Interpolatory and noninterpolatory Hermite subdivision schemes reproducing polynomials

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Abstract

In this study, we present a large family of Hermite subdivision schemes with tension parameters. The proposed schemes are quasi-interpolatory because they reproduce polynomials up to certain degrees. Depending on the choice of tension parameters, the corresponding schemes become interpolatory. The smoothness analysis has been performed by using the factorization framework of subdivision operators. Also, the approximation order of the proposed schemes is discussed. Some numerical examples are presented in order to demonstrate the performance of the proposed Hermite schemes.

Introduction

Let $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$ be an initial sequence of vectors attached to the integer grid. A Hermite subdivision scheme of (order $d+1$) computes recursively a new sequence of refined vectors $\mathbf{f}_k \in \ell^{d+1}(\mathbb{Z})$ by the rule

$$\mathbf{D}^{k+1}\mathbf{f}_{k+1}(i) = \sum_{j\in\mathbb{Z}} \mathbf{A}(i-2j)\mathbf{D}^k\mathbf{f}_k(j),$$

where $\mathbf{D} = \mathrm{diag}(1, 2^{-1}, \dots, 2^{-d})$. The sequence of matrices $\mathbf{A} := \{\mathbf{A}(i) : i \in \mathbb{Z}\} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ is called the *subdivision mask*, and it is assumed to be finitely supported, i.e., only finitely many elements are nonzero.

A Hermite subdivision scheme is said to be *convergent* if for any initial data $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$, there exists a uniformly continuous vector-valued function $\boldsymbol{\phi} = [\phi_0, \dots, \phi_d]^T \in C(\mathbb{R}, \mathbb{R}^{d+1})$ such that for any compact set $K \in \mathbb{R}$,

$$\lim_{k\to\infty} \max_{i\in K\cap 2^k\mathbb{Z}} \|\mathbf{f}_k(i) - \boldsymbol{\phi}(t_i^k)\|_\infty = 0.$$

and $\boldsymbol{\phi} \neq \mathbf{0}$ for some initial vector data $\mathbf{f}_0$. Moreover, the Hermite scheme is said to be C^L-convergent with $L \geq d$, if $\phi_0 \in C^L(\mathbb{R})$ and

$$\frac{d^m\phi_0}{dx^m} = \phi_m, \quad m = 0, \dots, d.$$

We call $\boldsymbol{\phi}$ a *limit function* of the Hermite subdivision scheme.

Subdivision schemes are, in some sense, classified into two main categories: interpolatory and approximating schemes. It is well-known that interpolating schemes are usually less smooth than the approximating schemes of the same order. Moreover, interpolation, in spite of being a very desirable property in curve (and surface) designs, often produces undesirable artifacts such as wiggles or undulations when the initial control points are irregular. From this view point, the aim of this study is to present a new class of quasi-interpolatory Hermite subdivision schemes.

Objectives

Present a new class of Hermite subdivision schemes of order two, which enable us to (1) reproduce polynomials, (2) generalize interpolation schemes, (3) provide good approximation orders and higher-order smoothness, and (4) give flexibilities in designs to accommodate the various design circumstances.

Hermite Subdivision Schemes Reproducing Polynomials

The proposed Hermite subdivision scheme H_{2M+1} reproduces polynomials up to degree $4M-1$ such that it guarantees the approximation order $4M$. The subdivision masks of the proposed schemes are represented in terms of the fundamental Hermite interpolating polynomials U_j and V_j defined by

$$U_j(x) = \ell_j(x)^2\big(1 - 2\ell_j'(j)(x-j)\big), \quad V_j(x) = \ell_j(x)^2(x-j),$$

where $\ell_j(x) := \prod_{m=-M+1,\, m\neq j}^{M} (x-m)/(j-m)$. We will use the notations

$$\mathbf{D} := \mathrm{diag}(1, 2^{-1}), \quad \mathbf{M}_j(x) := \begin{bmatrix} U_j^{(0)}(x) & V_j^{(0)}(x) \\ U_j^{(1)}(x) & V_j^{(1)}(x) \end{bmatrix}.$$

Construction of odd mask

For $r = 1, 2$, the rth rows of the matrices $\mathbf{A}(1-2i)$ in the odd mask $\{\mathbf{A}(1-2i) \in \mathbb{R}^{2\times 2} : i = -M+1, \dots, M\}$ are obtained by solving the linear system

$$2^{-(r-1)}p_n^{(r-1)}(\tfrac{1}{2}) = \sum_{j=-M+1}^{M} \big(\mathbf{A}_{r1}(1-2j)p_n^{(0)}(j) + \mathbf{A}_{r2}(1-2j)p_n^{(1)}(j)\big), \quad n = 0, \dots, 4M-1.$$

This linear system for each r is uniquely solvable. The solution can be written in the matrix form

$$\mathbf{A}(1-2j) = \mathbf{D}\mathbf{M}_j(\tfrac{1}{2}) \quad j = -M+1, \dots, M.$$

Construction of even mask

For $r = 1, 2$, the rth rows of the matrices $\mathbf{A}(2i)$ in the even mask $\{\mathbf{A}(2i) \in \mathbb{R}^{2\times 2} : i = -M, \dots, M\}$ are defined by solving the system of equations

$$2^{-(r-1)}p_n^{(r-1)}(0) = \sum_{j=-M}^{M} \big(\mathbf{A}_{r1}(-2j)p_n^{(0)}(j) + \mathbf{A}_{r2}(-2j)p_n^{(1)}(j)\big), \quad n = 0, \dots, 4M-1. \tag{1}$$

For each r, this is an underdetermined system of $4M+2$ unknowns in $4M$ equations so that there are two degrees of freedom which will be used as tension parameters. We set the tension parameters as $\mathbf{A}_{11}(\pm 2M) = 2^{-(4M-4)}\theta$ and $\mathbf{A}_{22}(\pm 2M) = 2^{-(4M-3)}\omega$.

Theorem. *Let* $\mathbf{S}$ *be the matrices defined by*

$$\mathbf{S} := \begin{bmatrix} 2^{-(4M-4)}\theta & -2^{-(4M-4)}\eta\theta \\ -2^{-(4M-3)}\omega\zeta & 2^{-(4M-3)}\omega \end{bmatrix},$$

with $\eta := (1 + U_M^{(0)}(-M))/U_M^{(1)}(-M)$ *and* $\zeta := (1 + V_M^{(1)}(-M))/V_M^{(0)}(-M)$. *Then, the solution to the system* (1) *can be written as*

$$\mathbf{A}(2M) = \mathbf{S},$$
$$\mathbf{A}(-2j) = \mathbf{D}\delta_{j,0} - \mathbf{S}\mathbf{M}_j(-M), \quad j = -M+1, \dots, M.$$

The scheme H_3 reproducing cubic polynomials

General form of mask: Setting the tension parameters as $\mathbf{A}_{11}(\pm 2) = \theta$ and $\mathbf{A}_{22}(\pm 2) = \frac{\omega}{2}$, the general form of the mask $\mathbf{A} = \{\mathbf{A}(i) : i = -2, \dots, 2\}$ is specifically given as

$$\mathbf{A} = \left\{ \begin{bmatrix} \theta & -\frac{\theta}{2} \\ -\frac{3\omega}{2} & \frac{\omega}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} 1-2\theta & 0 \\ 0 & \frac{1+4\omega}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \theta & \frac{\theta}{2} \\ \frac{3\omega}{2} & \frac{\omega}{2} \end{bmatrix} \right\}. \tag{2}$$

Special cases: (i) If the case $\theta = 0$ and $\omega = 0$, the scheme H_3 becomes the so-called Merrien interpolatory Hermite subdivision scheme of order 2 [6], whose mask is given by

$$\mathbf{A} = \left\{ \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{bmatrix} \right\}. \tag{3}$$

(ii) A repeated application of the de Rham transform [2] to the above Merrien scheme generates a Hermite scheme associated with the mask given by

$$\mathbf{A} = \left\{ \begin{bmatrix} \frac{1}{128} & -\frac{1}{256} \\ \frac{3}{32} & -\frac{1}{32} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{63}{64} & 0 \\ 0 & \frac{3}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{128} & \frac{1}{256} \\ -\frac{3}{32} & -\frac{1}{32} \end{bmatrix} \right\}.$$

This scheme is obtained by setting $\theta = \frac{1}{128}$ and $\omega = -\frac{1}{16}$ into the mask of H_3 in (2).

Symbol: Let $A_0(z)$ be the symbol of the Merrien scheme with the mask (3). Then the symbol $A(z)$ of H_3 is factored into

$$A(z) = \begin{bmatrix} \frac{8\theta z^2+(1-16\theta)z+8\theta}{z} & \frac{4\theta(z^2-1)}{z} \\ \frac{6\omega(z^2-1)}{z} & \frac{2\omega z^2+(1+8\omega)z+2\omega}{z} \end{bmatrix} A_0(z).$$

Approximation Order

For an integer $m \geq 0$ and a compact set K in $\mathbb{R}$, we denote by $W_\infty^m(K)$ the Sobolev space defined by $W_\infty^m(K) = \left\{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{m,K} := \sum_{\ell=0}^m \|f^{(\ell)}\|_{L_\infty(K)} < \infty \right\}$.

Theorem. *Assume that a* C^d*-convergent Hermite subdivision scheme of order* $d+1$ *reproduces polynomials of degree up to* $N \geq d$. *Let* K *be a compact set in* $\mathbb{R}$. *Then the Hermite scheme has the approximation order as follows:*

$$\|f^{(\ell)} - f_\infty^{(\ell)}\|_{L_\infty(K)} \leq c\|f\|_{N+1,K}2^{-k(N+1-\ell)}, \quad \ell = 0, \dots, d$$

with a positive constant c *independent of* f *and* k.

Smoothness & Numerical Examples

Subdivision scheme	H_3	H_5	H_7	H_9	H_{11}
Maximum smoothness	C^4	C^5	C^6	C^7	C^8

Table 1: Maximum smoothness of the scheme H_{2M+1} for $M = 1, \dots, 5$

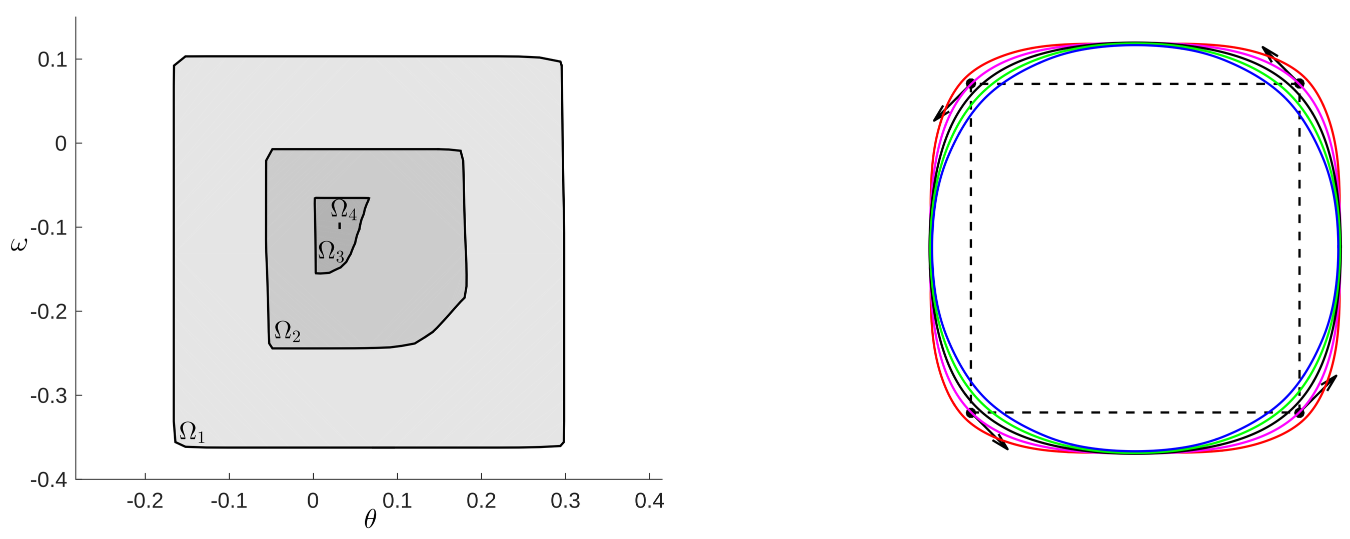

Figure 1: Left: Ranges Ω_n of θ and ω corresponding to C^n-smoothness of H_3 for $n = 1, \dots, 4$, Right: Limit curves of H_3 with $(\theta, \omega) = (-0.02875, -0.18), (0.00125, -0.14), (0.03125, -0.10), (0.06125, -0.06), (0.09125, -0.02)$

Figure 2: Limit curves generated from the irregularly spaced initial control points. Left: Merrien scheme, Right: H_3 with $\theta = 0.078$ and $\omega - 0.3$. The arrows indicate the tangent vectors.

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