Square Integrable Reps., an Invaluable Tool

From Coherent States to Quantum Mechanics on Phase Space

Paolo Aniello

Dipartimento di Fisica "E. Pancini", Università di Napoli "Federico II"
&
INFN – Sezione di Napoli (QUANTUM)

CIRM – Marseille, November 2016

Outline of the talk:

- (Generalized) coherent states and square integrable reps.
- Semidirect products
- Square integrable reps. and quantum mechanics on phase space
- Detour: *classical* states and functions of positive type (PTFs)
- *Quantum* states and functions of *quantum* positive type (QPTFs)
- Playing with functions of positive type: *classical-quantum semigroups*
- Introducing quantization into the game: from classical-quantum semigroups to *twirling semigroups* (open quantum systems)

(Generalized) coherent states and sq. integrable reps.

It is well known that the standard **coherent states**

$$|z\rangle = \mathsf{D}(z)\,|0\rangle\,, \quad z = \Big(q/\sqrt{2}, p/\sqrt{2}\Big), \tag{1}$$

are generated by a projective representation (**Weyl system**)

$$G = \mathbb{R}^n \times \mathbb{R}^n \ni (q,p) \mapsto U(q,p) := \exp(\mathrm{i}(p\cdot\widehat{q} - q\cdot\widehat{p})) = \mathsf{D}\Big(q/\sqrt{2}, p/\sqrt{2}\Big), \tag{2}$$

$$U(q+\tilde{q}, p+\tilde{p}) = \mathrm{e}^{\frac{\mathrm{i}}{2}(q\cdot\tilde{p}-p\cdot\tilde{q})}\, U(q,p)\, U(\tilde{q},\tilde{p}). \tag{3}$$

U is related to a unitary representation of the *central extension* (Heisenberg-Weyl group) $\mathbb{H}_n$, i.e., of the Lie group $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, with composition law

$$(\tau,q,p)(\tilde{\tau},\tilde{q},\tilde{p}) = (\tau+\tilde{\tau}+(q\cdot\tilde{p}-p\cdot\tilde{q})/2, q+\tilde{q}, p+\tilde{p}), \quad \tau,\tilde{\tau}\in\mathbb{R},\ q,\tilde{q}\in\mathbb{R}^n,\ p,\tilde{p}\in\mathbb{R}^n;$$

namely,

$$U(q,p) = \mathsf{S}(0,q,p), \tag{4}$$

where the *Schrödinger representation* S of $\mathbb{H}_n$ is defined by

$$\Big(\mathsf{S}(\tau,q,p)f\Big)(x) := \mathrm{e}^{-\mathrm{i}(\tau+q\cdot p/2)}\mathrm{e}^{\mathrm{i}p\cdot x} f(x-q), \quad f\in\mathsf{L}^2(\mathbb{R}^n). \tag{5}$$

One of the salient properties of coherent states, i.e.,

$$\frac{1}{\pi^n}\int \mathsf{d}^{2n}z\ |z\rangle\langle z| = \frac{1}{(2\pi)^n}\int \mathsf{d}^n q\,\mathsf{d}^n p\, U(q,p)\,|0\rangle\langle 0|\,U(q,p)^* = I, \tag{6}$$

can be regarded as a consequence of the fact that the projective representation U is *square integrable*; equivalently, that the unitary representation S is *square integrable modulo* the center of $\mathbb{H}_n$.

Let U be an irreducible (projective) representation of a *locally compact group* G in a separable complex Hilbert space $\mathcal{H}$. For every pair $\psi, \phi \in \mathcal{H}$, let us consider the (bounded, continuous) 'coefficient function'

$$c_{\psi\phi}\colon G \ni g \mapsto \langle U(g)\,\psi, \phi\rangle \in \mathbb{C} \tag{7}$$

and the set of 'admissible vectors for U'

$$\mathcal{A}(U) := \left\{\psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H}:\ \phi \neq 0,\, c_{\psi\phi} \in \mathsf{L}^2(G, \nu_G; \mathbb{C})\right\}. \tag{8}$$

Then, the representation U is said to be **square integrable** if

$$\mathcal{A}(U) \neq \{0\}. \tag{9}$$

Clearly, every irreducible unitary representation of a *compact* group is square integrable.

Square integrable representations are ruled by the following result (Schur; Weyl; Godement 1947; Duflo-Moore 1976; Grossmann-Morlet-Paul 1985):

Theorem 1 *Let* $U\colon G \to \mathcal{U}(\mathcal{H})$ *be a square integrable representation. Then, the set* $\mathcal{A}(U)$ *is a dense linear span in* $\mathcal{H}$*, stable under the action of* U*, and, for any pair of vectors* $\phi \in \mathcal{H}$ *and* $\psi \in \mathcal{A}(U)$*, the coefficient* $c_{\psi\phi}\colon G \to \mathbb{C}$ *is square integrable w.r.t. the left Haar measure* ν_G*.*
Moreover, there exists a unique positive selfadjoint, injective linear operator D_U *in* $\mathcal{H}$ *— the* 'the Duflo-Moore operator' *associated with* U *— such that*

$$\mathcal{A}(U) = \mathrm{Dom}\Big(D_U\Big) \tag{10}$$

and the following 'orthogonality relations' hold:

$$\int_G \overline{c_{\psi_1\phi_1}(g)}\, c_{\psi_2\phi_2}(g)\; \mathrm{d}\nu_G(g) = \langle \phi_1, \phi_2\rangle \, \langle D_U\, \psi_2, D_U\, \psi_1\rangle, \tag{11}$$

for all $\phi_1, \phi_2 \in \mathcal{H}$ *and all* $\psi_1, \psi_2 \in \mathcal{A}(U)$*.*
The operator D_U *is bounded if and only if* G *is unimodular —* $\triangle_G \equiv 1$ *— and, in such case, it is a multiple of the identity:* $D_U = d_U\, I$*,* $d_U > 0$*.*

Hence:

$$0 \neq \psi \in \mathcal{A}(U) \quad \Rightarrow \quad \|D_U\, \psi\|^{-2} \int_G \mathrm{d}\nu_G(g)\, |U(g)\psi\rangle\langle U(g)\psi| = I. \tag{12}$$

Semidirect products

Assume that a locally compact group G is the **semidirect product** of an *abelian*, closed normal subgroup $\mathbb{A}$ (normal factor) by a closed subgroup H (homogeneous factor):

$$G = \mathbb{A} \rtimes H. \tag{13}$$

The inner action of G determines an **action** of H on $\mathbb{A}$:

$$(\,\cdot\,)[\,\cdot\,] : H \times \mathbb{A} \ni (h, a) \mapsto h[a] = hah^{-1} \in \mathbb{A}. \tag{14}$$

The group G may also be thought of as the *cartesian product* of $\mathbb{A} \times H$, endowed with the *composition law* induced by the action of H on $\mathbb{A}$:

$$(a, h)(a', h') = (a + h[a'], hh'), \quad a, a' \in \mathbb{A},\ h, h' \in H. \tag{15}$$

Let $\widehat{\mathbb{A}}$ be the **Pontryagin dual** of $\mathbb{A}$ (the group of unitary characters) and

$$\langle\!\langle \cdot , \cdot \rangle\!\rangle : \ \mathbb{A} \times \widehat{\mathbb{A}} \ni (a, \hat{\mathtt{x}}) \mapsto \langle\!\langle a, \hat{\mathtt{x}} \rangle\!\rangle = \hat{\mathtt{x}}(a) \in \mathbb{C} \tag{16}$$

the *pairing* between $\mathbb{A}$ and $\widehat{\mathbb{A}}$. The **dual action** of H on $\widehat{A}$ is defined by:

$$\langle\!\langle a, h[\hat{\mathtt{x}}] \rangle\!\rangle := \langle\!\langle h^{-1}[a], \hat{\mathtt{x}} \rangle\!\rangle, \quad a \in \mathbb{A},\ h \in H,\ \hat{\mathtt{x}} \in \widehat{\mathbb{A}}. \tag{17}$$

A standard way for producing irreducible representations of G is Mackey's 'little group method' or **Mackey machine**. Choose an *orbit* $\mathscr{O}$ of the dual action of H on $\widehat{\mathbb{A}}$ through a certain point $\widehat{\mathtt{x}}_0$,

$$\mathscr{O} = H[\widehat{\mathtt{x}}_0], \quad \widehat{\mathtt{x}}_0 \in \widehat{\mathbb{A}}, \tag{18}$$

and an *irreducible representation* $\mathtt{J}\colon H_0 \to \mathcal{U}(\mathcal{J})$ of the *stability subgroup* H_0 of H at $\widehat{\mathtt{x}}_0$; namely:

$$H_0 = \{h \in H \mid h[\widehat{\mathtt{x}}_0] = \widehat{\mathtt{x}}_0\}. \tag{19}$$

The representation of G, *induced* by the representation $\widehat{\mathtt{x}}_0\mathtt{J}\colon G_0 \to \mathcal{U}(\mathcal{J})$ of $G_0 := \mathbb{A} \rtimes H_0$ defined by

$$\Big((\widehat{\mathtt{x}}_0\mathtt{J})\,(a,s)\Big)v := \langle\!\langle a, \widehat{\mathtt{x}}_0 \rangle\!\rangle\, \mathtt{J}(s)\, v, \quad a \in A,\ s \in H_0,\ v \in \mathcal{J}, \tag{20}$$

is *irreducible*. The *unitary equivalence classes of representations* of G that can be obtained via the Mackey machine are in *one-to-one correspondence with the pairs*

$$(\mathscr{O}, \mathtt{J}), \quad \mathscr{O} \subset \widehat{\mathbb{A}}, \tag{21}$$

where $\mathscr{O}$ is a H-orbit and J spans *a maximal set of mutually inequivalent irreducible representations of the stability subgroup* of H at a point arbitrarily fixed in $\mathscr{O}$.
If G is a **regular** semidirect product — i.e., if each orbit of H in $\widehat{\mathbb{A}}$ is *locally closed* — then *every* irreducible representation of G can be produced via the Mackey machine.

The *square-integrability* of these induced representations of *semidirect products* with an *abelian normal factor* is characterized by the following result (see P. A., G. Cassinelli, E. De Vito, A. Levrero, "Square-integrability of induced representations of semidirect products", *Rev. Math. Phys.* **10** (1998) 301):

Theorem 2 *The induced representation* $\mathrm{Ind}_{G_0}^{G}(\hat{\mathtt{x}}_0\mathtt{J})$ *is square integrable if and only if the following conditions hold:*

- *the* H*-orbit* $\mathscr{O} = H[\hat{\mathtt{x}}_0] \subset \widehat{\mathbb{A}}$ *is thick, namely, the Haar measure of* $\mathscr{O}$ *is not zero:* $\nu_{\widehat{\mathbb{A}}}(\mathscr{O}) \neq 0$*;*
- *the representation* $\mathtt{J}\colon H_0 \to \mathcal{U}(\mathcal{J})$ *of the stability subgroup* H_0 *at* $\hat{\mathtt{x}}_0$ *is square integrable.*

In the case where G *is a Lie group, if* $\widehat{\mathbb{A}}$ *is a Lie group on which* H *acts smoothly and the orbit* $\mathscr{O}$ *is locally closed, then*

$$\nu_{\widehat{\mathbb{A}}}(\mathscr{O}) \neq 0 \iff \textit{the orbit } \mathscr{O} \textit{ is open in } \widehat{\mathbb{A}} \iff \dim(H) - \dim(H_0) = \dim(\widehat{\mathbb{A}}).$$

Semidirect products that admit sq. int. reps. include the *affine group* $\mathbb{R}\rtimes\mathbb{R}_*^+$ or $\mathbb{R}\rtimes\mathbb{R}_*$ (wavelet transform), the *similitude group* $\mathbb{R}^n\rtimes(\mathrm{SO}(n)\times\mathbb{R}_*^+)$, the *shearlet group* $\mathbb{R}^{n+1}\rtimes(\mathbb{R}^n\rtimes\mathbb{R}_*^+)$ or $\mathbb{R}^{n+1}\rtimes(\mathbb{R}^n\rtimes\mathbb{R}_*)$ (shearlet transform) and the *reduced Heisenberg group* $\overline{\mathbb{H}}_n = \mathbb{H}_n/2\pi\mathbb{Z}$, whereas, e.g., the *euclidean* $\mathbb{R}^n\rtimes\mathrm{SO}(n)$ and the *Poincaré* $\mathbb{R}^4\rtimes\mathrm{SL}(2;\mathbb{C})$ groups do not admit such reps.

Sq. int. reps. and phase-space quantum mechanics

Denoting by $\mathcal{B}_2(\mathcal{H})$ the Hilbert space of **Hilbert-Schmidt operators** in $\mathcal{H}$, a *square integrable* (in general, projective) representation $U\colon G \to \mathcal{U}(\mathcal{H})$ allows one to define a **dequantization map**

$$\mathscr{D}\colon \mathcal{B}_2(\mathcal{H}) \to \mathsf{L}^2(G) \equiv \mathsf{L}^2(G, \nu_G; \mathbb{C}), \tag{22}$$

which is an *isometry*. If G is *unimodular* and $\widehat{\rho}$ is of *trace class*, $\mathscr{D}\widehat{\rho}$ is of the form

$$(\mathscr{D}\widehat{\rho})(g) = d_U^{-1}\, \mathrm{tr}(U(g)^* \widehat{\rho}), \quad d_U > 0. \tag{23}$$

The **quantization map** associated with U is the adjoint of the dequantization map; i.e., it is *the partial isometry* $\mathscr{Q}$ defined by

$$\mathscr{Q} := \mathscr{D}^*\colon \mathsf{L}^2(G) \to \mathcal{B}_2(\mathcal{H}). \tag{24}$$

Clearly, $\mathrm{Ker}(\mathscr{Q}) = \mathrm{Ran}(\mathscr{D})^{\perp}$. The **star product** is defined by

$$\mathsf{L}^2(G) \times \mathsf{L}^2(G) \ni (f_1, f_2) \mapsto f_1 \star f_2 := \mathscr{D}((\mathscr{Q}f_1)(\mathscr{Q}f_2)) \in \mathsf{L}^2(G). \tag{25}$$

For functions in $\mathrm{Ran}(\mathscr{D})$ this is the *'dequantized product of operators'*. One can provide explicit formulae for the star product (P. A., "Star products: a group-theoretical point of view", *J. Phys. A: Math. Theor.* **42** (2009) 475210).

In the case where G is **unimodular**, we have a simple result:

Theorem 3 *Let G be unimodular and $U\colon G \to \mathcal{U}(\mathcal{H})$ a square integrable projective representation, with multiplier* m*; i.e., $U(gh) = \mathsf{m}(g,h)U(g)U(h)$. Then, for any $f_1, f_2 \in \mathsf{L}^2(G)$, we have:*

$$\begin{aligned}\Big(f_1 \star f_2\Big)(g) \;&=\; d_U^{-1} \int_G \mathrm{d}\nu_G(h)\, f_1(h) \Big(\mathsf{P} f_2\Big)(h^{-1}g)\, \overline{\mathsf{m}(h, h^{-1}g)} \\ &=\; d_U^{-1} \int_G \mathrm{d}\nu_G(h) \Big(\mathsf{P} f_1\Big)(h) \Big(\mathsf{P} f_2\Big)(h^{-1}g)\, \overline{\mathsf{m}(h, h^{-1}g)}, \end{aligned} \tag{26}$$

where P *is the orthogonal projection onto* $\mathsf{Ran}(\mathscr{D})$*. Therefore, for any $f_1, f_2 \in \mathsf{Ran}(\mathscr{D})$, the following formula holds* (‘m-twisted convolution’)*:*

$$\Big(f_1 \star f_2\Big)(g) = d_U^{-1} \int_G \mathrm{d}\nu_G(h)\, f_1(h)\, f_2(h^{-1}g)\, \overline{\mathsf{m}(h, h^{-1}g)}. \tag{27}$$

Let G be the **group of translations on phase space** $\mathbb{R}^n \times \mathbb{R}^n$. Then, $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^n)$, U is the *Weyl system* — $U(q,p) = \exp(\mathrm{i}(p \cdot \widehat{q} - q \cdot \widehat{p}))$ — $\mathsf{Ran}(\mathscr{D}) = \mathsf{L}^2(G) = \mathsf{L}^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q\, \mathrm{d}^n p; \mathbb{C})$ and $d_U = 1$; moreover:

$$\mathsf{m}(q,p;q',p') = \exp(\mathrm{i}(q \cdot p' - p \cdot q')/2). \tag{28}$$

However, for every **state** $\widehat{\rho}$, the function $(\mathscr{D}\widehat{\rho})(q,p) = \mathrm{tr}(U(q,p)^*\widehat{\rho})$ is *not* the *Wigner distribution* ϱ, but the *quantum characteristic function* $\widetilde{\varrho}$ . . .

Detour: *classical* states and PTFs

Recall that the Banach space $\mathsf{L}^1(G)$ of $\mathbb{C}$-valued functions on G, integrable w.r.t. the left Haar measure ν_G, endowed with the *convolution product*,

$$(\varphi_1 \circledcirc \varphi_2)(g) := \int_G \varphi_1(h)\varphi_2(h^{-1}g)\, \mathrm{d}\nu_G(h), \tag{29}$$

and the *involution*,

$$\mathsf{I}\colon \varphi \mapsto \varphi^*, \quad \varphi^*(g) := \triangle_G(g^{-1})\,\overline{\varphi(g^{-1})}, \tag{30}$$

with $\triangle_G$ denoting the modular function, is a *Banach* $*$*-algebra* $\big(\mathsf{L}^1(G), \circledcirc, \mathsf{I}\big)$.

Definition 1 *A* **positive** *bounded linear functional on the Banach* $*$*-algebra* $\big(\mathsf{L}^1(G), \circledcirc, \mathsf{I}\big)$*, realized as a function in the Banach space of* ν_G*-essentially bounded functions* $\mathsf{L}^\infty(G)$*, is called a* **function of positive type** *on* G*. Namely, a function* $\chi \in \mathsf{L}^\infty(G)$ *is said to be of positive type if*

$$\int_G \chi(g)\,(\varphi^* \circledcirc \varphi)(g)\, \mathrm{d}\nu_G(g) \geq 0, \quad \textbf{(PTF condition)} \tag{31}$$

for all $\varphi \in \mathsf{L}^1(G)$.

A function of positive type $\chi \in \mathsf{L}^\infty(G)$ agrees ν_G-almost everywhere with a (bounded) continuous function and

$$\|\chi\|_\infty := \nu_G\text{-ess sup}_{g\in G}|\chi(g)| = \chi(e). \tag{32}$$

For a *bounded continuous* function $\chi\colon G \to \mathbb{C}$ the following facts are *equivalent*:

P1) χ is of positive type;

P2) χ satisfies the PTF condition (31), for all $\varphi \in \mathsf{C_c}(G)$;

P3) χ satisfies the condition

$$\int_G \int_G \chi(g^{-1}h)\overline{\varphi(g)}\varphi(h)\,\mathrm{d}\nu_G(g)\mathrm{d}\nu_G(h) \geq 0, \tag{33}$$

for all $\varphi \in \mathsf{C_c}(G)$;

P4) χ is a **positive definite function**, i.e.,

$$\sum_{j,k} \chi(g_j^{-1}g_k)\,\overline{c_j}\,c_k \geq 0, \tag{34}$$

for every finite set $\{g_1,\ldots,g_m\} \subset G$ and arbitrary c-numbers $c_1,\ldots,c_m$.

Let G be **abelian** and let $\widehat{G}$ be its **dual** group. By **Bochner's theorem**, denoting by $\mathsf{CM}(\widehat{G})$ the Banach space of *complex Radon measures* on $\widehat{G}$, we can add another item to the previous list of equivalent facts:

P5) χ is the Fourier transform of a positive measure $\mu \in \mathsf{CM}(\widehat{G})$.

Now, setting $G = \mathbb{R}^n \times \mathbb{R}^n$, the *physical relevance* of functions of positive type becomes evident. Indeed, a **classical state** is a normalized positive functional on the *commutative* C*-algebra of classical observables. By **Gelfand theory**, such an algebra is (isomorphic to) the algebra of *continuous functions vanishing at infinity* $\mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n)$, endowed with the *point-wise product*. The **dual** of $\mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n)$ is $\mathsf{CM}(\mathbb{R}^n \times \mathbb{R}^n)$, the space of **complex Radon measures**, and the associated *states* are the **probability measures** on $\mathbb{R}^n \times \mathbb{R}^n$. The **expectation value** of an **observable** $f \in \mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ in the **state** $\mu \in \mathsf{CM}(\mathbb{R}^n \times \mathbb{R}^n)$ is given by the pairing

$$\langle f \rangle_\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(q,p)\, \mathrm{d}\mu(q,p). \tag{35}$$

It is often useful to replace a state μ with its *symplectic Fourier transform*,

$$\chi(q,p) \equiv \widetilde{\mu}(q,p) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{e}^{\mathrm{i}\omega(q,p;q',p')}\, \mathrm{d}\mu(q',p') \tag{36}$$

$$\omega(q,p;q',p') := q \cdot p' - p \cdot q'. \tag{37}$$

Note that $\chi \equiv \widetilde{\mu}$ *is a continuous function of positive type on* $\mathbb{R}^n \times \mathbb{R}^n$*:* $\widetilde{\mu} \in \mathsf{P}_n$*.* The normalization condition $\mu(\widehat{G}) = 1$ corresponds to $\chi(0) = \|\chi\|_\infty = 1$; i.e., to *the normalization of* χ *as a functional*. In probability theory, χ is called the **characteristic function** of μ.

Quantum states and *quantum* PTFs

In the *phase space formulation* of QM, a *pure* state $\widehat{\rho}_\psi = |\psi\rangle\langle\psi|$ in $\mathsf{L}^2(\mathbb{R}^n)$ is replaced with a function (**Wigner function**):

$$\mathbb{R}^n \times \mathbb{R}^n \ni (q,p) \mapsto \varrho_\psi(q,p) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i} p\cdot x}\, \overline{\psi\Big(q - \frac{x}{2}\Big)}\, \psi\Big(q + \frac{x}{2}\Big)\ \mathrm{d}^n x. \quad (38)$$

This definition *extends* in a natural way to every *trace class operator*. One then obtains a separable *Banach space* of functions $\mathsf{LW}_n \subset \mathsf{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$, which contains a *convex cone* W_n, formed by those functions that are associated with *positive* trace class operators in $\mathsf{L}^2(\mathbb{R}^n)$. W_n contains the *convex set* $\bar{\mathsf{W}}_n$ of *Wigner functions* characterized by the *normalization condition*

$$\lim_{r\to+\infty} \int_{|q|^2+|p|^2 \le r^2} \varrho(q,p)\ \mathrm{d}^n q\, \mathrm{d}^n p = \mathrm{tr}(\widehat{\rho}) = 1, \quad (39)$$

where $\varrho \in \bar{\mathsf{W}}_n$ is the function associated with a certain state $\widehat{\rho}$. As in the classical setting, one can replace a Wigner distribution with its *symplectic Fourier(-Plancherel) transform*

$$\Big(\widehat{\mathcal{F}}_{\mathsf{sp}}\, \varrho\Big)(q,p) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} \varrho(q',p')\, \mathrm{e}^{\mathrm{i}(q\cdot p' - p\cdot q')}\ \mathrm{d}^n q' \mathrm{d}^n p'. \quad (40)$$

Then, the space LW_n is mapped onto a dense subspace LQ_n of $\mathsf{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$,

$$\mathsf{LQ}_n := \widehat{\mathcal{F}}_{\mathsf{sp}}\, \mathsf{LW}_n, \tag{41}$$

and the convex cone $\mathsf{W}_n \subset \mathsf{LW}_n$ is mapped onto a convex cone $\mathsf{Q}_n \subset \mathsf{LQ}_n$.

By analogy with the classical case, we may call a function

$$\widetilde{\varrho} := (2\pi)^n\, \widehat{\mathcal{F}}_{\mathsf{sp}}\, \varrho, \quad \varrho \in \breve{\mathsf{W}}_n, \tag{42}$$

the *quantum characteristic function* associated with the *quasi-probability distribution* ϱ. Similarly to the classical case, the **quantum characteristic functions**, are those functions in Q_n satisfying the *normalization condition*

$$\widetilde{\varrho}(0) = 1. \tag{43}$$

These functions form a convex subset $\breve{\mathsf{Q}}_n$ of LQ_n. Moreover:

$$\widetilde{\varrho}(q,p) = \mathrm{tr}(U(q,p)^* \widehat{\rho}) = (\mathscr{D}\widehat{\rho})(q,p), \tag{44}$$

where U is the *Weyl system*, i.e., $U(q,p) = \exp(\mathrm{i}(p \cdot \widehat{q} - q \cdot \widehat{p}))$.

Natural problem: Is it possible to characterize *intrinsically* the convex set of *Wigner functions* $\breve{\mathsf{W}}_n$ or the convex set $\breve{\mathsf{Q}}_n$ of *quantum characteristic functions*? The analysis of this problem leads to the notion of **function of quantum positive type**.

As in the classical setting, we consider a $*$-*algebra of functions*, and then define the functions of positive type as suitable *functionals* on this algebra. The Hilbert space $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ becomes a $*$-algebra — more precisely, a H*-algebra — once endowed with the **twisted convolution**

$$\left(\mathcal{A}_1 \circledast \mathcal{A}_2\right)(q,p) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{A}_1(q',p')\, \mathcal{A}_2(q-q',p-p')\, \mathrm{e}^{\frac{\mathrm{i}}{2}(q\cdot p' - p\cdot q')}\, \mathrm{d}^n q' \mathrm{d}^n p',$$

$\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$, and with the *involution* $\mathsf{J}\colon \mathcal{A} \mapsto \mathcal{A}^*$,

$$\mathcal{A}^*(q,p) := \overline{\mathcal{A}(-q,-p)}, \quad \mathcal{A} \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{45}$$

Notice:
$$\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n) \circledast \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n) = \mathrm{LQ}_n \quad \text{and} \quad \mathsf{J}\,\mathrm{LQ}_n = \mathrm{LQ}_n. \tag{46}$$

(The twisted convolution is the star product associated with the Weyl system: it realizes the of operator product in terms of phase-space functions.)

Definition 2 *A* **function of quantum positive type** *is a positive bounded linear functional on the* H*-*algebra* $\left(\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n), \circledast, \mathsf{J}\right)$. *Thus, we say that a function* $\mathcal{Q} \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ *is of quantum positive type if*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{Q}(q,p)\, (\mathcal{A}^* \circledast \mathcal{A})(q,p)\, \mathrm{d}^n q\, \mathrm{d}^n p \geq 0, \quad \textbf{(QPTF condition)} \tag{47}$$

for all $\mathcal{A} \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$.

(P. A., “Playing with functions of positive type . . .”, *Phys. Scr.* **90** (2015) 074042)

If a *continuous* function $\mathcal{Q}$ is of quantum positive type, then it is bounded and

$$\|\mathcal{Q}\|_\infty = \mathcal{Q}(0). \quad \text{(compare with (32))} \tag{48}$$

Moreover, for a *continuous* function $\mathcal{Q}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ the following facts are **equivalent** ($z \equiv (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathrm{d}z \equiv \mathrm{d}^n q\, \mathrm{d}^n p$, $\omega(z,z') \equiv q \cdot p' - p \cdot q'$):

Q1) $\mathcal{Q}$ is of quantum positive type;

Q2) $\mathcal{Q}$ satisfies the QPTF condition (47), for all $\mathcal{A} \in \mathsf{C_c}(\mathbb{R}^n \times \mathbb{R}^n)$;

Q3) $\mathcal{Q}$ satisfies the condition

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{Q}(z-z')\,\overline{\mathcal{A}(z')}\,\mathcal{A}(z)\,\mathrm{e}^{\mathrm{i}\omega(z',z)/2}\,\mathrm{d}z\,\mathrm{d}z' \geq 0, \tag{49}$$

for all $\mathcal{A} \in \mathsf{C_c}(\mathbb{R}^n \times \mathbb{R}^n)$;

Q4) $\mathcal{Q}$ is a **quantum positive definite function**, i.e.,

$$\sum_{j,k} \mathcal{Q}(z_k - z_j)\,\mathrm{e}^{\mathrm{i}\omega(z_j,z_k)/2}\,\overline{c_j}\,c_k \geq 0, \tag{50}$$

for every finite set $\{z_1,\ldots,z_m\} \subset \mathbb{R}^n \times \mathbb{R}^n$ and arbitrary c-numbers $c_1,\ldots,c_m$;

Q5) $\mathcal{Q}$ *is* — up to the normalization: $\mathcal{Q}(0) = 1$ — *the Fourier-Plancherel transform of a Wigner quasi-probability distribution*.

Playing with functions of positive type

The *convolution* $\mu_1 \circledcirc \mu_2$ of a pair of *probability measures* $\mu_1, \mu_2 \in \mathsf{CM}(G)$,

$$\int_G \varphi(g)\, \mathrm{d}\mu_1 \circledcirc \mu_2(g) := \int_G \int_G \varphi(gh)\, \mathrm{d}\mu_1(g)\mathrm{d}\mu_2(h), \quad \varphi \in \mathsf{C_c}(G), \qquad (51)$$

is a probability measure too. Endowed with convolution the convex set $\mathsf{PM}(G)$ of **probability measures** on G becomes a **semigroup**, with *identity* δ_e. If G is *abelian*, to the convolution of probability measures corresponds — via the FT — the *point-wise multiplication of characteristic functions*. Hence, the point-wise product $\chi_1\chi_2$ of two continuous functions of positive type on G is a continuous function of positive type too.

Let us take $G = \mathbb{R}^n \times \mathbb{R}^n$. Endowed with the point-wise product the set $\bar{\mathsf{P}}_n \subset \mathsf{P}_n$ of **normalized functions of (classical) positive type** on $\mathbb{R}^n \times \mathbb{R}^n$ is a **semigroup**, with the identity $\chi \equiv 1$.

What happens with the point-wise multiplication of a function of *classical* positive type by a continuous function of *quantum* positive type?

Theorem 4 *The point-wise product $\chi\mathcal{Q}$ of $\chi \in \mathsf{P}_n$ by $\mathcal{Q} \in \mathsf{Q}_n$ belongs to Q_n; in particular, to the convex set of quantum characteristic functions $\bar{\mathsf{Q}}_n$ if χ and $\mathcal{Q}$ are normalized.*

Consider then a *multiplication semigroup of functions of positive type*

$$\{\chi_t\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}\}_{t\in\mathbb{R}^+}\subset\breve{\mathsf{P}}_n,\quad \chi_t\,\chi_s=\chi_{t+s},\ t,s\geq 0,\quad \chi_0\equiv 1 \tag{52}$$

(continuous w.r.t. the the topology of uniform convergence on compact sets on $\breve{\mathsf{P}}_n$). Such semigroups can be classified: the FT of a multiplication semigroup of functions of positive type on $\mathbb{R}^n\times\mathbb{R}^n$ is a *convolution semigroup of probability measures* (characterized by the Lévy-Kintchine formula).

As χ_t is a bounded continuous function, we can define a *bounded operator* $\widehat{\mathfrak{C}}_t$ in $\mathsf{L}^2(\mathbb{R}^n\times\mathbb{R}^n)$:

$$\Big(\widehat{\mathfrak{C}}_t f\Big)(q,p):=\chi_t(q,p)\,f(q,p),\quad f\in\mathsf{L}^2(\mathbb{R}^n\times\mathbb{R}^n),\ \ t\geq 0. \tag{53}$$

The set $\{\widehat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of operators:

1. $\widehat{\mathfrak{C}}_t\widehat{\mathfrak{C}}_s=\widehat{\mathfrak{C}}_{t+s}$, $t,s\geq 0$;
2. $\widehat{\mathfrak{C}}_0=\mathbb{I}$;
3. $\lim_{t\downarrow 0}\|\widehat{\mathfrak{C}}_t f-f\|=0$, $\forall f\in\mathsf{L}^2(\mathbb{R}^n\times\mathbb{R}^n)$.

It is natural to consider the *restriction* of the semigroup of operators $\{\widehat{\mathfrak{C}}_t\}_{t\in\mathbb{R}+}$ to a linear subspace of $\mathsf{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$. Indeed, by complex linear superpositions one can extend the convex cone Q_n of functions of quantum positive type on $\mathbb{R}^n \times \mathbb{R}^n$ to the dense subspace LQ_n of $\mathsf{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$. A semigroup of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}+}$ in LQ_n is then defined as follows.
Since, by Theorem 4, the point-wise product of a continuous function of *classical* positive type by a continuous function *quantum* positive type is a function of the latter type, we can set

$$\mathfrak{C}_t \colon \mathsf{LQ}_n \to \mathsf{LQ}_n, \quad \Big(\mathfrak{C}_t\,\mathcal{Q}\Big)(q,p) := \chi_t(q,p)\,\mathcal{Q}(q,p). \tag{54}$$

It is clear that we have:

$$\mathfrak{C}_t\mathsf{Q}_n \subset \mathsf{Q}_n, \quad \mathfrak{C}_t\bar{\mathsf{Q}}_n \subset \bar{\mathsf{Q}}_n. \tag{55}$$

We will call the semigroups of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}+}$ a **classical-quantum semigroup**. The introduction of this semigroup of operators may be regarded as a mere mathematical *divertissement*, based on the properties of functions of positive type. But it turns out that it has a precise *physical interpretation*.

The relation with *open quantum systems*

The Weyl system U gives rise to an **isometric representation** of $\mathbb{R}^n \times \mathbb{R}^n$ in $\mathcal{B}_1(\mathcal{H})$:

$$U \vee U(q,p) \colon \mathcal{B}_1(\mathcal{H}) \ni \widehat{\rho} \mapsto U(q,p)\,\widehat{\rho}\,U(q,p)^*, \quad \mathcal{H} = \mathsf{L}^2(\mathbb{R}^n). \tag{56}$$

Given a **convolution semigroup** $\{\mu_t\}_{t\in\mathbb{R}^+}$ of measures on $\mathbb{R}^n \times \mathbb{R}^n$, a **semigroup of operators** $\{\mu_t[U]\}_{t\in\mathbb{R}^+}$ in $\mathcal{B}_1(\mathcal{H})$ is defined by setting

$$\mu_t[U]\,\widehat{\rho} := \int_{\mathbb{R}^n\times\mathbb{R}^n} \Big(U \vee U(q,p)\,\widehat{\rho}\Big)\,\mathrm{d}\mu_t(q,p). \tag{57}$$

This semigroup of operators — a *twirling semigroup* (classical-noise sem.)— is a **quantum dynamical semigroup** (completely positive, trace-preserving).

Theorem 5 *Let* $\{\chi_t\}_{t\in\mathbb{R}^+}$ *be the multiplication semigroup of functions of positive type associated with* $\{\mu_t\}_{t\in\mathbb{R}^+}$*,*

$$\chi_t(q,p) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(q\cdot p' - p\cdot q')}\,\mathrm{d}\mu_t(q',p'), \tag{58}$$

and let $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ *be the proper classical-quantum semigroup generated by* $\{\chi_t\}_{t\in\mathbb{R}^+}$*. The quantization map* $\mathscr{Q}$ *intertwines* $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ *with the quantum dynamical semigroup* $\{\mu_t[U]\}_{t\in\mathbb{R}^+}$*:*

$$\mathscr{Q}\,(\mathfrak{C}_t\,\mathcal{Q}) = \mu_t[U]\,(\mathscr{Q}\,\mathcal{Q}), \quad \mathcal{Q} \in \mathsf{LQ}_n, \quad t \geq 0. \tag{59}$$

Thank you for your attention

and

many thanks to the organizers!

References:

P. A., G. Cassinelli, E. De Vito, A. Levrero, "Square-integrability of induced representations of semidirect products", *Rev. Math. Phys.* **10** (1998) 301

P. A., "Square integrable projective representations and square integrable representations modulo a relatively central subgroup", *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 233

P. A., "Extended wavelet transforms", *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 341

P. A., "Star products: a group-theoretical point of view", *J. Phys. A: Math. Theor.* **42** (2009) 475210

P. A., A. Kossakowski, G. Marmo, F. Ventriglia, "Brownian motion on Lie groups and open quantum systems", *J. Phys. A: Math. Theor.* **43** (2010) 265301

P. A., "On a certain class of semigroups of operators", *Open Syst. & Inf. Dynamics* **18** (2011) 129

P. A., "Quantum dynamical semigroups, group representations and convolution semigroups", *Phys. Scr.* **T153** (2013) 014003

P. A., "Playing with functions of positive type, classical and quantum", *Phys. Scr.* **90** (2015) 074042

P. A., D. Chruściński, "Characterizing the dynamical semigroups that do not decrease a quantum entropy", *J. Phys. A: Math. Theor.* **49** (2016) 345301