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Variational Bézier or B-spline curves and surfaces

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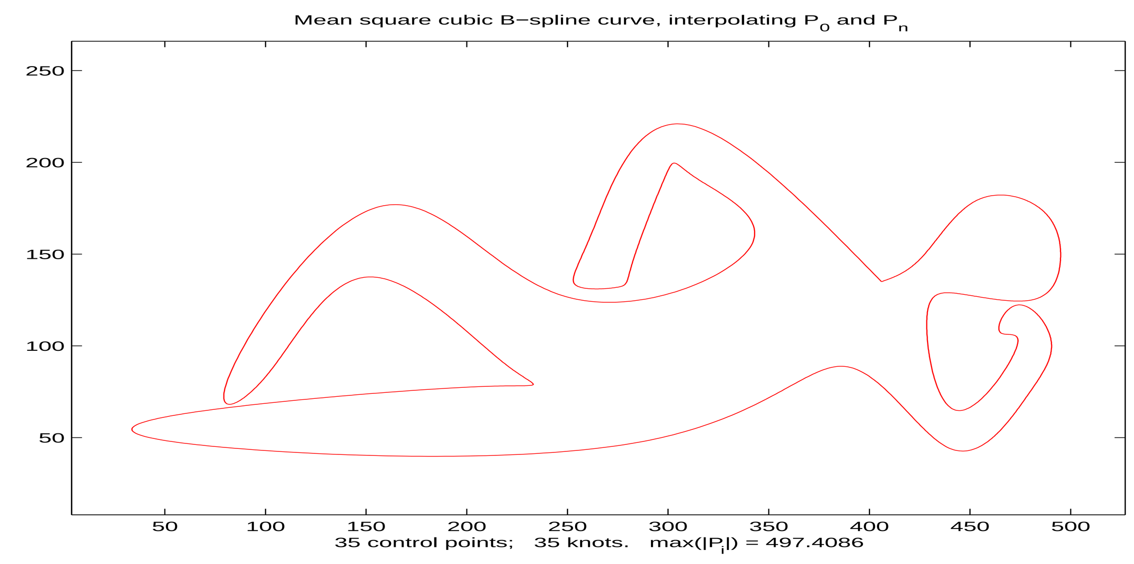
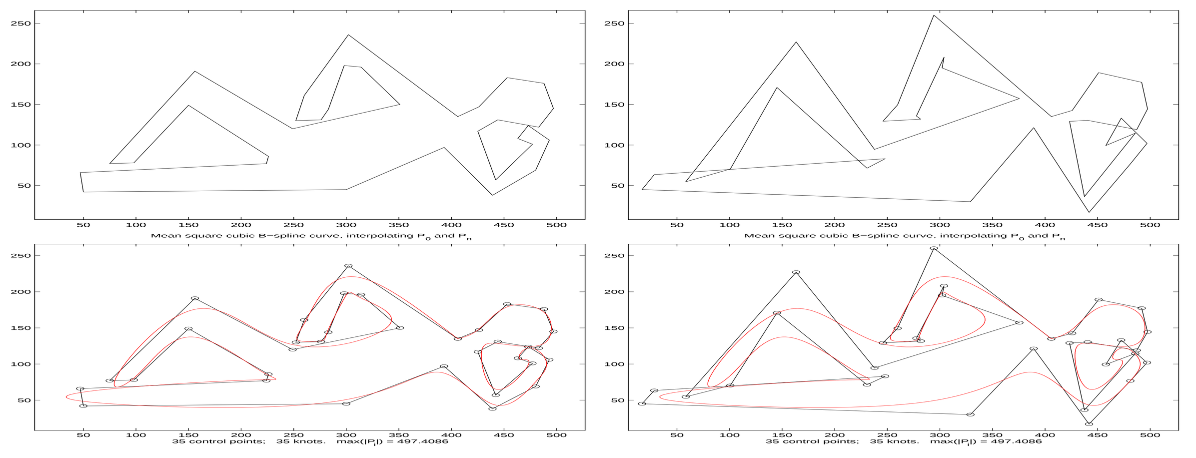


Question:

To generate the above curve, which control polygon do you prefer?



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Aim

Obtain a curve closer to the control polygon (resp. "flatter"), in order that building a control polygon for obtaining a curve of a certain shape is more intuitive, still being in the same vectorial space.

Or on the opposite, obtain a curve "flatter"

Means

1. **Minimize the L^2 distance from the B-curve to the control polygon**
2. Minimize the ℓ^2 distance from the B-curve to the control points.
3. Minimize $\int_0^1 (C''(t))^2 dt$ (flatter)

And possibly increase or decrease the number of knots of the obtained B-curve (i.e. $n \neq k$).

Any mix between these criteria: minimize $E = \rho_1 E_1 + \rho_2 E_2 + \rho_3 E_3$

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Notations

Original control points: $(P_j)_{j=0:k}$

Hat function (step $\frac{1}{k}$, centered in $\frac{j}{k}$): b_j^k

Original control polygon: $P(t) = \sum_{j=0:k} P_j\, b_j^k(t)$

"B-functions": Bernstein, B-spline, exponential B-splines, ECC systems...: $(B_j^k)_{=0:k}$

Usual "B-curve": $C(t) = \sum_{j=0:k} P_j\, B_j^k(t)$

Computed control points: $(Q_i)_{j=0:n}$

"Variational B-curve" ("New B-curve") $CV(t) = \sum_{i=0:n} Q_i\, B_i^n(t)$

"Variational B-functions" ("new" B-functions): $(BV_j^k)_{j=0:k}$....

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Easy mathematics! (example)

Minimize $E1(Q) \;=\; \int_0^1 \left(\sum_{i=0:n} Q_i\, B_i^n(t) \;-\; \sum_{j=0:k} P_j\, b_j^k(t)\right)^2 dt$

$\frac{1}{2}\frac{\partial E1}{\partial Q_{i_0}^n} \;=\; \int_0^1 \left(\sum_{i=0:n} Q_i\, B_i^n(t) \;-\; \sum_{j=0:k} P_j\, b_j^k(t)\right) B_{i_0}^n(t)\, dt \;=\; 0$

$$\Longrightarrow \sum_{i=0:n} Q_i \int_0^1 B_i^n(t)\, B_{i_0}^n(t)\, dt \;=\; \sum_{j=0:k} P_j \int_0^1 b_j^k(t) B_{i_0}^n(t)\, dt$$

A linear system to be solved (order $n+1$).

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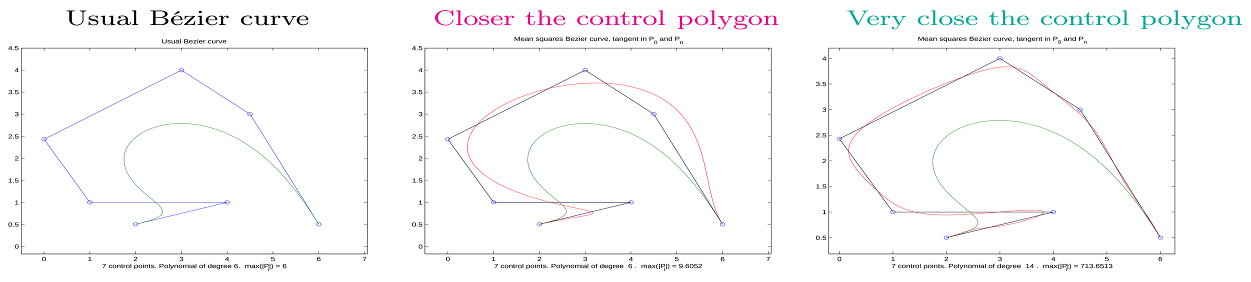
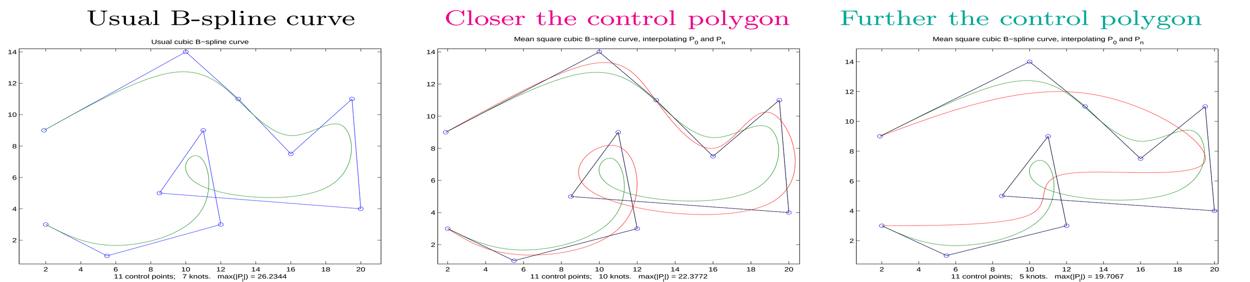
A glance on some results

Usual Bézier curve | Closer the control polygon | Very close the control polygon



Usual B-spline curve | Closer the control polygon | Further the control polygon



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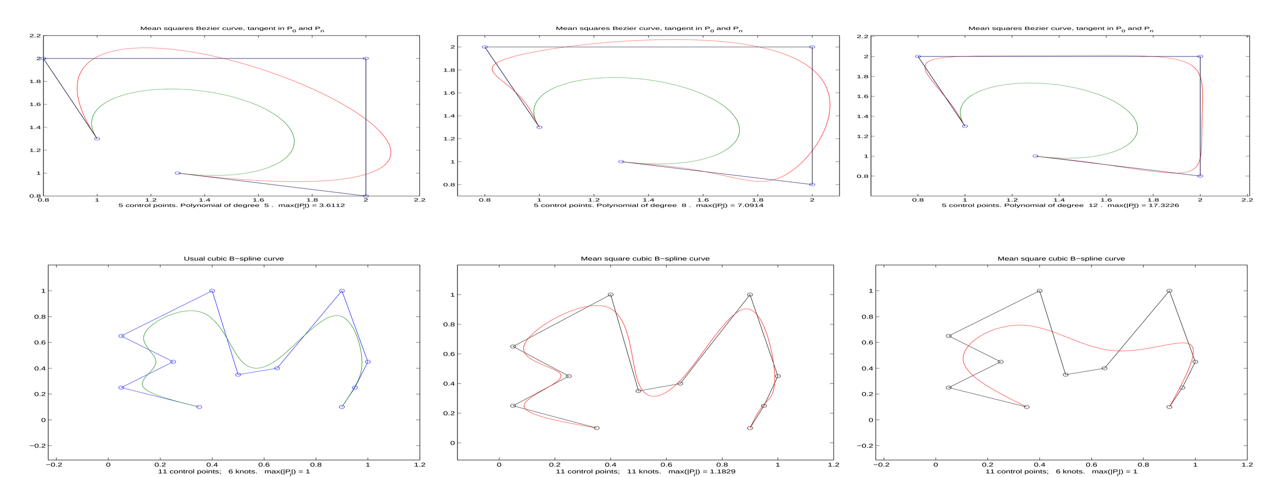
Even closer (further) to the control polyg.:

Increase (decrease) the degree of the polynomial (still keeping same data P): $n > k$.



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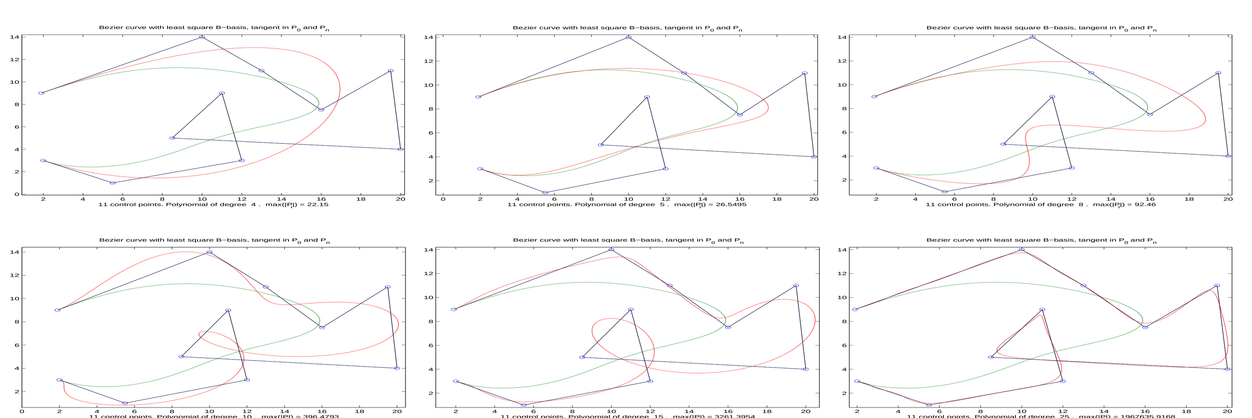
Convergence towards control polygon

Since polynomials and splines are dense on the space of continuous functions, the B-curve converges towards the control polygon when n tends to infinity.

However be careful with the condition number for polynomials!



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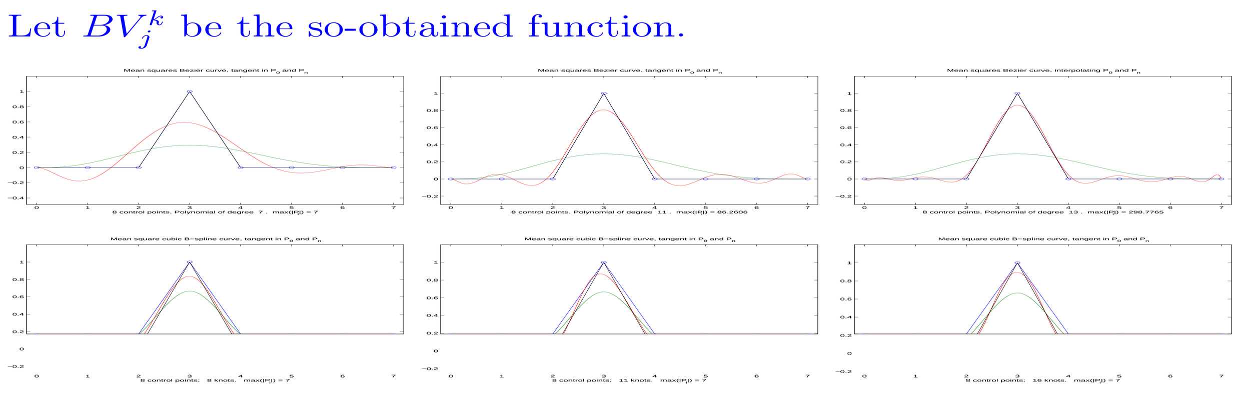
"Variational B-functions"

Apply the above minimization with the hat functions $(b_j^k)_{j=0:k}$

Let BV_j^k be the so-obtained function.



Then $CV(t)$ is also equal to $\sum_{j=0:k} P_j\, BV_j^k(t)$, which means:
the optimal B-curve is also the BV-curve of the original polyg.

The BV's are "new B-functions".

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How to design curves or surfaces

- **Choose a functional space and associated B-functions**
  - Polynomials (Bernstein)
  - Polynomial splines, NURBS, polyharmonic splines
  - Fractional (polynomial or polyharmonic) splines
  - ECC spaces (hyperbolic or circular, and polynomials, sum of monomials,...)
  - ...
- Choose a control polygon (polyhedron for surfaces)
- Choose a level of distance to the control polygon
- Compute the associated B-curve
- Go back to any above item if necessary

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What to remember from this poster

DISCONNECT the form of the curves (or surfaces) (ie the functional space in which they are) from the distance of the curve (surface) to the control points.

It is easy to **CHOOSE AND MINIMIZE A "DESIRED DISTANCE"** for given B-functions (space) and control polygon.

Doing a global minimization is equivalent to using "new B-functions".

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