Lyapunov Theorem for continuous frames

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Abstract

Akemann and Weaver (2014) have shown an interesting generalization of Weaver's KS_2 Conjecture (2004) in the form of approximate Lyapunov theorem. This was made possible thanks to the breakthrough solution of the Kadison-Singer problem by Marcus, Spielman, and Srivastava (2015). In this talk we show a similar type of Lyapunov theorem for continuous frames. In contrast with discrete frames, the proof of this result does not rely on the recent solution of the Kadison-Singer problem.

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Let $\mathcal{H}$ be a separable Hilbert spaces and let (X, μ) be a measure space. A family of vectors $\{\phi_t\}_{t \in X}$ is a *continuous frame* over X for $\mathcal{H}$ if:

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(i) for each $f \in \mathcal{H}$, the function $X \ni t \mapsto \langle f, \phi_t\rangle \in \mathbb{C}$ is measurable, and

(ii) there are constants $0 < A \le B < \infty$, called *frame bounds*, such that

$$A||f||^2 \le \int_X |\langle f, \phi_t\rangle|^2 d\mu(t) \le B||f||^2 \qquad \text{for all } f \in \mathcal{H}. \quad (1)$$

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When $A = B$, the frame is called *tight*, and when $A = B = 1$, it is a *continuous Parseval frame*. More generally, if only the upper bound holds in (1), that is $A = 0$, we say that $\{\phi_t\}_{t \in X}$ is a *continuous Bessel family* with bound B.

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2. *Since $\mathcal{H}$ is separable, by the Pettis measurability theorem, the weak measurability (i) is equivalent to (Bochner) strong measurability on σ-finite measure spaces X. That is, $t \mapsto \phi_t$ is a pointwise a.e. limit of simple measurable functions. Moreover, every measurable function $\phi : X \to \mathcal{H}$ is a.e. uniform limit of a sequence of countably-valued measurable functions.*

Proposition

Suppose that $\{\phi_t\}_{t\in X}$ *is a continuous Bessel family, then its support* $\{t \in X : \phi_t \neq 0\}$ *is a* σ*-finite subset of* X*.*

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Proof.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$, where the index set I is at most countable. For any $n \in \mathbb{N}$ and $i \in I$, by Chebyshev's inequality the Bessel bound yields

$$\mu(\{t \in X : |\langle e_i, \phi_t\rangle|^2 > 1/n\}) \leq Bn < \infty.$$

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Hence, the set

$$\{t \in X : \phi_t \neq 0\} = \bigcup_{i\in I} \bigcup_{n\in\mathbb{N}} \{t \in X : |\langle e_i, \phi_t\rangle|^2 > 1/n\}$$

is a countable union of sets of finite measure. □

Definition

Suppose that $\{\phi_t\}_{t \in X}$ is a continuous Bessel family. For any measurable function $\tau : X \to [0,1]$, define a *modified frame operator*

$$S_{\sqrt{\tau}\phi, X} f = \int_X \tau(t) \langle f, \phi_t \rangle \phi_t \, d\mu(t) \qquad f \in \mathcal{H}.$$

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Remark

A quick calculations shows that $\{\sqrt{\tau(t)}\phi_t\}_{t\in X}$ *is also a continuous Bessel family with the same bound as* $\{\phi_t\}_{t\in X}$. *Hence, a modified frame operator is merely the usual frame operator associated to* $\{\sqrt{\tau(t)}\phi_t\}_{t\in X}$.

Approximation result for continuous frames

Lemma

Let (X, μ) *be a measure space and let* $\mathcal{H}$ *be a separable Hilbert space. Suppose that* $\{\phi_t\}_{t \in X}$ *is a continuous Bessel family in* $\mathcal{H}$*. Then for every* $\varepsilon > 0$*, there exists a continuous Bessel family* $\{\psi_t\}_{t \in X}$*, which takes only countably many values, such that for any measurable function* $\tau : X \to [0, 1]$ *we have*

$$||S_{\sqrt{\tau}\phi, X} - S_{\sqrt{\tau}\psi, X}|| < \varepsilon.$$

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Remark

A continuous Bessel family $\{\psi_t\}_{t\in X}$, which takes only countably many values, is essentially a discrete Bessel sequence.

Proof.

By Proposition 1 we can assume that (X, μ) is σ-finite. Define the sets $X_0 = \{t \in X : ||\phi_t|| < 1\}$ and

$$X_n = \{t \in X : 2^{n-1} \le ||\phi_t|| < 2^n\}, \qquad n \ge 1.$$

Then, for any $\varepsilon > 0$, we can find a partition $\{X_{n,m}\}_{m \in \mathbb{N}}$ of each X_n such that $\mu(X_{n,m}) \le 1$. Then, we can find a countably-valued measurable function $\{\psi_t\}_{t \in X}$ such that

$$||\psi_t - \phi_t|| \le \frac{\varepsilon}{4^n 2^m} \qquad \text{for } t \in X_{n,m}.$$

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Take any $f \in \mathcal{H}$ with $||f|| = 1$. Then, for any $t \in X_{n,m}$,

$$\begin{aligned} ||\langle f, \psi_t\rangle|^2 - |\langle f, \phi_t\rangle|^2| &= |\langle f, \psi_t - \phi_t\rangle||\langle f, \psi_t + \phi_t\rangle| \\ &\leq ||\psi_t - \phi_t||(||\psi_t|| + ||\phi_t||) \\ &\leq \frac{\varepsilon}{4^n 2^m}(2^n + \varepsilon + 2^n) \leq \frac{3\varepsilon}{2^n 2^m}. \end{aligned}$$

Proof continued.

Integrating over $X_{n,m}$ and summing over $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ yields

$$\int_X ||\langle f,\psi_t\rangle|^2 - |\langle f,\phi_t\rangle|^2| d\mu(t) \le \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{3\varepsilon}{2^n 2^m}\mu(X_{n,m}) \le 6\varepsilon.$$

Using the fact that $S_{\sqrt{\tau}\phi,X}$ is self-adjoint, we have

$$\begin{aligned}||S_{\sqrt{\tau}\phi,X} - S_{\sqrt{\tau}\psi,X}|| &= \sup_{||f||=1} |\langle (S_{\sqrt{\tau}\phi,X} - S_{\sqrt{\tau}\psi,X})f, f\rangle| \\ &= \sup_{||f||=1} \left| \int_X \tau(t)(|\langle f,\psi_t\rangle|^2 - |\langle f,\phi_t\rangle|^2) d\mu(t)\right| \\ &\le 6\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

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Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. Let $\mathcal{D}$ be a discrete maximal abelian self-adjoint subalgebra (MASA) of $B(\mathcal{H})$. Say, $\mathcal{H} = \ell^2(\mathbb{N})$ and $\mathcal{D}$ is the algebra of diagonal operators. Does every pure state on $\mathcal{D}$ extend to a **unique** *pure state on $B(\mathcal{H})$?*

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Finally, Marcus-Spielman-Srivastava solved the problem in 2013.

Solution of Kadison-Singer Problem

Theorem (Marcus–Spielman–Srivastava 2015)

If $\epsilon > 0$ and $v_1, \ldots, v_m$ are independent random vectors in $\mathbb{C}^d$ with finite support. Then,

$$\mathbb{E}\left[\sum_{i=1}^{m} v_i v_i^*\right] = \mathbf{I} \qquad \text{and} \qquad \mathbb{E}\left[\|v_i\|^2\right] \le \epsilon \qquad \text{for all } i$$

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Corollary (Weaver's KS_r Conjecture holds)

$\{u_i\}_{i=1}^M \subset \mathbb{C}^d$ *Bessel seq. with bound* 1 *and* $\|u_i\|^2 \le \delta$ *for all* i

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$\Longrightarrow$ $\forall r \in \mathbb{N}$ $\exists$ *partition* $I_1, \ldots, I_r$ *of* $\{1, \ldots, M\}$ *such that each* $\{\phi_i\}_{i \in I_j}$ *is a Bessel sequence with bound* $\left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2$.

Theorem (B.-Casazza-Marcus-Speegle 2016+)

If $0 < \epsilon < 1/2$ *and* $v_1, \dots, v_m$ *are independent random vectors in* $\mathbb{C}^d$ *with support of size* 2. *Then,*

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$$\implies \qquad \mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \le 1 + 2\sqrt{\epsilon}\sqrt{1-\epsilon}\right) > 0.$$

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Corollary (Weaver KS_2 Conjecture holds)

$\{u_i\}_{i=1}^M \subset \mathbb{C}^d$ *Bessel seq. with bound* 1 *and* $\|u_i\|^2 \le \delta < 1/4$

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$\Longrightarrow$ $\exists$ *partition* I_1, I_2 *of* $\{1, \ldots, M\}$ *such that*
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Improves KS_2 constant from $1/(2+\sqrt{2})^2 \approx 0.085$ to $1/4 = 0.25$.

Approximate Lyapunov theorem for discrete frames

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Lemma (Akemann, Weaver (2014))

There exists a universal constant $C > 0$ such that the following holds. Suppose $\{\phi_i\}_{i \in I}$ is a Bessel family in a separable Hilbert space $\mathcal{H}$, which consists of vectors of norms $\|\phi_i\|^2 \le \varepsilon$, where $\varepsilon > 0$. Let

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$$\left\| \sum_{i \in I_0} \phi_i \otimes \phi_i - \tau S \right\| \le C ||S|| \varepsilon^{1/4}.$$

Theorem (Akemann, Weaver (2014))

Suppose $\{\phi_i\}_{i\in I}$ *is a Bessel family with bound* B *in a separable Hilbert space* $\mathcal{H}$*, which consists of vectors of norms* $\|\phi_i\|^2 \le \varepsilon$*, where* $\varepsilon > 0$*. Suppose that* $0 \le \tau_i \le 1$ *for all* $i \in I$*. Consider the modified frame operator*

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$$S_{\sqrt{\tau_\cdot}\phi_\cdot, I} = \sum_{i\in I} \tau_i \phi_i \otimes \phi_i.$$

Then, there exists a subset of indices $I_0 \subset I$ *such that*

$$\left\| \sum_{i\in I_0} \phi_i \otimes \phi_i - S_{\sqrt{\tau_i}\phi_i, i\in I} \right\| \le CB\varepsilon^{1/8}.$$

Theorem (Akemann, Weaver (2014))

Suppose $\{\phi_i\}_{i\in I}$ is a Bessel family with bound B in a separable Hilbert space $\mathcal{H}$, which consists of vectors of norms $\|\phi_i\|^2 \le \varepsilon$, where $\varepsilon > 0$. Suppose that $0 \le \tau_i \le 1$ for all $i \in I$. Consider the modified frame operator

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Theorem (Lyapunov (1940))

The range of a vector-valued measures with values in a finite dimensional space $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a compact and convex subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Theorem (B. (2016))

Let (X,μ) be a non-atomic σ-finite measure space. Suppose that $\{\phi_t\}_{t\in X}$ is a continuous Bessel family in $\mathcal{H}$. For any measurable function $\tau : X \to [0,1]$, consider a modified frame operator

$$S_{\sqrt{\tau}\phi,X} f = \int_X \tau(t)\langle f, \phi_t\rangle \phi_t d\mu(t) \qquad f \in \mathcal{H}.$$

Lyapunov theorem for continuous frames

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Then, for any $\varepsilon > 0$, there exists a measurable set $E \subset X$ such that

$$||S_{\phi,E} - S_{\sqrt{\tau}\phi, X}|| < \varepsilon. \tag{2}$$

Proof.

Let $\{\psi_t\}_{t\in X}$ be continuous Bessel family from approximation lemma. Since $\{\psi_t\}_{t\in X}$ takes only countably many values, there exists a sequence $\{\tilde{\psi}_n\}_{n\in N}$ in $\mathcal{H}$ and a partition $\{X_n\}_{n\in\mathbb{N}}$ of X such that

$$\psi_t = \tilde{\psi}_n \quad \text{for all } t \in X_n,\ n \in \mathbb{N}.$$

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Since $\{\psi_t\}_{t\in X}$ is Bessel, we have $\mu(X_n) < \infty$ for all n such that $\tilde{\psi}_n \neq 0$. Moreover, by subdividing sets X_n if necessary we can assume that

$$||\tilde{\psi}_n||^2 \mu(X_n) \le \varepsilon^2 \qquad \text{for all } n \in \mathbb{N}.$$

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$$||\tilde{\psi}_n||^2 \mu(X_n) \le \varepsilon^2 \qquad \text{for all } n \in \mathbb{N}.$$

This is possible since the measure μ is non-atomic. Then, the continuous frame $\{\psi_t\}_{t\in X}$ is equivalent to a discrete frame

$$\phi_n = \sqrt{\mu(X_n)}\psi_n \qquad n \in \mathbb{N}.$$

Proof (continued).

More precisely, for any measurable function $\tau : X \to [0,1]$, the frame operator $S_{\sqrt{\tau}\psi,X}$ of a continuous Bessel family $\{\sqrt{\tau(t)}\psi_t\}_{t\in X}$ coincides with the frame operator of a discrete Bessel sequence

$$\{\sqrt{\tau_n}\phi_n\}_{n\in\mathbb{N}} \qquad \text{where } \tau_n = \int_{X_n} \tau(t)d\mu(t).$$

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Let $E_n \subset X_n$ be such that $\mu(E_n) = \tau_n\mu(X_n)$. Define $E = \bigcup_{n\in I} E_n$. Then,

$$\begin{aligned} &||S_{\phi,E} - S_{\sqrt{\tau}\phi,X}|| \\ &\le ||S_{\phi,E} - S_{\psi,E}|| + ||S_{\psi,E} - S_{\sqrt{\tau}\psi,X}|| + ||S_{\sqrt{\tau}\psi,X} - S_{\sqrt{\tau}\phi,X}|| \\ &\le \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows (2). □

Main theorem

Previous theorem implies the following variant of Lyapunov theorem in a spirit of a result of Uhl (1969).

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Let (X, μ) be a non-atomic measure space. Suppose that $\{\phi_t\}_{t \in X}$ is a continuous Bessel family in $\mathcal{H}$. Let $\mathcal{S}$ be the set of all partial frame operators

$$\mathcal{S} = \{S_{\phi,E} : E \subset X \text{ is measurable}\},$$

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Remark

Taking closure in the above theorem is necessary.

Proof.

Note that set

$$\mathcal{T} = \{S_{\sqrt{\tau}\phi,X} : \tau \text{ is any measurable } X \to [0,1]\}$$

is a convex subset of $B(\mathcal{H})$. Hence, its operator name closure $\overline{\mathcal{T}}$ is also closed. Since $\mathcal{S} \subset \mathcal{T}$, by previous theorem their closures are the same $\overline{\mathcal{T}} = \overline{\mathcal{S}}$. □

Lyapunov property

Theorem (Uhl (1969))

Suppose a vector-valued measure μ with values in a Banach space $\mathcal{X}$ is such that:

- *$\mathcal{X}$ is either reflexive or has separable dual,*
- *μ has bounded variation, $||\mu|| = \sup \sum_n ||F(E_n)|| < \infty$.*

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Remark

1. *The positive operator valued measure $E \mapsto S_{\phi,E}$ in general does not have bounded variation. Moreover, the closure of $\mathcal{S}$ might not be compact.*
2. *Kadets and Shechtman (1992) introduced the Lyapunov property of a Banach space: "the closure of a range of every non-atomic vector measure is convex". They have shown that c_0 and ℓ^p spaces for $1 \leq p < \infty$, $p \neq 2$, satisfy the Lyapunov property. However, ℓ^2 fails this property.*

Example

Consider a continuous Bessel family $\{\phi_t\}_{t\in[0,1]}$ with values in $L^2([0,1])$ given by $\phi_t = \chi_{[0,t]}$. We claim that there is no measurable set $E \subset [0,1]$ such that $S_{\phi,E} = \frac{1}{2} S_{\phi,[0,1]}$.

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For any $0 \le a < b \le 1$, define $f_n(t) = n\chi_{[a,a+1/n]} - n\chi_{[b-1/n,b]}$. Then, $g_n(t) = \int_0^t f_n(s)ds$ is a piecewise linear function with knots at $(a,0)$, $(a+1/n,1)$, $(b-1/n,1)$, and $(b,0)$, where $n > 2/(b-a)$. Applying the above and taking the limit as $n \to \infty$ yields

$$\frac{b-a}{2} = \frac{1}{2}\lambda([a,b]) = \lambda(E \cap [a,b]).$$

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$$\frac{b-a}{2} = \frac{1}{2}\lambda([a,b]) = \lambda(E \cap [a,b]).$$

Since $[a,b]$ is an arbitrary subinterval of $[0,1]$, this contradicts the Lebesgue Differentiation Theorem.

Problem

Does the main theorem generalize to positive operator valued measures (POVM)?

THE END