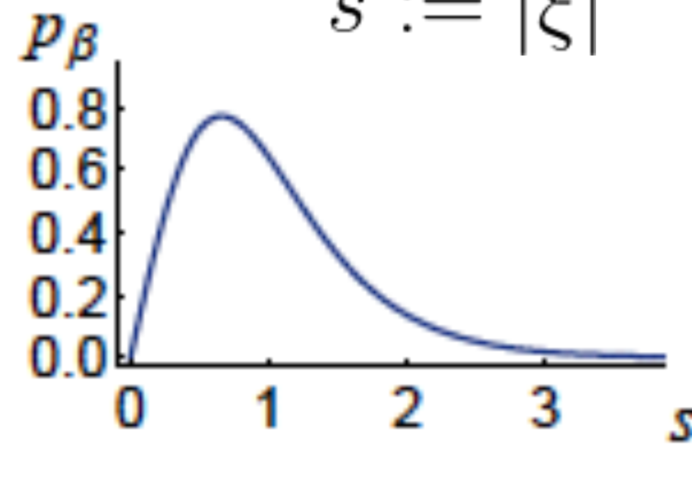
$$|\xi\rangle := S(\xi)\,|0\rangle\,, \qquad S(\xi) = e^{\xi\left(a^\dagger\right)^2 - \bar{\xi} a^2}$$

produce M copies of the same state.

Probability distribution: $p^S_\beta(\xi) = \frac{p_\beta(s)}{2\pi}\,, \quad \text{with} \;\; p_\beta(s) = \frac{\beta \sinh s}{(\cosh s)^{\beta+1}}$

Probabilistic benchmark: $F_{MP}(\beta) = \frac{N+\beta}{N+M+\beta}$

$s := |\xi|$



Example 6: squeezed one-photon states

State transformation game:
given N copies an unknown squeezed one-photon state $S(\xi)|1\rangle$
produce M copies of the same state.

Probabilistic distribution: same as before

Probabilistic benchmark: $F_{MP}(\beta) = \dfrac{3N+\beta}{3(N+M)+\beta}$

Application: teleportation of cat states.

Example 7: single-mode Gaussian states

State transformation game:

given N copies an unknown single-mode Gaussian state

$$|\alpha,\xi\rangle := D(\alpha)S(\xi)\,|0\rangle$$

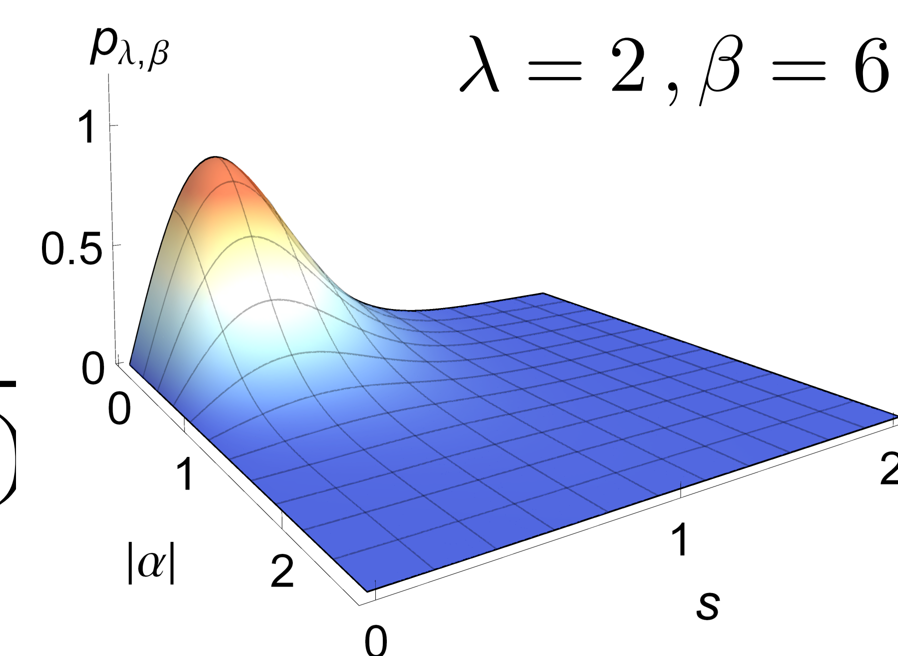
produce M copies of the same state.

Probability distribution:

$$p^G_{\lambda,\beta}(\alpha,s,\theta) = \frac{\lambda\beta}{2\pi^2}\frac{e^{-\lambda|\alpha|^2+\lambda\mathrm{Re}(e^{-i\theta}\alpha^2)\tanh s}\sinh s}{(\cosh s)^{\beta+2}}$$

Probabilistic benchmark:

$$F_{\mathrm{c}}^{(1cs)}(\lambda,\beta) = \frac{(N+\lambda)(N+\beta)}{(N+M+\lambda)(N+M+\beta)}$$



CONCLUSIONS
&
OPEN DIRECTIONS

Conclusion

- State transformation games:
  a general framework for many quantum tasks

  GPCS appearing naturally in many of them

- Quantum benchmarks:
  how to certify quantum advantages.

  general expressions for GPCS that are coherent to each other
  Benchmarks for teleportation, cloning, storage, and transmission.

- Open problems: gate simulation games
  different type of "coherent states" playing a role there.

THANKS FOR YOUR ATTENTION