Helmholtz-Hodge decomposition, divergence-free Wavelets and Applications [1]

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Outline

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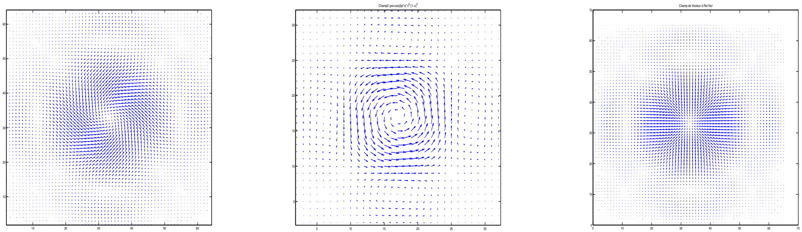
1 (i) - The Helmholtz decomposition



Velocity $\mathbf{v}$ $=$ *div-free part* $\mathbf{v}_{\text{div}} = \mathit{curl}\ \varphi$ $+$ *curl-free part* ∇q

- The sum is orthogonal in $L^2(\mathbb{R}^2)^2$: $\int \mathbf{v}_{\text{div}} \cdot \nabla q = -\int q\ \mathbf{div}\ \mathbf{v}_{\text{div}} = 0.$
- The div-free part $\mathbf{v}_{\text{div}}$ can be written as the *curl* of a *scalar* stream function φ :

$$\mathbf{v}_{\text{div}} = \mathit{curl}\ \varphi = (\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x})$$

(one has $\mathbf{div}\,\mathit{curl}\ \varphi = 0$)
- The term ∇q is curl-free since $\mathbf{curl}(\nabla q) = \nabla \times \nabla q = 0.$

1 (i) - Application 1 : Maxwell equations

Following Helmholtz (1858), the electric field E, that vanishes suitably quickly at infinity can be decomposed as :

$$E = E_{rot} + E_{sol}$$

where $\nabla \cdot E_{rot} = 0$ (rotational component) and $curl\ E_{sol} = \nabla \times E_{sol} = 0$ (solenoidal component).

E and B (magnetic field) are related to a scalar potential V and a vector potential A :

$$\left\{ \begin{array}{lcl} E & = & -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ B & = & \nabla \times A \end{array} \right. \tag{1}$$

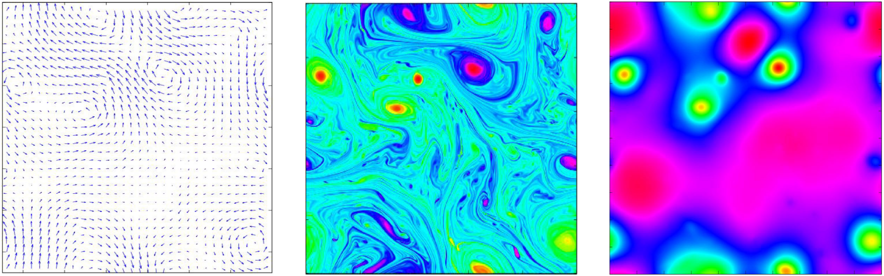
By Maxwell equations $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, $\nabla \times E = \frac{\partial \mathbf{B}}{\partial t}$, $\nabla \cdot B = 0, \ldots$ rewrites :

$$E = \nabla \times F - \nabla V$$

(V charge density potential)

1 (i) - Application 2 : Incompressible fluids

2D turbulent velocity/vorticity/pressure fields



velocity $\mathbf{u}_t(\mathbf{x}) = (u_1, u_2)$, *vorticity* $\omega = curl\ \mathbf{u} = (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})$, *pressure* $p_t(\mathbf{x})$

- Incompressibility condition : $\nabla.\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$

$\rightarrow$ $\mathbf{u} = curl\ \varphi$ where φ is the stream function (and $\mathbf{u} \perp \nabla p$)

1 (i) - Application 3 : visualization of 3D vector fields

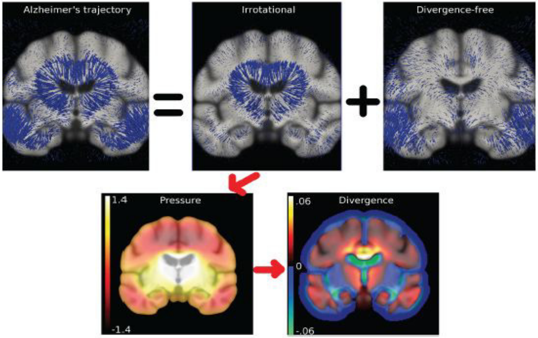
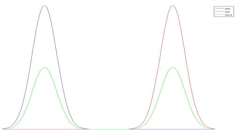
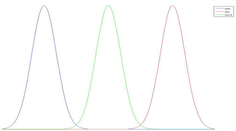


Fig. 1: Helmholtz decomposition of a longitudinal trajectory in Alzheimer's disease, and pressure potential and divergence maps associated to the irrotational component. The divergence describes the critical areas of local expansion and contraction.

[Lorenzi-Ayache-Pennec, MICCAI2012]

1 (i) - Application 5 : Optimal Transport

Linear interpolation between two densities ρ_0 and ρ_1 ($\int \rho_0 = \int \rho_1$) vs *interpolation by transport*

Monge-Kantorovitch problem (MKP)

Find a transport M from ρ_0 to ρ_1 that realize the infimum of the Wasserstein distance :

$$d_2(\rho_0, \rho_1)^2 = \inf_M \int |M(x) - x|^2 \rho(x) dx$$

1 (i) - Application 5 : Optimal Transport

Benamou-Brenier formulation (2000)

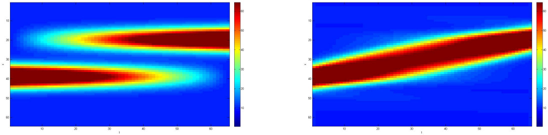
Continuum mechanic framework : $Q = (0, T) \times [0,1]^n$

- Time-dependent densities $\rho(t,x) \geq 0$ s.t. $\rho(0,x) = \rho_0(x)$, $\rho(T,x) = \rho_1(x)$,
- Velocities $v(t,x)$ s.t. $\partial_t \rho + \nabla.(\rho v) = 0$

Let $m = \rho v$, $V(Q) = \{f = (\rho, m) \in (L^2(Q))^{1+n},\ \mathrm{div}_{t,x} f = 0\}$.

Convex problem with linear constraints

$$d_2(\rho_0, \rho_1)^2 = \inf_{(\rho,m)\in V(Q)} T \int_0^T \int_{[0,1]^n} \frac{|m|^2}{\rho}\, dxdt \qquad \left(= \int\int \rho v^2\right)$$



Linear interpolation between ρ_0 and ρ_1 vs *interpolation by transport*

1 (ii) - Helmholtz decomposition in the whole space

$$\boxed{\mathbf{u} = \mathbf{u}_{div} + \mathbf{u}_{curl} = curl\ \varphi + \nabla q \quad \text{in} \quad \mathbb{R}^n} \quad (n = 2, 3)$$

Remark that :

$$curl\ \varphi \perp \nabla q \ , \quad \text{div}\ \mathbf{u} = \nabla \cdot \mathbf{u} = \ \text{div}(\mathbf{u}_{div}) + \text{div} \nabla q = \Delta q \quad (Poisson)$$

- **In the whole space** $\Omega = \mathbb{R}^2$ we have the orthogonal splitting :

$$(L^2(\mathbb{R}^2))^2 = \mathcal{H}_{div}(\mathbb{R}^2) \oplus \mathcal{H}_{div}^{\perp}(\mathbb{R}^2)$$

where

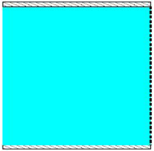
$$\begin{aligned} \mathcal{H}_{div}(\mathbb{R}^2) &= \{\mathbf{u}_{\text{div}} \in (L^2(\mathbb{R}^2))^2 \ ; \ \mathbf{div}\ \mathbf{u}_{\text{div}} = 0\} \\ &= \{\mathbf{u}_{\text{div}} = curl\ \varphi \ ; \ \varphi \in H^1(\mathbb{R}^2)\} \end{aligned}$$

is the space of *divergence-free* vector functions on $\mathbb{R}^2$.
Its orthogonal complement is the space of *curl-free* vector functions :

$$\begin{aligned} \mathcal{H}_{div}^{\perp}(\mathbb{R}^2) &= \{\mathbf{u}_{\text{curl}} \in (L^2(\mathbb{R}^2))^2 \ ; \ \mathbf{curl}\ \mathbf{u}_{\text{curl}} = 0\} \\ &= \{\mathbf{u}_{\text{curl}} = \nabla q \ ; \ q \in H^1(\mathbb{R}^2)\} \end{aligned}$$

1 (ii) - Helmholtz-Hodge decomposition in the square/cube

- Expected boundary conditions on $\Gamma = \partial\Omega$:
$\mathbf{u} = 0$ (Dirichlet) or $\mathbf{u} \cdot \nu = 0$ (free-slip)
(ν outward normal to Γ)



- Now we want to write $\mathbf{u} \in (L^2(\Omega))^2$ as follows :

$$\boxed{\mathbf{u} = curl\ \varphi + \nabla q \quad \text{in} \quad \Omega}$$

By Green's formula, the sum is orthogonal if :

$$\int_\Omega curl\ \varphi \cdot \nabla q = -\int_\Omega q\ \mathbf{div}(curl\ \varphi) + \int_\Gamma q\ curl\ \varphi \cdot \nu = \int_\Gamma q\ curl\ \varphi \cdot \nu = 0$$

Two possibilities :

$$\boxed{curl\ \varphi \cdot \nu = 0 \quad \text{on} \quad \Gamma}$$

or

$$\boxed{q = 0 \quad \text{on} \quad \Gamma}$$

(iii) More general : Helmholtz-Hodge Decomposition

[Girault-Raviart 86]

- **For** $\mathbf{u} \in (L^2(\Omega))^n$, $\Omega \subset \mathbb{R}^n$ a *regular* open subset, we have :

$$\mathbf{u} = curl\ \varphi + \nabla q + h \ \rightarrow \textbf{unique} \quad \varphi, q \in H_0^1(\Omega), h$$

with

$$\nabla \cdot (curl\ \varphi) = 0, \qquad curl\ (\nabla q) = 0, \qquad \nabla \cdot h = 0 \quad \text{and} \quad curl\ h = 0$$

- **In terms of spaces**, we obtain :

$$(L^2(\Omega))^n = \mathcal{H}_{div}(\Omega) \oplus \mathcal{H}_{curl}(\Omega) \oplus \mathcal{H}_{har}(\Omega) \ \rightarrow \textbf{orthogonal sum}$$

where

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^n \ ; \ \nabla \cdot \mathbf{u} = 0 \ \underline{\text{and}} \ \mathbf{u} \cdot \nu = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{H}_{curl}(\Omega) = \{\nabla q \ ; \ q \in H_0^1(\Omega)\} \ \text{and} \ \mathcal{H}_{har}(\Omega) = \{\nabla q \ ; \ q \in H^1(\Omega) \ \underline{\text{and}} \ \Delta q = 0\}$$

(iii) Practical computation of the Helmholtz-Hodge decomposition

$$\mathbf{u} = curl\ \varphi + \nabla q = \mathbf{u}_{div} + \nabla q$$

- In the whole space $\mathbb{R}^n$:

$$\text{div}\nabla q (= \Delta q) = \text{div}\ \mathbf{u}$$

$\mathbf{u}_{div} = \mathbf{u} - \nabla(\Delta)^{-1}\text{div}\mathbf{u}$
$\longrightarrow$ all computations can be done in Fourier domain

- On a subdomain Ω : one has first to **solve the Poisson equation**

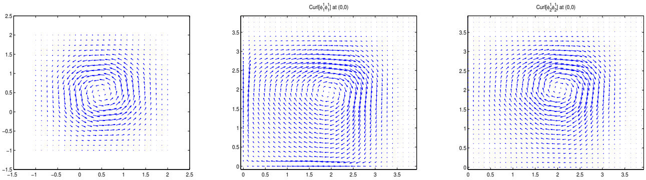
$$\Delta q = \text{div}\ \mathbf{u}$$

with suitable B.C. $\frac{\partial q}{\partial \nu} = \mathbf{u} \cdot \nu$ on $\partial\Omega$, and then compute $\mathbf{u}_{div} = \mathbf{u} - \nabla q$
$\longrightarrow$ needs for Compatible Discrete Operators
(grad(curl) = 0 and div(grad) = 0) on staggered grids.

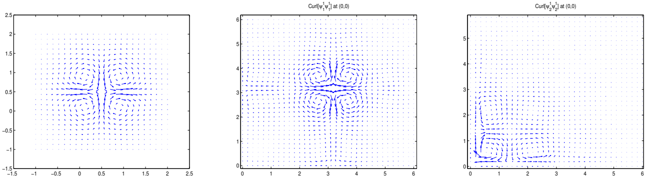
- If $\Omega = [0,1]^n$, use **divergence free/curl free wavelets** !

Divergence-free function basis of $\mathcal{H}_{div}(\Omega) = curl(H_0^1(\Omega))$

Scaling functions : $\Phi^{div}_{j_0,\mathbf{k}} = \mathbf{curl}[\varphi^D_{j_0,k_1} \otimes \varphi^D_{j_0,k_2}]$



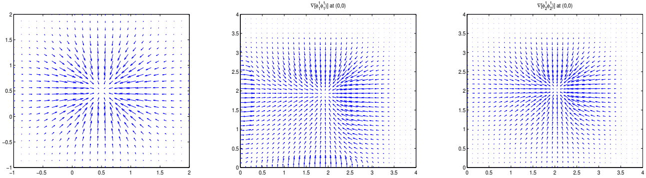
Wavelets : $\Psi^{div}_{\mathbf{j},\mathbf{k}} = \mathbf{curl}[\psi^D_{j_1,k_1} \otimes \psi^D_{j_2,k_2}]$



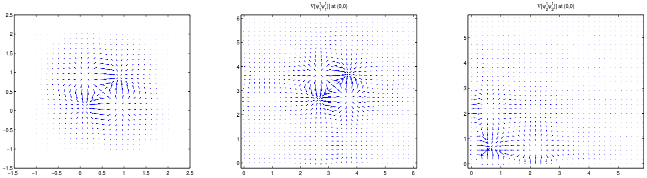
[Kadri-Harouna Perrier 2012]

Curl-free function basis of $\mathcal{H}_{curl}(\Omega) = grad(H_0^1(\Omega))$

Scaling functions : $\Phi^{curl}_{j_0,\mathbf{k}} = \nabla[\varphi^D_{j_0,k_1} \otimes \varphi^D_{j_0,k_2}]$



Wavelets : $\Psi^{curl}_{\mathbf{j},\mathbf{k}} = \nabla[\psi^D_{j_1,k_1} \otimes \psi^D_{j_2,k_2}]$



Helmholtz-Hodge decomposition by wavelets

$$\mathbf{u} = \mathbf{u}_{\text{div}} + \mathbf{u}_{\text{curl}} + \mathbf{u}_{\text{har}}$$

Then : $\langle \mathbf{u} / \Psi^{\text{div}}_{\mathbf{j},\mathbf{k}} \rangle = \langle \mathbf{u}_{\text{div}} / \Psi^{\text{div}}_{\mathbf{j},\mathbf{k}} \rangle$ and $\langle \mathbf{u} / \Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}} \rangle = \langle \mathbf{u}_{\text{curl}} / \Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}} \rangle$

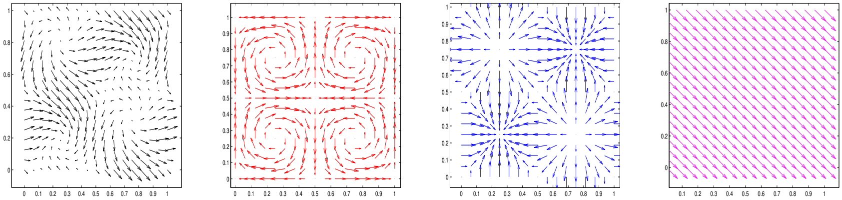
Searching

$$\mathbf{u}_{\text{div}} = \sum_{\mathbf{j},\mathbf{k}} d^{\text{div}}_{\mathbf{j},\mathbf{k}} \; \Psi^{\text{div}}_{\mathbf{j},\mathbf{k}} \quad \text{and} \quad \mathbf{u}_{\text{curl}} = \sum_{\mathbf{j},\mathbf{k}} d^{\text{curl}}_{\mathbf{j},\mathbf{k}} \; \Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}}$$

leads to :

$$(d^{\text{div}}_{\mathbf{j},\mathbf{k}}) = \mathbb{M}^{-1}_{\text{div}}(\langle \mathbf{u} / \Psi^{\text{div}}_{\mathbf{j},\mathbf{k}} \rangle) \quad \text{and} \quad (d^{\text{curl}}_{\mathbf{j},\mathbf{k}}) = \mathbb{M}^{-1}_{\text{curl}}(\langle \mathbf{u} / \Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}} \rangle)$$

($\mathbb{M}_{\text{div}}$ and $\mathbb{M}_{\text{curl}}$: Gram matrices). Finally $\mathbf{u}_{\text{har}} = \mathbf{u} - \mathbf{u}_{\text{div}} - \mathbf{u}_{\text{curl}}$



$$\mathbf{u} \quad = \quad \mathbf{u}_{\text{div}} \quad + \quad \mathbf{u}_{\text{curl}} \quad + \quad \mathbf{u}_{\text{har}}$$

Outline

(i) Incompressible Navier-Stokes equations

$$(NS) \quad \begin{cases} \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \mathbf{p} = \mathbf{f}, & x \in \Omega, \ t \in [0, T] \\ \nabla \cdot \mathbf{v} = 0, & x \in \Omega, \ t \in [0, T] \\ \mathbf{v}(0, x) = \mathbf{v}_0(x), & x \in \Omega \\ \mathbf{v} = 0, & x \in \partial\Omega, \ t \in [0, T] \end{cases}$$

Unknowns : velocity $\mathbf{v}(t, x)$ and pressure $\mathbf{p}(t, x)$

Projecting (NS) onto $\mathcal{H}_{div}(\Omega)$ yields :

$$\partial_t \mathbf{v} = \mathbb{P}[\nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{f}] \qquad (NSP)$$

with $\mathbb{P} : (L^2(\Omega))^n \to \mathcal{H}_{div}(\Omega)$ orthogonal projector.

The pressure $\mathbf{p}$ is recovered through the Helmholtz-Hodge decomposition :

$$\nabla \mathbf{p} = \nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{f} - \mathbb{P}[\nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{f}]$$

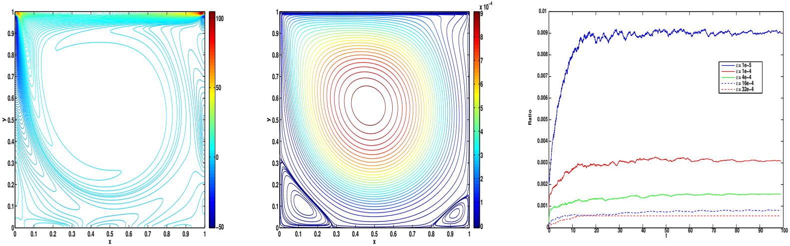
$\longrightarrow$ Modified projection method with spacial approximation :

$$\mathbf{v}(t, x) = \sum_{\mathbf{j}, \mathbf{k}} d^{div}_{\mathbf{j}, \mathbf{k}}(t) \ \Psi^{div}_{\mathbf{j}, \mathbf{k}}(x)$$

`[Kadri-Harouna Perrier 2014]`

Lid Driven Cavity, Re=1000

Divergence-free scaling coefficients :



Vorticity contour (left) and divergence-free scaling function coefficients contour (middle). Steady state for $Re = 1000$ *and* $j = 7$*.*
Evolution in time of the ratio of divergence-free wavelet coefficients up to a fixed ϵ *(right), for* $Re = 1000$ *and* $j = 8$*.*

1 (iii) - Optimal Transport Computation

Set $\rho_0, \rho_1 \in L^\infty$.
$V(Q) = \{f = (\rho, m) \in (L^2(Q))^{1+n},\ \rho \in L^\infty,\ v = \frac{m}{\rho} \in L^1,\ \text{div}_{t,x} f = 0\}$.

$$(BB) \qquad \inf_{(\rho,m)\in V(Q)} \mathcal{J}(\rho, m) \quad with \quad \mathcal{J}(\rho, m) = \int_0^T \int_{[0,1]^n} \frac{|m|^2}{\rho} dxdt$$

- **Helmholtz-Hodge decomposition** of $(\rho, m) \in V(Q)$

$$(\rho, m) = \nabla \times \phi + \nabla h$$

with $\nabla = \nabla_{t,x}$, $\phi = 0$ on ∂Q, and

$$\begin{cases} \Delta h = 0 \text{ in } Q, \\ \frac{\partial h}{\partial n} = (\rho, m) \cdot \nu \text{ on } \partial Q, \end{cases}$$

New functional (h being fixed)

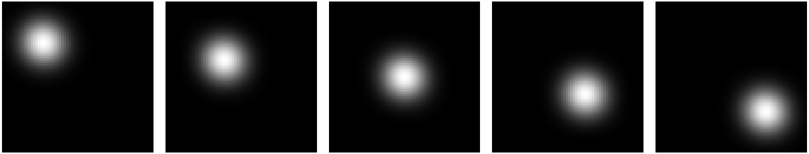
$$J_h(\phi) = \int_0^T \int_{(0,1)^n} F(\nabla \times \phi + \nabla h) dxdt$$

where $F : (X, Y) \mapsto \frac{|Y|^2}{X}$.

Proposition : The functional J_h has better convexity properties than $\mathcal{J}(\rho, m) \longrightarrow$ Primal dual algorithm to J_h [Henry, Maître, P. 2016]

Application to 2D+t (test case)

[Henry, Maître, P. 2016]



t = 0 t = 0.25 t = 0.5 t = 0.75 t = 1

Comparaison between the primal-dual method using HH decomposition (PDHH) and the same primal-dual method of [Papadakis, Peyré, Oudet 2014] *(PDPOP^{gh}), needing a projection at each iteration on a* $64 \times 64 \times 64$ *grid .*

$\|\|\rho_i - \rho_s\|\|$	*PDHH*	*PDPOP^{gh}*	Speedup
10^{-2}	1243 (3'11")	514 (3'31")	9%
10^{-3}	3985 (10'10")	3761 (25'46")	61%
10^{-4}	30569 (1 :21'56")	30349 (3 :27'30")	61%

TABLE – Performance evaluation

Outline

1 - Helmholtz-Hodge decomposition and boundary conditions
- (i) The Helmholtz decomposition, applications
- (ii) Helmholtz/Helmholtz-Hodge decomposition
- (iii) Practical computation of the Helmholtz-Hodge decomposition

2 - Applications
- (i) Navier-Stokes simulation
- (ii) Optimal transport computation

Conclusion

Conclusion

- **Helmholtz-Hodge decomposition occurs in various contexts**
- **Divergence-free and curl-free wavelets allows its practical computation on square/cubic domains**
- **Used for Navier-Stokes simulation (2D, 3D)**
- **Optimal Transport** : (in progress) use of divergence-free wavelets, to provide a new functional in terms of div free wavelet coefficients (preliminary studies with periodic divergence free wavelets using a gradient descent method)