Methods for Constructing Multivariate Tight Wavelet Frames

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Tight wavelet frames

A tight wavelet frame of $L^2(\mathbb{R}^d)$ is a family

$$X(\psi_1, \dots, \psi_N) =$$

$$\{\psi_{n;j,k}(x) = |\det M|^{j/2} \psi_n(M^j x - k) \;:\; 1 \le n \le N,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$$

obtained by dilations by powers of the matrix $M \in \mathbb{Z}^{d \times d}$ and shifts by $\mathbb{Z}^d$ of the functions $\psi_n \in L^2(\mathbb{R}^d)$, such that

$$\|f\|_2^2 = \sum_{n=1}^{N} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{n;j,k} \rangle|^2$$

for all $f \in L^2(\mathbb{R}^d)$.

UEP = Unitary Extension Principle

$\mathbb{T}^d = \{z \in \mathbb{C}^d : |z_1| = \cdots |z_d| = 1\}$ is the d-dimensional torus.

Special types of these frames can be constructed by solving the following matrix-extension (or matrix factorization) problem (Ron, Shen 1997):

Given a vector

$$F(z) = (F_1(z), \ldots, F_m(z))^T$$

of trigonometric polynomials $F_j \in \mathbb{C}[\mathbb{T}^d]$,

$$F_j(z) = \sum_{0 \le |\alpha| \le r} c_\alpha z^\alpha, \quad z = (z_1, \ldots, z_d) \in \mathbb{T}^d,$$

find a matrix $G(z)$ of trigonometric polynomials such that

$$\boxed{I_{m \times m} - F(z)F(z)^* = G(z)G(z)^*.}$$

UEP = Unitary Extension Principle

Find a matrix $G \in (\mathbb{C}[\mathbb{T}^d])^{m\times N}$ such that

$$\boxed{I_{m\times m} - F(z)F(z)^* = G(z)G(z)^*.}$$

Warm-up: Can we have $G \in (\mathbb{C}[\mathbb{T}^d])^{m\times m}$?

Yes, but ...
this requires

$$\det(I_{m\times m} - F(z)F(z)^*) = 1 - F(z)^*F(z) = |\det G(z)|^2$$

be a single square modulus of a trigonometric polynomial.

This property is very restrictive for multivariate trignometric polynomials!

UEP = Unitary Extension Principle

Observations from Linear Algebra (with Lai 2006) reduce the matrix extension problem to a scalar problem:

$$\boxed{I - F(z)F(z)^* = G(z)G(z)^*} \qquad \text{matrix extension}$$

$$H^* = (1 - F^*F, F^*G) \qquad \downarrow \quad \uparrow \qquad G = (I - FF^*, FH^*)$$

$$\boxed{1 - F(z)^*F(z) = H(z)^*H(z)} \qquad \text{scalar extension}$$

Proof: If $\begin{bmatrix} F \\ H \end{bmatrix} \in \mathbb{C}^{m+N}$ is a vector of norm 1 in the scalar extension, then

$$I_{m+N} - \begin{bmatrix} F \\ H \end{bmatrix} \begin{bmatrix} F \\ H \end{bmatrix}^* = \left(I_{m+N} - \begin{bmatrix} F \\ H \end{bmatrix} \begin{bmatrix} F \\ H \end{bmatrix}^* \right)^2,$$

so taking the first m rows of the left-hand side gives a proper matrix G.

UEP = Unitary Extension Principle

Connection to Algebraic Geometry:

1. Existence of "sum-of-squares" decompositions

$$1 - \sum_{j=1}^{m} |F_j(z)|^2 = \sum_{n=1}^{N} |H_n(z)|^2, \qquad z \in \mathbb{T}^d,$$

   with trigonometric polynomials H_j is related to Hilbert's 17th problem.

2. Even if "sum-of-squares" decompositions exist, there are no a-priori bounds on the number N and the degree of H_j, in general.

M. Marshall, *Positive Polynomials and Sums of Squares*, 2010.

M. Charina, M. Putinar, C. Scheiderer, J. S.:
An algebraic perspective on multivariate tight wavelet frames, part I (Constr. Approx. 2013) and II (Appl. Comput. Harmon. Anal. 2015).

Examples where UEP works

Box-splines on $\mathbb{R}^d$ with direction set $\Xi \subset \mathbb{Z}^d \setminus \{0\}$ with $d + d_0$ distinct directions:

- The scaling-symbol is a product of univariate trigonometric polynomials

$$F_1(z) = \prod_{k=1}^{d+d_0} \left(\frac{1 + z^{\xi_k}}{2} \right)^{r_k}.$$

  The vector $F(z)$ has components $F_1, \ldots, F_{2^d}$, where F_j's come from putting negative signs to some/all coordinates z_k.

- The scalar sos-decomposition

$$1 - F(z)^* F(z) = \sum_{j=1}^{N} |H_j(z)|^2$$

  exists with $N = d + d_0 2^d$ trigonometric polynomials H_j.

  The main step of the proof uses the Riesz-Féjer Lemma.

Examples where UEP works

Improvements for box-splines: Semi-Definite Programming (SDP)

- The number of terms in the sos-decomposition can be reduced by a standard method for positive polynomials.
  1. Take a monomial vector
     $$X(z) = [z^\alpha; \alpha \in A],$$
     where A contains all monomials that appear in $H_1, \ldots, H_N$, and write
     $$1 - F(z)^* F(z) = X(z) B X(z)^*.$$
     with a hermitian positive semi-definite matrix $B \in \mathbb{C}^{|A| \times |A|}$ which is computed from the coefficients of H_j's.
  2. Using SDP, find another representation
     $$1 - F(z)^* F(z) = X(z) C X(z)^*$$
     where C is hermitian and positive semi-defnite with smaller rank. Then find new H_j's from C.
- For the piecewise linear box-spline in $\mathbb{R}^2$ we reduce the number of frame generators from 10 to 6. (with M. Charina, JAT 2010)

Examples where UEP works

Examples of tight frames based on the refinable function of subdivision schemes:

- dimension 2: For the butterfly scheme (N. Dyn, J. Gregory, D. Levin, 1990), we find a tight frame with 13 frame generators; an earlier approach gave 18 generators.
- dimension 3: For the butterfly scheme (Chang, McDonnell, Qin, 2003), we find a tight frame with 31 frame generators. Improvements using the SDP approach were not attempted.
- Several other examples by Antolin and Zalik, Lai and Nam, since 2006.
- extension to irregular subdivision with the Loop scheme

General results from Algebraic Geometry

Motivation came from work of C. Scheiderer, *Sums of squares on real algebraic surfaces*, Manuscripta Math. 119 (2006), 395-410:

- For dimension $d = 2$, the sos-decomposition always exists. But there are no bounds on the number N and the degree r of the trigonometric polynomials H_j.
- For dimension $d \geq 3$, there exist non-negative trigonometric polynomials which are NOT sum-of-squares of trigonometric polynomials.

Another Perspective: System Theory

The connection to System Theory is established, when we consider z as a complex variable in the polydisk

$$\mathbb{D}^d = \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_k| < 1 \text{ for } 1 \leq k \leq d\}.$$

The multivariate theory was developed by Agler and McCarthy, Bose, Ball and Trent since 1990.

Another Perspective: System Theory

For dimensions $n_1, \dots, n_d \in \mathbb{N}$ we define a block diagonal matrix

$$Z = \mathrm{diag}(z_1 I_{n_1}, \dots, z_d I_{n_d}).$$

Theorem (Agler 1990, Cole, Wermer 1999)

Assume that the polynomial vector $\begin{bmatrix} F \\ H \end{bmatrix} : \mathbb{D}^d \to \mathbb{C}^{\tilde{m}}$ satisfies

$$F(z)^* F(z) + H(z)^* H(z) = 1 \qquad \text{for all} \quad z \in \mathbb{T}^d$$

and is an element of the *Schur-Agler class*. Then there exist $n_1, \dots, n_d \in \mathbb{N}$, $N = n_1 + \cdots + n_d$, and a contraction

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(\tilde{m}+N)\times(1+N)}$$

such that

$$\begin{bmatrix} F(z) \\ H(z) \end{bmatrix} = A + BZ(I - DZ)^{-1} C, \qquad z \in \mathbb{D}^d.$$

This is called a realization of $[F, H]^T$ as the transfer function of a linear system.

Another Perspective: System Theory

- *Schur-Agler class:* subset of all holomorphic function vectors with

  $$F(z)^* F(z) \leq 1, \qquad z \in \mathbb{T}^d,$$

  which satisfies the von Neumann inequality

  $$\|F(T_1, \dots, T_d)\|_{\mathrm{op}} \leq 1$$

  for every d-tupel of commuting contractions on an arbitrary Hilbert space.
- Similar obstacles as for sum-of-squares:
  For dimension $d \geq 3$, not all polynomial vectors with $F^* F \leq 1$ on $\mathbb{T}^d$ are in the Schur-Agler class (Varopoulos 1974)
- Algorithms for $d = 1$ and $d = 2$: Kummert (1989), Basu (2000)

Our "benchmark" example of piecewise linear box-spline frame in $\mathbb{R}^2$:

Improvement from 6 frame generators (SDP approach) to 5 generators.
This is the minimum!

More general OEP = Oblique Extension Principle

More tight wavelet frames are obtained by finding the factorization

$$\boxed{K(z) - F(z)L(z)F(z)^* = G(z)G(z)^*}$$

where $K(z)$ is a given diagonal matrix with trig. polynomials $K_{jj}(z) > 0$, and the trig. polynomial $L(z) > 0$ is related to $K_{11}(z)$ by some scaling operation.

K is used to increase the number of vanishing moments of wavelet frames.

More general OEP = Oblique Extension Principle

Basic assumption: K has a factorization

$$\boxed{K(z) = R(z)R(z)^*}$$

where R is an $m \times r$-matrix of trig. polynomials.

Then the scalar extension

$$\boxed{\frac{1}{L(z)} - F^*(z)K(z)^{-1}F(z) = H(z)^*H(z)}$$

with rational trig. vector $H = (H_1, \dots, H_N)$ leads to

$$\boxed{K(z) - F(z)L(z)F(z)^* = G(z)G(z)^*}$$

where

$$G = (R - FLF^*(R^\dagger)^*,\ FLH^*)$$

and $R^\dagger(z) = R(z)^*K(z)^{-1}$ is the Moore-Penrose pseudoinverse of $R(z)$.

More general OEP = Oblique Extension Principle

Work to be done:

- Results for OEP with trig. polynomials instead of rational functions are only known for special examples.
- The connection to System Theory has not been explored in full generality.

THANK YOU!