$\mathfrak{su}(1,1)$ coherent states for Landau levels: physical and mathematical description

Isiaka Aremua[a,b]

(Joint work with **Mahouton Norbert Hounkonnou**[b] **and Ezinvi Baloïtcha**[b])

[a]Université de Lomé, Faculté des Sciences, Département de Physique,
Laboratoire de Physique des Matériaux et de Mécanique Appliquée,
01 BP 1515 Lomé, Togo
[b]ICMPA-UNESCO Chair
University of Abomey-Calavi
072 B.P. 50 Cotonou, Republic of Benin

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Introduction

Coherent states (CS) were introduced for the first time by Schrödinger[a] in 1926.

[a] E. Schrödinger, Naturwissenschaften **14** 664 (1926)

They were rediscovered by Klauder[a] in a mathematical physics application and by Glauber[b] and Sudarshan[c] in the context of quantum optics at the beginning of the 1960's .

[a] J.R. Klauder , Ann. Phys. **11**, 123-168 (1960)
[b] R. J. Glauber, Phys. Rev. **130** 2529, ibid. **131** 2766 (1963)
[c] E. C. G. Sudarshan, Phys. Rev. Lett. **10** 277 (1963)

• The system of charged quantum particles interacting with a constant magnetic field is undoubtedly one of the most thoroughly investigated systems in quantum mechanics, mainly inspired by condensed matter physics and quantum optics.

Introduction-Motivations

• A family of coherent states (CS) adapted to such a system was first proposed in[a]. In[b], the behavior of the transverse motion of electrons in an external uniform magnetic field was studied, and the constructed CS were investigated to obtain Landau diamagnetism for a free electron gas[c].

• In[d] generalized Klauder-Perelomov and Gazeau-Klauder[e] CS of Landau levels were constructed using two different representations for the Lie algebra $\mathfrak{h}_4$.

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[a] I. A. Malkin and V. I. Man'ko: *Coherent States of a Charged Particle in a Magnetic Field*, *Zh. Eksp. Teor. Fiz.* **55**, 1014 (1968).

[b] A. Feldman and A. H. Kahn: *Landau diamagnetism from the coherent states of an electron in a uniform magnetic field*, *Phys. Rev. B* **1**, 4584 (1970).

[c] L. D. Landau: *Diamagnetismus der Metalle*, *Z. Phys.* **64**, 629 (1930).

[d] H. Fakhri: *Generalized Klauder-Perelomov and Gazeau-Klauder coherent states for Landau levels*, *Phys. Lett. A* **313**, 243-251 (2003).

[e] J. P. Antoine, J. P. Gazeau, P. Monceau, J. R. Klauder and K. A. Penson: *Temporally stable coherent states for infinite well and Pöschl-Teller potentials*, *J. Math. Phys.* **42**, 2349-2387 (2001).

Introduction-Motivations

• In[a], the Landau levels were reorganized into two different hidden symmetries, namely $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. The representation of $\mathfrak{su}(1,1)$ by the Landau levels then led to the construction of the Barut-Girardello CS[b] (BGCS). In[c], Klauder-Perelomov CS of $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ algebras, for Landau levels, minimizing the Heisenberg uncertainty relation, and their statistical properties were discussed.

[a] H. Fakhri: *su(1,1)-Barut-Girardello coherent states for Landau levels*, *J. Phys. A: Math. Gen.* **37**, 5203-5210 (2004).

[b] A. O. Barut and L. Girardello: *New "coherent"states associated with non compact groups*, *Commun. Math. Phys.* **21**, 41 (1971).

[c] A. Dehghani, H. Fakhri and B. Mojaveri: *The minimum-uncertainty coherent states for Landau levels*, *J. Math. Phys.* **53**, 123527 (2012); A. Dehghani and B. Mojaveri: *New physics in Landau levels*, *J. Phys. A: Math. Theor.* **46**, 385303 (2013).

Outline

Preliminaries

Consider the Lie algebra $\mathfrak{su}(1,1)$ [a] corresponding to the $\mathrm{SU}(1,1)$ group, spanned by the three group generators $\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ such that

$$[\mathcal{K}_1, \mathcal{K}_2] = -i\mathcal{K}_3, \quad [\mathcal{K}_2, \mathcal{K}_3] = i\mathcal{K}_1, \quad [\mathcal{K}_3, \mathcal{K}_1] = i\mathcal{K}_2. \tag{1}$$

Conveniently, define the raising and lowering operators $\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2$ as the following second-order differential operators:

$$[\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_3, \qquad [\mathcal{K}_3, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm. \tag{2}$$

[a] R. Gilmore: *Lie Groups, Lie algebras, and Some of their Applications*, Wiley, New York 1974.

Preliminaries

There results the following representation of the Lie algebra $\mathfrak{su}(1,1)$ in $\mathfrak{H}$, spanned by the states $|\upsilon, k\rangle, (\upsilon \geq 0, k > 1)$,

$$\langle \upsilon', k|\upsilon, k\rangle = \delta_{\upsilon\upsilon'}, \qquad \sum_{\upsilon=0}^{\infty} |\upsilon, k\rangle\langle \upsilon, k| = 1, \tag{3}$$

given by:

$$\begin{aligned} \mathcal{K}_{+}|\upsilon, k\rangle &= \sqrt{(\upsilon+1)(\upsilon+2k)}|\upsilon+1, k\rangle \\ \mathcal{K}_{-}|\upsilon, k\rangle &= \sqrt{\upsilon(\upsilon+2k-1)}|\upsilon-1, k\rangle \\ \mathcal{K}^2|\upsilon, k\rangle &= k(k-1)|\upsilon, k\rangle \end{aligned} \tag{4}$$

where $\mathcal{K}^2$ and the real number k satisfy

$$\mathcal{K}^2 = \mathcal{K}_3^2 - \mathcal{K}_1^2 - \mathcal{K}_2^2 = k(k-1)\mathcal{I}. \tag{5}$$

Preliminaries

The Barut-Girardello coherent states (BGCS) $|z,k\rangle_{BG}$ are defined as eigenstates of the lowering operator $\mathcal{K}_-$ as follows[a]

$$\mathcal{K}_-|z,k\rangle_{BG} = z|z,k\rangle_{BG} \tag{6}$$

$$|z,k\rangle_{BG} = \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}}\sum_{\upsilon=0}^{\infty}\frac{|z|^{\upsilon}}{\sqrt{\upsilon!\Gamma(\upsilon+2k)}}|\upsilon,k\rangle \tag{7}$$

where $\Gamma(\dots)$ is the Euler gamma function and $I_{2k-1}(\dots)$-the Bessel function of the first kind.

[a] A. O. Barut and L. Girardello: *New "coherent" states associated with non compact groups*, *Commun. Math. Phys.* **21**, 41 (1971).

Preliminaries

Consider a system of spinless charged particles confined to two-dimensional (x, y)-space, with a magnetic field **B** applied along the z-direction. When a harmonic potential is introduced and the Coulomb interactions are neglected, this system is described by the Fock-Darwin Hamiltonian [1]

$$\mathcal{H} = \frac{1}{2M}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 + \frac{M\omega_0^2}{2}\mathbf{r}^2 \tag{8}$$

where M is the particle mass, e the particle charge, **p** is the kinetic momentum and **A** is the vector potential. We study the problem by considering the transverse motion of the electrons in the (x, y)-space[2].

[1] V. Fock: *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld*, *Z. Phys.* **47**, 446 (1928); C. G. Darwin: *The diamagnetism of the free electron*, *Proc. Camb. Phil. Soc.* **27**, 86 (1930).

[2] A. Feldman and A. H. Kahn: *Landau diamagnetism from the coherent states of an electron in a uniform magnetic field*, *Phys. Rev. B* **1**, 4584 (1970).

Preliminaries

In the symmetric gauge

$$\mathbf{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right), \tag{9}$$

the classical Hamiltonian $\mathcal{H}$ in (8) becomes

$$\tilde{\mathcal{H}}(\mathbf{p},\mathbf{r}) \equiv \tilde{\mathcal{H}} = \frac{1}{2M}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2\right] + \frac{M\omega_0^2}{2}(x^2+y^2). \tag{10}$$

Introduce the coordinate operators $(\hat{R}_1, \hat{R}_2) = (\hat{X}, \hat{Y})$, and momentum operators $(\hat{P}_1, \hat{P}_2) = (\hat{P}_x, \hat{P}_y)$, satisfying the canonical commutation relations

$$[\hat{R}_i, \hat{P}_j] = i\hbar\delta_{ij}. \tag{11}$$

Let

$$\hat{Z} = \hat{X} + i\hat{Y}, \quad \hat{P}_z = \frac{1}{2}(\hat{P}_x - i\hat{P}_y), \quad \hat{P}_{\bar{z}} = \frac{1}{2}(\hat{P}_x + i\hat{P}_y) \tag{12}$$

and consider the set of energy raising operator π_+ and lowering operator π_- defined by[3]

$$\pi_+ = 2\hat{P}_{\bar{z}} + i\frac{M\Omega}{2}\hat{Z}, \quad \pi_- = 2\hat{P}_z - i\frac{M\Omega}{2}\hat{\bar{Z}}, \tag{13}$$

with $\Omega = \sqrt{\omega_c^2 + 4\omega_0^2}$ and the cyclotron frequency $\omega_c = \frac{eB}{Mc}$, such that

$$[\pi_-, \pi_+] = 2M\Omega\hbar. \tag{14}$$

Defining the operators $\hat{\pi}_x$ and $\hat{\pi}_y$ as:

$$\hat{\pi}_x := \hat{P}_x - \frac{M\Omega}{2}\hat{Y},$$

$$\hat{\pi}_y := \hat{P}_y + \frac{M\Omega}{2}\hat{X}, \tag{15}$$

[3] A. Feldman and A. H. Kahn: *Landau diamagnetism from the coherent states of an electron in a uniform magnetic field*, *Phys. Rev. B* **1**, 4584 (1970).

we can formulate the angular momentum raising operator X_+ and lowering operator X_- as follows:

$$X_+ = \left(\hat{X} - \frac{\hat{\pi}_y}{M\Omega}\right) + i\left(\hat{Y} + \frac{\hat{\pi}_x}{M\Omega}\right) = \frac{1}{2}\hat{Z} + \frac{2i}{M\Omega}\hat{P}_{\bar{z}}, \tag{16}$$

$$X_- = \left(\hat{X} - \frac{\hat{\pi}_y}{M\Omega}\right) - i\left(\hat{Y} + \frac{\hat{\pi}_x}{M\Omega}\right) = \frac{1}{2}\hat{\bar{Z}} - \frac{2i}{M\Omega}\hat{P}_z, \tag{17}$$

with their commutation relation given by $[X_+, X_-] = 2l^2$.
The quantity $l := \sqrt{\frac{\hbar}{M\Omega}}$ is taken as the classical radius of the ground-state's Landau orbit. The Hamiltonian (10) takes the form[4]

$$\tilde{H} = \frac{1}{2}\left[\frac{\pi_+\pi_-}{2M}\left(1 + \frac{\omega_c}{\Omega}\right) + \frac{M\Omega^2}{2}\left(1 - \frac{\omega_c}{\Omega}\right)X_-X_+ + \hbar\Omega\right]. \tag{18}$$

[4] A. Feldman and A. H. Kahn: *Landau diamagnetism from the coherent states of an electron in a uniform magnetic field*, *Phys. Rev. B* **1**, 4584 (1970)

Preliminaries

Where the operators π_+ and π_- act on the eigenstates $|n, m\rangle$ as[a]

$$\begin{aligned} \pi_+|n-1, m-1\rangle &= \sqrt{2M\hbar\Omega n}|n, m\rangle, \\ \pi_-|n, m\rangle &= \sqrt{2M\hbar\Omega n}|n-1, m-1\rangle, \end{aligned} \tag{19}$$

and the operators X_+ and X_- act as

$$\begin{aligned} X_+|n, m\rangle &= \sqrt{\frac{2\hbar}{M\Omega}}(n-m)^{1/2}|n, m+1\rangle, \\ X_-|n, m\rangle &= \sqrt{\frac{2\hbar}{M\Omega}}(n-m+1)^{1/2}|n, m-1\rangle. \end{aligned} \tag{20}$$

[a] A. Feldman and A. H. Kahn: *Landau diamagnetism from the coherent states of an electron in a uniform magnetic field*, *Phys. Rev. B* **1**, 4584 (1970)

Preliminaries

The eigenvalues corresponding to (10) are given by[a]

$$\mathcal{E}_{n,m} = \hbar\Omega\left(n + \frac{1}{2}\right) - \frac{\hbar}{2}(\Omega - \omega_c)m. \tag{21}$$

The eigenstates $|n,m\rangle$ are determined by two quantum numbers: $n \in \mathbb{N}$ associated to the energy, and $m \in \mathbb{Z}$ associated to the $z-$ projection of the angular momentum, where the constraint $n \geq m$ holds.
The Hilbert space spanned by the states $|n,m\rangle$ is given by

$$\mathfrak{H} = span\{|n,m\rangle,\ n \geq m,\ n = 0,1,2,\cdots,+\infty; m = n, n-1,\cdots,0,-1,\cdots,-\infty\}. \tag{22}$$

[a] A. Feldman and A. H. Kahn: *Landau diamagnetism from the coherent states of an electron in a uniform magnetic field*, *Phys. Rev. B* **1**, 4584 (1970).

Preliminaries

Introduce the decomposition, for non positive integers m',

$$\mathfrak{H} = \left(\bigoplus_{m'<0} \mathfrak{H}'_{m'}\right) \bigoplus \left(\bigoplus_{m=0}^{+\infty} \mathfrak{H}_m\right) \tag{23}$$

such that the identity operator $I_{\mathfrak{H}}$ on the whole Hilbert space $\mathfrak{H}$ writes as follows:

$$I_{\mathfrak{H}} = \left(\bigoplus_{m'<0} I_{\mathfrak{H}'_{m'}}\right) \bigoplus \left(\bigoplus_{m=0}^{+\infty} I_{\mathfrak{H}_m}\right) \tag{24}$$

with

$$I_{\mathfrak{H}'_{m'}} = \sum_{n=0}^{+\infty} |n, m'\rangle\langle n, m'|, \qquad I_{\mathfrak{H}_m} = \sum_{n=m}^{+\infty} |n, m\rangle\langle n, m|. \tag{25}$$

Preliminaries

Rewrite now the step and orbit-center coordinate operators, denoted by $\hat{\pi}_\pm$ and $\hat{X}_\pm$, with the help of dimensionless variables as follows:

$$\hat{\pi}_+ = l\sqrt{2}\left[\frac{i}{4l^2}\hat{Z} + \frac{1}{\hbar}\hat{P}_{\bar{z}}\right], \quad \hat{\pi}_- = l\sqrt{2}\left[-\frac{i}{4l^2}\hat{\bar{Z}} + \frac{1}{\hbar}\hat{P}_z\right], \tag{26}$$

$$\hat{X}_+ = \frac{\sqrt{2}}{l}\left[\frac{1}{4}\hat{Z} + \frac{il^2}{\hbar}\hat{P}_{\bar{z}}\right], \quad \hat{X}_- = \frac{\sqrt{2}}{l}\left[\frac{1}{4}\hat{\bar{Z}} - \frac{il^2}{\hbar}\hat{P}_z\right]. \tag{27}$$

These operators satisfy the following canonical commutation relations

$$[\hat{\pi}_-, \hat{\pi}_+] = \mathbb{I}, \qquad [\hat{X}_+, \hat{X}_-] = \mathbb{I}. \tag{28}$$

The application of X_+ on the highest weight vector $|n, n\rangle$ yields 0.

The physical system $\mathfrak{su}(1,1)$ Lie algebra generators

Consider the Lie algebra[5] $\mathfrak{su}(1,1)$ corresponding to the $\mathrm{SU}(1,1)$ group, spanned by the three group generators $\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ such that

$$[\mathcal{K}_1, \mathcal{K}_2] = -i\mathcal{K}_3, \quad [\mathcal{K}_2, \mathcal{K}_3] = i\mathcal{K}_1, \quad [\mathcal{K}_3, \mathcal{K}_1] = i\mathcal{K}_2. \tag{29}$$

As matter of convenience, let us define the raising and lowering operators $\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2$ as the following second-order differential operators[a]:

$$\mathcal{K}_+ := \hat{\pi}_+ \hat{X}_-, \quad \mathcal{K}_- := \hat{\pi}_- \hat{X}_+, \tag{30}$$

and the operator

$$\mathcal{K}_3 := \frac{1}{2}(\hat{\pi}_+ \hat{\pi}_- + \hat{X}_+ \hat{X}_-). \tag{31}$$

[a] H. Fakhri: *su(1,1)-Barut-Girardello coherent states for Landau levels*, *J. Phys. A: Math. Gen.* **37**, 5203-5210 (2004).

[5] R. Gilmore: *Lie Groups, Lie algebras, and Some of their Applications*, Wiley, New York 1974.

The physical system $\mathfrak{su}(1,1)$ Lie algebra generators

They satisfy the commutation relations of the Lie algebra $\mathfrak{su}(1,1)$:

$$[\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_3, \qquad [\mathcal{K}_3, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm. \tag{32}$$

The operators $\hat{\pi}_+$ and $\hat{\pi}_-$ act as follows:

$$\hat{\pi}_+|n-1, m-1\rangle = \sqrt{n}|n, m\rangle, \quad \hat{\pi}_-|n, m\rangle = \sqrt{n}|n-1, m-1\rangle, \tag{33}$$

$$\hat{X}_+|n, m\rangle = \sqrt{(n-m)}|n, m+1\rangle, \quad \hat{X}_-|n, m\rangle = \sqrt{(n-m+1)}|n, m-1\rangle. \tag{34}$$

The state $|m, m\rangle$ is annihilated by the lowering operator $\mathcal{K}_-$, i.e.

$$\mathcal{K}_-|m, m\rangle = 0. \tag{35}$$

The physical system $\mathfrak{su}(1,1)$ Lie algebra generators

There results the following realization of the Lie algebra $\mathfrak{su}(1,1)$ in $\mathfrak{H} = span\{|n,m\rangle,\ n \geq m,\ n = 0,1,2,\cdots,+\infty; m = n, n-1,\cdots,0,-1,\cdots,-\infty\}$ given in (22):

$$\mathcal{K}_+|n-1,m\rangle = \sqrt{n(n-m)}|n,m\rangle$$
$$\mathcal{K}_-|n,m\rangle = \sqrt{n(n-m)}|n-1,m\rangle$$
$$\mathcal{K}_3|n,m\rangle = \frac{1}{2}(2n-m+1)|n,m\rangle. \tag{36}$$

We deduce the expression of an arbitrary state $|n,m\rangle$ using the first equation of (36) as follows:

$$|n,m\rangle = \sqrt{\frac{\Gamma(m+1)}{\Gamma(n-m+1)\Gamma(n+1)}}\,\mathcal{K}_+^{n-m}|m,m\rangle. \tag{37}$$

CS for the $\mathfrak{su}(1,1)$ algebra-BGCS

We construct the $\mathfrak{su}(1,1)$ algebra CS for the quantum system on the Hilbert subspace $\mathfrak{H}_m := span\{|n,m\rangle\}_{n\geq m, m\geq 0}$ and on the subspace $\mathfrak{H}'_{m'} := span\{|n,m'\rangle\}_{n\geq m', m'<0}$.

The BGCS in the Hilbert subspace $\mathfrak{H}_m$ can be defined as eigenstates of the lowering generator $\mathcal{K}_-$ of the Lie algebra[a] $\mathfrak{su}(1,1)$, i.e.

$$\mathcal{K}_-|z\rangle_m = z|z\rangle_m \tag{38}$$

where z is an arbitrary complex variable of the form

$$z = \rho e^{i\phi}, 0 \leq \rho < \infty, 0 \leq \phi < 2\pi.$$

[a] A. O. Barut and L. Girardello: *New "coherent" states associated with non compact groups*, *Commun. Math. Phys.* **21**, 41 (1971).

CS for the $\mathfrak{su}(1,1)$ algebra-BGCS

The eigenstates $|z\rangle_m$ can be represented as the superposition of the complete orthonormal basis $|n,m\rangle$ of $\mathfrak{H}_m$ as follows:

$$|z\rangle_m = \sum_{n=m}^{+\infty} \langle n,m|z\rangle_m \, |n,m\rangle. \tag{39}$$

The state $|z\rangle_m$ in the Hilbert subspace $\mathfrak{H}_m$ can be rewritten as:

$$|z\rangle_m = |\langle m,m|z\rangle_m| \sum_{n=m}^{+\infty} z^{n-m} \sqrt{\frac{\Gamma(m+1)}{\Gamma(n-m+1)\Gamma(n+1)}} |n,m\rangle. \tag{40}$$

The normalization factor is given by the term

$$|\langle m, m|z\rangle_m| = \sqrt{\frac{|z|^m}{I_m(2|z|)\,\Gamma(m+1)}}. \tag{41}$$

Finally, the eigenstates $|z\rangle_m$ become

$$|z\rangle_m = \frac{|z|^{m/2}}{\sqrt{I_m(2|z|)}} \sum_{n=m}^{+\infty} \frac{z^{n-m}}{\sqrt{\Gamma(n-m+1)\Gamma(n+1)}} |n, m\rangle \tag{42}$$

where $I_m(2|z|)$ is the modified Bessel function of the first kind given by [a]

$$I_m(2|z|) = \sum_{n=0}^{+\infty} \frac{|z|^{2n+m}}{\Gamma(n+1)\Gamma(n+m+1)}. \tag{43}$$

[a] W. Magnus, F. Oberhettinger and R. P. Soni: *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York 1966.

CS for the $\mathfrak{su}(1,1)$ algebra-BGCS

These states satisfy the following resolution of the identity[6]

$$\int_{\mathbb{C}} |z\rangle_{m\ m}\langle z| d\varrho(z) = \sum_{n=m}^{+\infty} |n,m\rangle\langle n,m| = I_{\mathfrak{H}_m} \tag{44}$$

on $\mathfrak{H}_m$; with $d\varrho(z) = \frac{2}{\pi} I_m(2|z|)K_m(2|z|)d^2z, \quad d^2z = d(Re\, z)d(Im\, z)$.

The BGCS for $m' < 0$, denoted by $|z\rangle_{m'}$, corresponding to the constraint $\mathbf{B}.\mathbf{L} \leq 0$ with $\mathbf{L}$ the angular momentum, are given in the subspace $\mathfrak{H}'_{m'} = span\{|n,m'\rangle\}_{n\geq m', m'<0}$ through the relation

$$|z\rangle_{m'} = \frac{|z|^{-m'/2}}{\sqrt{I_{-m'}(2|z|)}} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{\Gamma(n-m'+1)\Gamma(n+1)}} |n,m'\rangle, \quad m' < 0. \tag{45}$$

[6] A. O. Barut and L. Girardello: *New "coherent" states associated with non compact groups*, *Commun. Math. Phys.* **21**, 41 (1971).

CS for the $\mathfrak{su}(1,1)$ algebra-BGCS

The resolution of the identity on the Hilbert subspace $\mathfrak{H}'_{m'} = span\{|n, m'\rangle\}_{n\geq m', m'<0}$ is given by

$$\int_{\mathbb{C}} |z\rangle_{m'\ m'}\langle z| d\tilde{\varrho}(z) = \sum_{n=0}^{+\infty} |n, m'\rangle\langle n, m'| = I_{\mathfrak{H}'_{m'}}. \tag{46}$$

The resolution of the identity on the entire Hilbert space $\mathfrak{H}$ is deduced from (44) and (46) as

$$I_{\mathfrak{H}} = \sum_{m'<0} \int_{\mathbb{C}} |z\rangle_{m'\ m'}\langle z| d\tilde{\varrho}(z) \bigoplus \sum_{m=0}^{+\infty} \int_{\mathbb{C}} |z\rangle_{m\ m}\langle z| d\varrho(z). \tag{47}$$

Reproducing kernel

The resolution of the identity on the subspace $\mathfrak{H}_m := span\{|n,m\rangle\}_{n\geq m, m\geq 0}$ (or on $\mathfrak{H}'_{m'} = span\{|n,m'\rangle\}_{n\geq m', m'<0}$),

$$\int_{\mathbb{C}} |z\rangle_{m\ m}\langle z| d\varrho(z) = \sum_{n=m}^{+\infty} |n,m\rangle\langle n,m| = I_{\mathfrak{H}_m} \tag{48}$$

suggests to discuss the BGCS relation with the reproducing kernels given by $\mathcal{K}(z,z') := {}_m\langle z'|z\rangle_m$ satisfying:

Proposition

The following properties

- hermiticity $\quad \mathcal{K}(z,z') = \overline{\mathcal{K}(z',z)}$,
- positivity $\quad \mathcal{K}(z,z) > 0$,
- idempotence $\quad \int_{\mathbb{C}} d\varrho(z'')\mathcal{K}(z,z'')\mathcal{K}(z'',z') = \mathcal{K}(z,z')$

are satisfied by the function $\mathcal{K}$ on $\mathfrak{H}_m$.

Reproducing kernel

The reproducing kernel $\mathcal{K}(z, z')$ is given by

$$\mathcal{K}(z, z') =_m \langle z'|z\rangle_m = \left(\frac{|zz'|}{\bar{z}'z}\right)^{m/2} \frac{I_m\left(2\sqrt{\bar{z}'z}\right)}{\sqrt{I_m(2|z'|)I_m(2|z|)}}. \tag{49}$$

From the resolution of the identity, for any $|\Psi\rangle \in \mathfrak{H}_m$, we have

$$|\Psi\rangle = \int_{\mathbb{C}} d\varrho(z)\Psi(z)|z\rangle_m \tag{50}$$

where $\Psi(z) := {}_m\langle z|\Psi\rangle$, such that the following reproducing property

$$\Psi(z) = \int_{\mathbb{C}} d\varrho(z')\mathcal{K}(z, z')\Psi(z') \tag{51}$$

is also verified.

Proof

- Positivity

$$
\begin{aligned}
\mathcal{K}(z,z) &= \left(\frac{|z'z|}{\bar{z}'z}\right)^{m/2} \frac{I_m\left(2\sqrt{\bar{z}'z}\right)}{\sqrt{I_m(2|z'|)I_m(2|z|)}} \\
&= \left(\frac{|z^2|}{|z|^2}\right)^{m/2} \frac{I_m\left(2\sqrt{|z|^2}\right)}{\sqrt{I_m(2|z|)I_m(2|z|)}} \\
\mathcal{K}(z,z) &= \left(\frac{|z^2|}{|z|^2}\right)^{m/2} > 0.
\end{aligned} \tag{52}
$$

- Idempotence

$$
\begin{aligned}
\int_{\mathbb{C}} d\varrho(z'')\mathcal{K}(z,z'')\mathcal{K}(z'',z') &= \int_{\mathbb{C}} d\varrho(z'') \left[\frac{|z''z||z'z''|}{\bar{z}''z\bar{z}'z''}\right]^{m/2} \times \\
&\times \frac{I_m\left(2\sqrt{\bar{z}''z}\right) I_m\left(2\sqrt{\bar{z}'z''}\right)}{\sqrt{I_m(2|z''|)I_m(2|z|)}\sqrt{I_m(2|z'|)I_m(2|z''|)}}
\end{aligned}
$$

$$\begin{aligned}
\int_{\mathbb{C}} d\varrho(z'')\mathcal{K}(z,z'')\mathcal{K}(z'',z') &= \int_{\mathbb{C}} \left[\frac{2}{\pi}K_m(2|z''|)I_m(2|z''|)d^2z''\right] \\
&\times \left(\frac{|z'z|}{\bar{z}'z}\right)^{m/2}\left(\frac{|z''||z''|}{\bar{z}''z''}\right)^{m/2} \\
&\frac{I_m\left(2\sqrt{\bar{z}''z}\right)I_m\left(2\sqrt{\bar{z}'z''}\right)}{I_m(2|z''|)\sqrt{I_m(2|z|)}\sqrt{I_m(2|z'|)}} \\
&= \left(\frac{|z'z|}{\bar{z}'z}\right)^{m/2}\frac{1}{\sqrt{I_m(2|z'|)I_m(2|z|)}}\times \\
&\times \int_{\mathbb{C}} d^2z''\frac{2}{\pi}K_m(2|z''|)I_m\left(2\sqrt{\bar{z}''z}\right)I_m\left(2\sqrt{\bar{z}'z''}\right) \\
&= \left(\frac{|z'z|}{\bar{z}'z}\right)^{m/2}\frac{I_m\left(2\sqrt{\bar{z}'z}\right)}{\sqrt{I_m(2|z'|)I_m(2|z|)}}\times \\
&\times \int_{\mathbb{C}} d^2z''\frac{2}{\pi}K_m(2|z''|)I_m(2|z''|) \\
&= \mathcal{K}(z,z') \qquad (53)
\end{aligned}$$

Observables of the physical system mean values in the BGCS

The operators π_+, π_- and X_+, X_- of the system, provided in (13), (16) and (17), are used to express the coordinate operators $\hat{X}$, $\hat{Y}$ and momentum operators $\hat{P}_x, \hat{P}_y$ as follows:

$$\hat{X} = \frac{1}{2}\left[X_+ + X_-\right] - \frac{i}{2M\Omega}\left[\pi_+ - \pi_-\right], \quad \hat{Y} = -\frac{i}{2}\left[X_+ - X_-\right] - \frac{1}{2M\Omega}\left[\pi_+ + \pi_-\right], \tag{54}$$

$$\hat{P}_x = -\frac{i}{4}M\Omega\left[X_+ - X_-\right] + \frac{1}{4}\left[\pi_+ + \pi_-\right], \quad \hat{P}_y = -\frac{1}{4}M\Omega\left[X_+ + X_-\right] - \frac{i}{4}\left[\pi_+ - \pi_-\right]. \tag{55}$$

The mean values of $\hat{X}$, $\hat{Y}$ and $\hat{P}_x$, $\hat{P}_y$ in the states $|z\rangle_m$ are given in terms of the classical dynamical variables p, q with $z = \frac{q+ip}{\sqrt{2}}$ and the modified Bessel functions of the first kind, where the notation $\xi_{a,b}(z) = I_a(2|z|)/I_b(2|z|)$ has been introduced, by

$$\langle \hat{X} \rangle_{z,m} = \sqrt{\frac{\hbar}{M\Omega}} q \sqrt{\frac{\xi_{m,m-1}(z)}{|z|}}, \tag{56}$$

$$\langle \hat{Y} \rangle_{z,m} = \sqrt{\frac{\hbar}{2M\Omega}} \left[p \sqrt{\frac{2\xi_{m,m-1}(z)}{|z|}} - 2\sqrt{|z|\xi_{m,m+1}(z)} \right], \tag{57}$$

$$\langle \hat{P}_x \rangle_{z,m} = \frac{1}{2}\sqrt{\frac{\hbar M\Omega}{2}} \left[p \sqrt{\frac{2\xi_{m,m-1}(z)}{|z|}} + 2\sqrt{|z|\xi_{m,m+1}(z)} \right], \tag{58}$$

$$\langle \hat{P}_y \rangle_{z,m} = -\frac{1}{2}\sqrt{\frac{\hbar M\Omega}{2}} q \sqrt{\frac{2\xi_{m,m-1}(z)}{|z|}}. \tag{59}$$

Observables of the physical system mean values

The observable dispersions lead for $|z| \gg 1$, to the following relation:

$$(\Delta\hat{X})^2_{z,m}(\Delta\hat{P}_x)^2_{z,m} = \frac{\hbar^2}{4}(m+1)^2 + \left(\frac{4\hbar}{M\Omega} + \frac{2\hbar}{M\Omega}\frac{1}{|z|} - \frac{\hbar^2}{2}\right) \times \left(\sqrt{2}(m+1)p + \frac{2}{|z|}p^2\right) + \frac{\hbar^2}{4}\frac{\sqrt{2}(m+1)}{|z|}p. \quad (60)$$

The relation (60) yields, in the limit $p \to 0$,

$$(\Delta\hat{X})^2_{z,m}(\Delta\hat{P}_x)^2_{z,m} = \frac{\hbar^2}{4}(m+1)^2 \geq \frac{\hbar^2}{4}, \qquad m \geq 0, \quad (61)$$

satisfying $(\Delta\hat{X})_{z,m}(\Delta\hat{P}_x)_{z,m} = \frac{\hbar}{2}$ at $m = 0$, which means that the uncertainty relation is saturated.

Probability density and time evolution in the BGCS

We analyse how the states $|z\rangle_m$ do evolve in time under the action of the time evolution operator provided by the physical Hamiltonian describing the quantum system.

From the quantity

$$ {}_m\langle z'|z\rangle_m = \left(\frac{|z'z|}{\bar{z}'z}\right)^{m/2} \frac{1}{\sqrt{I_m(2|z'|)I_m(2|z|)}} \sum_{\upsilon=0}^{+\infty} \frac{(\sqrt{\bar{z}'z})^{2\upsilon+m}}{\Gamma(\upsilon+1)\Gamma(\upsilon+m+1)}, \quad (62) $$

given a normalized state $|z_0\rangle_m$, the phase space distribution is defined by the probability density as follows:

$$ z \mapsto \varrho_{z_0}(z) := |{}_m\langle z|z_0\rangle_m|^2 = \frac{I_m(2\sqrt{z_0\bar{z}})I_m(2\sqrt{\bar{z}_0 z})}{I_m(2|z|)I_m(2|z_0|)}. \quad (63) $$

Probability density and time evolution in the BGCS

Its time evolution behavior is then provided by

$$z \mapsto \varrho_{z_0}(z,t) := |{}_m\langle z|e^{-\frac{i}{\hbar}\tilde{H}t}|z_0\rangle_m|^2. \tag{64}$$

By acting the evolution operator $U(t) = e^{-\frac{i}{\hbar}\tilde{H}t}$ on the state $|z_0\rangle_m$, we get

$$\begin{aligned} |z_0;t\rangle_m &= e^{-\frac{i}{\hbar}\tilde{H}t}|z_0\rangle_m \\ &= e^{-\frac{i}{2}[m(\Omega+\omega_c)+\Omega]t}\left[\frac{|z_0|^{m/2}}{\sqrt{I_m(2|z_0|)}}\sum_{\nu=0}^{+\infty}\frac{(z_0(t))^\nu}{\sqrt{\Gamma(\nu+1)\Gamma(\nu+m+1)}}|\nu,m\rangle\right] \\ &= e^{-\frac{i}{2}[m(\Omega+\omega_c)+\Omega]t}|z_0(t)\rangle_m \end{aligned} \tag{65}$$

where $z_0(t) := e^{-i\Omega t}z_0$.

Probability density and time evolution in the BGCS

Therefore

$$\varrho_{z_0}(z,t) := |{}_m\langle z|e^{-\frac{i}{\hbar}\tilde{H}t}|z_0\rangle_m|^2 = \frac{I_m(2\sqrt{z_0(t)\bar{z}})I_m(2\sqrt{\bar{z}_0(t)z})}{I_m(2|z|)I_m(2|z_0(t)|)}. \quad (66)$$

It comes that the time dependence of a given BGCS $|z\rangle_m$ is furnished by

$$|z;t\rangle_m = e^{-\frac{i}{\hbar}\tilde{H}t}|z\rangle_m = e^{-\frac{i}{2}[m(\Omega+\omega_c)+\Omega]t}|z(t)\rangle_m, \quad z(t) := e^{-i\Omega t}z. \quad (67)$$

The relation (67) shows that the time evolution of the CS $|z\rangle_m$ reduces to a rotation in the complex plane given by $z \mapsto z(t) = e^{-i\Omega t}z$ up to a phase, namely, $e^{-\frac{i}{2}[m(\Omega+\omega_c)+\Omega]t}$. Therefore, the semi-classical feature of the CS is given by (63), while the temporal stability property is highlighted by the relation (67). The latter asserts that the temporal evolution of any CS always remains a CS, and fixes the phase behavior of the CS $|z\rangle_m$ with the factor $e^{-i\Omega t}$.

Quantization of elementary classical observables

We establish in this section the correspondence (quantization[7]) between classical and quantum quantities.

For an operator $f(z,\bar{z})$, the Berezin-Klauder-Toeplitz quantization is obtained by corresponding to the operator $f(z,\bar{z})$ the operator valued integral given by

$$A_f = \int_{\mathbb{C}} |z\rangle_m f(z,\bar{z})\, {}_m\langle z| d\varrho(z). \tag{68}$$

Then, for the elementary classical variables z and $\bar{z}$, is realized via the maps $z \longmapsto A_z$ and $\bar{z} \longmapsto A_{\bar{z}}$ defined on the Hilbert subspaces $\mathfrak{H}_m$ and $\mathfrak{H}'_{m'}$ by

$$A_{z_{|\mathfrak{H}_m}} := \int_{\mathbb{C}} z\, |z\rangle_m\, {}_m\langle z| d\varrho(z), \qquad A_{z_{|\mathfrak{H}'_{m'}}} := \int_{\mathbb{C}} z\, |z\rangle_{m'}\, {}_{m'}\langle z| d\tilde{\varrho}(z) \tag{69}$$

[7] J. P. Gazeau: *Coherent States in Quantum Physics*, Wiley-VCH, Berlin 2009; I. Aremua, J. P. Gazeau and M. N. Hounkonnou: *Action-angle coherent states for quantum systems with cylindric phase space*, J. Phys. A: Math. Theor. **45**, 335302 (2012).

providing on the whole complex plane,

$$A_z := \sum_{m'<0} \int_{\mathbb{C}} z\,|z\rangle_{m'\ m'}\langle z| d\tilde{\varrho}(z) \bigoplus \sum_{m=0}^{+\infty} \int_{\mathbb{C}} z\,|z\rangle_{m\ m}\langle z| d\varrho(z),$$

$$A_{\bar{z}} := \sum_{m'<0} \int_{\mathbb{C}} \bar{z}\,|z\rangle_{m'\ m'}\langle z| d\tilde{\varrho}(z) \bigoplus \sum_{m=0}^{+\infty} \int_{\mathbb{C}} \bar{z}\,|z\rangle_{m\ m}\langle z| d\varrho(z). \qquad (70)$$

This gives, using the equations (36) the following relations:

$$A_z = \sum_{m'<0} \sum_{n=0}^{+\infty} \sqrt{(n-m'+1)(n+1)}|n,m'\rangle\langle n+1,m'|$$

$$\bigoplus \sum_{m=0}^{+\infty} \sum_{n=m}^{+\infty} \sqrt{(n-m+1)(n+1)}|n,m\rangle\langle n+1,m| = \mathcal{K}_-, \qquad (71)$$

Quantization of elementary classical observables

$$A_{\bar{z}} = \sum_{m'<0} \sum_{n=0}^{+\infty} \sqrt{n(n-m')}|n,m'\rangle\langle n-1,m'|$$

$$\bigoplus \sum_{m=0}^{+\infty} \sum_{n=m}^{+\infty} \sqrt{n(n-m)}|n,m\rangle\langle n-1,m| = \mathcal{K}_+. \quad (72)$$

Their commutator $[A_z, A_{\bar{z}}]$ is reduced to

$$[A_z, A_{\bar{z}}] = 2\mathcal{K}_3 \quad (73)$$

reminding the $\mathfrak{su}(1,1)$ commutation rules (32).

Quantization of elementary classical observables

Other interesting results emerging from this context are the following mean values:

$$ {}_m\langle z|A_{z_{|\mathfrak{H}_m}}|z\rangle_m = z = {}_{m'}\langle z|A_{z_{|\mathfrak{H}'_{m'}}}|z\rangle_{m'}, \; {}_m\langle z|A_{\bar{z}_{|\mathfrak{H}_m}}|z\rangle_m = \bar{z} = {}_{m'}\langle z|A_{\bar{z}_{|\mathfrak{H}'_{m'}}}|z\rangle_{m'}, \tag{74} $$

$$ {}_m\langle z|A^2_{z_{|\mathfrak{H}_m}}|z\rangle_m = z^2 = {}_{m'}\langle z|A^2_{z_{|\mathfrak{H}'_{m'}}}|z\rangle_{m'}, \; {}_m\langle z|A^2_{\bar{z}_{|\mathfrak{H}_m}}|z\rangle_m = \bar{z}^2 = {}_{m'}\langle z|A^2_{\bar{z}_{|\mathfrak{H}'_{m'}}}|z\rangle_{m'}, \tag{75} $$

Quantization of elementary classical observables

$$ {}_m\langle z|A_{\bar{z}_{|\mathfrak{H}_m}} A_{z_{|\mathfrak{H}_m}}|z\rangle_m = |z|^2 = {}_{m'}\langle z|A_{\bar{z}_{|\mathfrak{H}'_{m'}}} A_{z_{|\mathfrak{H}'_{m'}}}|z\rangle_{m'} \quad (76) $$

$$ {}_m\langle z|A_{z_{|\mathfrak{H}_m}} A_{\bar{z}_{|\mathfrak{H}_m}}|z\rangle_m = |z|^2 + 2\langle\mathcal{K}_3\rangle_{z,m}, \;\; {}_{m'}\langle z|A_{z_{|\mathfrak{H}'_{m'}}} A_{\bar{z}_{|\mathfrak{H}'_{m'}}}|z\rangle_{m'} = |z|^2 + 2\langle\mathcal{K}_3\rangle_{z,m'} \quad (77) $$

where $\langle\mathcal{K}_3\rangle_{z,m}$ and $\langle\mathcal{K}_3\rangle_{z,m'}$ are given by

$$ \langle\mathcal{K}_3\rangle_{z,m} = |z|\frac{I_{m'+1}(2|z|)}{I_{m'}(2|z|)} + \frac{m+1}{2}, \quad (78) $$

$$ \langle\mathcal{K}_3\rangle_{z,m'} = |z|\frac{I_{-m'+1}(2|z|)}{I_{-m'}(2|z|)} + \frac{-m'+1}{2}. \quad (79) $$

Mandel parameter

Using the following means values of the number operator and of its square:

$$\begin{aligned} \langle N\rangle_{z,m} &= \left\langle \mathcal{K}_3 - \frac{m+1}{2}\right\rangle_{z,m} = |z|\frac{I_{m+1}(2|z|)}{I_m(2|z|)}, \\ \langle N^2\rangle_{z,m} &= \left\langle \mathcal{K}_3^2 - (m+1)\mathcal{K}_3 + \left(\frac{m+1}{2}\right)^2\right\rangle_{z,m} \\ &= |z|^2\frac{I_{m+2}(2|z|)}{I_m(2|z|)} + |z|\frac{I_{m+1}(2|z|)}{I_m(2|z|)} \end{aligned} \tag{80}$$

from the expression of the Mandel parameter given by

$$\mathcal{Q} = \frac{(\Delta N)^2}{\langle N\rangle} - 1 \tag{81}$$

Mandel parameter

such that for $\mathcal{Q} \leq 0$, the emitted light is referred to as sub-Poissonian (corresponding to nonclassical states); $\mathcal{Q} = 0$ corresponds to the Poisson distribution (case of standard CS), whereas for $\mathcal{Q} > 0$ the light is called super-Poissonian (corresponding to classical states) [8], we obtain:

- for $|z| \ll 1$

$$\mathcal{Q} \simeq -\frac{|z|^2}{(m+1)(m+2)} < 0 \tag{82}$$

indicating that the BGCS $|z\rangle_m$ have sub-Poissonian statistics as discussed;

- for $|z| \gg 1$

$$\mathcal{Q} \simeq 0. \tag{83}$$

implying that the BGCS $|z\rangle_m$ have Poissonian statistics for $|z| \gg 1$. Therefore, for $|z| \gg 1$, the sates $|z\rangle_m$ coincide with the standard CS.

[8] L. Mandel and E. Wolf: *Optical coherence and quantum optics*, Cambridge University Press, Cambridge 1995.

The corresponding normalized density operator for a fixed number m, for a quantum gas of the system in thermodynamic equilibrium with a reservoir at temperature T, satisfying the quantum canonical distribution,examining the physical and chemical properties [a], is given by

$$\rho_m = \frac{1}{Z}\sum_{\nu=0}^{+\infty} e^{-\beta E_{\nu,m}}|\nu, m\rangle\langle\nu, m|, \quad E_{\nu,m} = \hbar\Omega\nu + \frac{\hbar}{2}\left[\Omega + (\Omega+\omega_c)m\right], \tag{84}$$

where the partition function Z is taken as the normalization constant.

[a] J. P. Gazeau: *Coherent States in Quantum Physics*, Wiley-VCH, Berlin 2009; D. Popov: *Barut-Girardello coherent states of the pseudo-harmonic oscillator, J. Phys. A: Math. Gen.* **34**, 1-14 (2001).

Further, this representation will allow to calculate the mean value or the *pseudo*-thermal average of a given operator as follows:

$$\langle\mathcal{O}\rangle_m = Tr(\rho_m\mathcal{O}) = \int_{\mathbb{C}} d\varrho(z)P_m(z)\,{}_m\langle z|\mathcal{O}|z\rangle_m. \tag{85}$$

The partition function Z, taken as the normalization constant, is given by

$$Z = e^{-\frac{\beta\hbar}{2}m(\Omega+\omega_c)}\left\{2\sinh\left\{\frac{\beta\hbar}{2}\Omega\right\}\right\}^{-1}, \tag{86}$$

such that the Q-Husimi function is

$$_m\langle z|\rho_m|z\rangle_m = \frac{1}{Z}e^{-\frac{\beta\hbar}{2}\Omega}e^{-\frac{\beta\hbar}{2}m\omega_c}\,\frac{I_m(2|z|e^{-\frac{\beta\hbar}{2}\Omega})}{I_m(2|z|)} \tag{87}$$

with the normalization condition:

$$Tr\rho_m = \int_{\mathbb{C}} d\varrho(z)\,_m\langle z|\rho|z\rangle_m = 1 \tag{88}$$

satisfied. The diagonal representation of the normalized density operator takes the form

$$\rho_m = [e^{\beta\hbar\Omega} - 1]e^{\frac{\beta\hbar}{2}\Omega m}\int_{\mathbb{C}} d\varrho(z)\frac{K_m(2|z|e^{\frac{\beta\hbar}{2}\Omega})}{K_m(2|z|)}|z\rangle_m\,_m\langle z|. \tag{89}$$

Thermal properties of the BGCS

Then, the *pseudo*-thermal expectation values of the number operator N and of its square are obained as

$$\langle N\rangle_m = Tr(\rho_m N) = \int_{\mathbb{C}} d\varrho(z) P_m(z)\, {}_m\langle z|N|z\rangle_m = \frac{1}{e^{\beta\hbar\Omega}-1}, \tag{90}$$

$$\langle N^2\rangle_m = Tr(\rho_m N^2) = \int_{\mathbb{C}} d\varrho(z) P_m(z)\, {}_m\langle z|N^2|z\rangle_m = \frac{1}{e^{\beta\hbar\Omega}-1} + 2\frac{1}{(e^{\beta\hbar\Omega}-1)^2}, \tag{91}$$

as respectively.

One remarks that both thermal expectation values $\langle N\rangle_m$ and $\langle N^2\rangle_m$ are independent of the Bargmann index given here by m. One can therefore define the thermal intensity correlation function, which is also independent of the index m, by

$$\langle g\rangle_m \equiv \frac{\langle N^2\rangle - \langle N\rangle}{\langle N\rangle^2} = \langle g^2\rangle = 2. \tag{92}$$

The Wehrl entropy is given in the BGCS as follows:

$$W = -\int_{\mathbb{C}} d\varrho(z)\, {}_m\langle z|\rho_m|z\rangle_m \ln[{}_m\langle z|\rho_m|z\rangle_m] \tag{93}$$

Setting ${}_m\langle z|\tilde{\rho}|z\rangle_m = \left(\frac{2\pi l^2}{A}\right)\, {}_m\langle z|\rho|z\rangle_m$,

$$\tilde{W} \simeq -\ln\left[1 - e^{-\beta\hbar\Omega}\right] - \ln\left(\frac{2\pi l^2}{A}\right) \equiv -1 + W_{calc}. \tag{94}$$

where $W_{calc} = 1 - \ln\left[1 - e^{-\beta\hbar\Omega}\right] - \ln\left(\frac{2\pi l^2}{A}\right)$ is the Wehrl entropy calculated for Landau's diamagnetism for a spinless electron in a uniform magnetic field. The thermal harmonic oscillator[9] frequency is denoted by Ω, and $l = \sqrt{\frac{\hbar}{M\Omega}}$. Hence, the studied model is an approximation of the problem of a thermal harmonic oscillator with frequency Ω.

[9] F. Pennini, A. Plastino and S. Curilef: *Fisher information, Wehrl entropy, and Landau Diamagnetism*, *Phys. Rev. B* **71**, 024420 (2005).

Concluding remarks

- We investigated the Fock-Darwin Hamiltonian describing a gas of spinless charged particles, subject to a perpendicular magnetic field **B**, confined in a harmonic potential. We used a set of step and orbit-center coordinate operators. Then we showed that the studied system possesses $\mathfrak{su}(1,1)$ Lie algebra. As a consequence, CS were constructed as the eigenstates of the $\mathrm{SU}(1,1)$ group generator $\mathcal{K}_-$.
- The mean values of $\mathrm{SU}(1,1)$ group generators and of the physical system observables, the probability density and the time dependence of the BGCS were discussed. Using these CS, the Berezin - Klauder - Toeplitz quantization were performed.
- Statistical properties of a gas in thermodynamic equilibrium with a reservoir at temperature T, satisfying the quantum canonical distribution, were investigated and discussed. The use of the density matrix expression in the BGCS representation led to the calculation of the Wehrl entropy showing that the considered system can be identified to a model of a thermal harmonic oscillator.

Thank you for your attention