Sampling for Solutions of the Heat Equation

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Problem

Place sensors in a swamp and measure pollution level.

1. Interpolate and determine pollution everywhere.
2. Determine source of pollution.
3. Determine original intensity of pollution.

$u(x, t)$ pollution level at $x \in \mathbb{R}^d$ at time $t > 0$.

Diffusion Equation (Heat Equation)

Tool: $u(x,t)$ is driven by the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \qquad && \text{on } \mathbb{R}^d \times (0,\infty) \\ u(\cdot,0) &= f \qquad && f \in L^p(\mathbb{R}^d). \end{aligned}$$

Its solution is

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-\frac{(x-y)^2}{4t}}\, dy = (f * G_t)(x)$$

with heat kernel

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{x^2}{4t}}.$$

Pollution Scenario

$f(x)$ ... initial *pollution* level at time $t = 0$ and $f \in L^p(\mathbb{R}^d)$
$u(x,t)$ ... pollution at x at time t
Assumption: $\operatorname{supp} f \subseteq B(0,R)$ (compact)

Possible types of given data acquired by the sensors at

$$x_j \in B(0, 2R)$$

(i) Samples at a fixed time: $y_j = u(x_j, t)$ for $j = 1, \dots, n$
(ii) Arbitrary samples: $y_j = u(x_j, t_j)$
(iii) Dynamic sampling: $y_j(t) = u(x_j, t)$ for all $t \in [t_0, t_1]$ (Aldroubi etc.)

Data are samples of a solution of the heat equation.

Possible Estimation Problems

(i) Determine the *spread of pollution*: estimate pollution $u(x,t)$ from samples $u(x_j,t)$ at fixed time t

(ii) Estimate *source* of pollution $\Leftrightarrow$ estimate initial condition f

(iii) Determine *total initial pollution* $\Leftrightarrow$ estimate norm $\|f\|_1$ or $\|f\|_2$

- (i) ... sampling in space of smooth functions (reasonably stable)
- (ii) and (iii) require backwards solution of heat equation: this is extremely ill-conditioned inverse problem.

Error estimate with mesh-width

Lemma (Lack of uniqueness)

Assume that $\delta > 0$ is such that

$$B(0, 2R) \subseteq \bigcup_{j=1}^{n} B(x_j, \delta)$$

and $y_j = (f_1 * G_t)(x_j) = (f_2 * G_t)(x_j)$ for $f_1, f_2 \in L^p(\mathbb{R}^d)$ and $j = 1, \dots, n$. *Two initial conditions yield the same data.* *Then*

$$\|f_1 * G_t - f_2 * G_t\|_p \leq C(t)\, \|f_1 - f_2\|_p \;\; \delta$$

Covering requires $n \approx |B(0, 2R)|/|B(0, \delta)| \asymp \delta^{-d}$ samples and yields

$$\|u(\cdot, t) - f * G_t\|_p = \mathcal{O}(n^{-1/d})$$

Aspects

- Positivity
- Sparsity
- Random sampling

because deterministic results in higher dimensions are hard (and usually weak).

- Linear measurements

$$y_j = u(x_j, t_j) = (f * G_{t_j})(x) = \langle f, G_{t_j}(\cdot - x_j)\rangle \qquad j = 1, \dots, n$$

(use radial basis functions?)

Positivity

Important aspect: pollution level is a non-negative quantity.

$$\boxed{f(x) \geq 0 \text{ and } u(x,t) = f * G_t \geq 0}$$

- crucial for physical model
- not usually assumed in sampling and interpolation problems

Is there Sparsity?

Recipe to introduce sparsity
Initial condition f is taken from a compact set $\mathcal{B}$ in $L^p(B(0,R))$

(i) $\mathcal{B}$ is a compact subset of

$$\{f \in L^1 : f \geq 0, \operatorname{supp} f \subset B(0,R), \|f\|_1 = 1\}$$

and $\mathcal{B}^* = \{\alpha f : \alpha > 0, f \in \mathcal{B}\}$.

(ii) Or

$$\mathcal{B}_0 = \{f \in L^2(B(0,R)) : 1/2 \leq \|f\|_2 \leq 1, \|\nabla f\|_2 \leq 1, f \geq 0\}$$

$$\mathcal{B}_t = \{f * G_t\Big|_{B(0,2R)} : f \in \mathcal{B}_0\} \qquad \text{is compact}$$

Random sampling of solutions I — L^1-theory

- Sample neighborhood $B(0, 2R)$ of $B(0, R)$ randomly at points $x_j \in B(0, 2R)$.

$\{x_j : j = 1, \dots n\}$ i.i.d. random variables and uniformly distributed in $B(0, 2R)$.

- Compact set for initial conditions:

Let $\mathcal{B}$ be a compact subset of

$$\{f \in L^1 : f \geq 0, \operatorname{supp} f \subset B(0, R), \|f\|_1 = 1\}.$$

Let $\mathcal{B}^* = \{\alpha f : \alpha > 0, f \in \mathcal{B}\}$.

Random sampling of solutions I

Theorem (R. Bass, K.G.)

Suppose $x_1, \dots, x_n$ are i.i.d. random variables uniformly distributed on $B(0, 2R)$. Suppose $0 < a \le t_1, \dots, t_r \le b < \infty$ are arbitrary. There exist $A, B, c_1, c_2 > 0$ such that with probability at least

$$1 - c_1 \exp(-c_2 n)$$

we have the sampling inequality

$$nA\|f\|_1 \le \sum_{j=1}^{n} u(x_j, t_j) \le nB\|f\|_1, \qquad f \in \mathcal{B}^*.$$

- Norm of initial condition (= intensity of pollution) can be reliably estimated by sampling heat equation (parameter estimation of $\alpha > 0$).

Idea

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} u(x_j,t) &\approx \mathbb{E}(u(\cdot,t)) \\
&= \frac{1}{\mathrm{vol}(B(0,2R))}\int_{B(0,2R)} u(x,t)\,dx \\
&\approx \frac{1}{\mathrm{vol}(B(0,2R))}\int_{\mathbb{R}^d} u(x,t)\,dx \\
&= \frac{1}{\mathrm{vol}(B(0,2R))}\int_{\mathbb{R}^d} (f * G_t)(x)\,dx \\
&= \frac{1}{\mathrm{vol}(B(0,2R))}\int_{\mathbb{R}^d} f(x)\,dx
\end{aligned}
$$

- probabilistic Bernstein inequality for sums of random variables
- covering numbers

Lemma for comparison of local and global norm

Lemma

Let $R > 0$ be fixed and $1 \le p < \infty$. There exists $b \in (0,1)$ such that if $f \in L^p$, $\operatorname{supp} f \subset B(0,R)$, $f \ge 0$, and $u = f * G_t$. Then

$$\int_{B(0,2R)} u(x,t)^p \, dx \ge b \int_{\mathbb{R}^d} u(x,t)^p \, dx.$$

Comment: $b \asymp \left(CK^{-d} e^{pK^2/t} t^{pd/2} + 1 \right)^{-1}$ for constant $C = \mathcal{O}(1)$.

All constants in Theorems depend explicitly on R, t etc.

Random sampling of solutions II — L^2-Theory

Compact set for sparsity is

$$\mathcal{B}_0 = \{f \in L^2(B(0,R)) : 1/2 \le \|f\|_2 \le 1, \|\nabla f\|_2 \le 1, h \ge 0\}$$

Theorem

Suppose $x_1, \dots, x_n$ are i.i.d. random variables uniformly distributed on $B(0,2R)$. There exist $A, B, c_1, c_2 > 0$ such that with probability at least

$$1 - c_1 e^{-c_2 n}$$

the sampling inequality holds:

$$An\|f\|_2^2 \le \sum_{j=1}^{n} |u(x_j,t)|^2 \le Bn\|f\|_2^2, \qquad u = f * G_t \in \mathcal{B}_0.$$

Summary

- New sampling and interpolation problems coming from PDE
- Interaction between PDEs and sampling theory
- Probabilistic methods
- Still a lots to do ...

Some Ideas for Numerical Approximation and/or Interpolation

Data given by linear measurements

$$y_j = u(x_j, t_j) = \langle f, G_{t_j}(\cdot - x_j)\rangle \qquad j = 1, \dots, n.$$

Possible *least square approximations*:

(i) Smallest solution at time t:

$$\operatorname{argmin}\{\|f * G_t\|_2 : f \in L^2(K)\}$$

(ii) Smallest initial condition:

$$\operatorname{argmin}\{\|f\|_2 : f \in L^2(K)\}$$

(iii) Interpolation of given data:

$$\operatorname{argmin}\Big\{\sum_{j=1}^{n} |y_j - h(x_j)|^2 : h \in \operatorname{span}\{G_{t_j}(\cdot - x_j) : j = 1, \dots, n\}\Big\}$$

Second Approach to Sparsity

Observation: solution $Tf = f * G_t\Big|_{\tilde{B}}$ is self-adjoint and compact.

As we sample $u(x,t) = f * G_t$ on n points in $\tilde{B}$, only eigenfunctions of large eigenvalues should be relevant relevant.

Possible approach:

(a) Choose an (orthonormal) basis of eigenfunctions

$$\{\psi_k, k \in \mathbb{N} : T\psi_k = \lambda_k \psi_k\}$$

(b) Solve

$$\operatorname{argmin}\{\sum_{j=1}^{n} |y_j - h(x_j)|^2 : h \in \operatorname{span}\{\psi_k : k = 1, \dots, N_0\}\}$$

for suitable dimension N_0.

$\rightarrow$ functionals for sampling are different from basis in reconstruction spaces — *generalized sampling* (Adcock-Hansen)