Directional time-frequency analysis via continuous frames

or

Frame recycling

How to re-use 1D frames in higher dimensions

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1. How to generate frames in higher dimensions?

Possible approaches:

- Tensor products of 1D frames
  $\rightarrow$ no directional or other geometric information.
- Native 2D constructions with additional structures:
  ridgelets, curvelets, shearlets, ...
- Idea by Grafakos and Sansing in the setting of continuous Gabor frames:
  Ridges $\rightarrow$ directional information for $\mathbb{R}^n$

Aim of this talk:

Generalizing the ridge idea to more general continuous and discrete frames, e.g. wavelet frames.

[L. Grafakos and C. Sansing. Gabor frames and directional time-frequency analysis. *Appl. Comp. Harm. Anal.*, 25:47–67, 2008.]

Outline

2. Ridge functions

Notation:

For functions $f, g : \mathbb{R}^m \to \mathbb{C}$, $m \in \mathbb{N}$,

- $\widehat{f}$ Fourier transform and $f^\vee$ inverse Fourier transform of f, if defined.
- $\langle f, g \rangle := \int_{\mathbb{R}^m} f(x)\overline{g(x)}\, dx$
  whenever the right hand side converges.

Let $h \in \mathcal{S}(\mathbb{R})$ (or h in a Sobolov space).
Let $u \in \mathbb{S}^{n-1}$, $n \in \mathbb{N}$, be a direction.

- Ridge function h_u on $\mathbb{R}^n$:

$$h_u(x) := h(u \cdot x), \quad x \in \mathbb{R}^n.$$

[A. Pinkus. Interpolation by ridge functions. J. Approx. Theory, 73:218–236, 1993.]

2. Ridge functions

Definition

Consider any function $g \in \mathcal{S}(\mathbb{R})$.

(i) *For given $\alpha > 0$, consider the differential operator*

$$\mathcal{D}^\alpha g := (\widehat{g}(\cdot)|\cdot|^\alpha)^\vee.$$

(ii) *Let*

$$G(s) := \mathcal{D}^{\frac{n-1}{2}} g(s), \quad s \in \mathbb{R}.$$

(iii) *For $u \in \mathbb{S}^{n-1}$, define the weighted ridge function G_u by*

$$G_u(x) := G(u \cdot x), \quad x \in \mathbb{R}^n.$$

These definitions make sense for a large class of non-differentiable functions and for Sobolov spaces $H^\beta(\mathbb{R})$ with $\beta > \alpha$.

3. Continuous frames

A generalization of the widely known (discrete) frames:

Definition (Continuous Frame)

Let $\mathcal{H}$ be a complex Hilbert space and M a measure space with a positive measure μ. A ***continuous frame*** *is a family of vectors $\{f_k\}_{k\in M}$ for which the following hold:*

(i) For all $f \in \mathcal{H}$, the mapping $k \mapsto \langle f, f_k \rangle$ is a measurable function on M.

(ii) There exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \int_M |\langle f, f_k \rangle|^2 \, d\mu(k) \leq B \|f\|^2, \ \forall f \in \mathcal{H}.$$

The continuous frame $\{f_k\}_{k\in M}$ is ***tight*** *if we can choose $A = B$.*

[S. Twareque Ali et al. Continuous frames in Hilbert space. Ann. Physics, 222(1):1–37, 1993.
G. Kaiser. A Friendly Guide to Wavelets. Birkhäuser, Boston, 1994.]

3. Continuous frames

Example:
For a countable set M equipped with the counting measure: Classical frames.

Theorem (Dual continuous frame)

For every continuous frame $\{f_k\}_{k \in M}$, there exists at least one **dual continuous frame** $\{g_k\}_{k \in M}$ *such that each $f \in \mathcal{H}$ has the representation*

$$f = \int_M \langle f, f_k \rangle g_k \, d\mu(k);$$

the integral should be interpreted in the weak sense.

Example:
If $\{f_k\}_{k \in M}$ is a continuous tight frame with bound A, then $\{A^{-1} f_k\}_{k \in M}$ is a dual continuous frame.

3. Continuous frames

The usual notation for translation, modulation, dilation

- $T_a f(x) := f(x - a)$,
- $E_b f(x) := e^{2\pi ibx} f(x)$,
- $D_c f(x) := c^{1/2} f(cx)$,

where $a, b \in \mathbb{R}, c > 0$.

Example: Continuous Gabor Frames

- Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle f_1, E_bT_ag_1\rangle\overline{\langle f_2, E_bT_ag_2\rangle}\,db\,da = \langle f_1, f_2\rangle\langle g_2, g_1\rangle.$$

- Let $g \in L^2(\mathbb{R}) \setminus \{0\}$.

  $\{E_bT_ag\}_{a,b\in\mathbb{R}}$ is a continuous tight frame for $L^2(\mathbb{R})$.

  ($M = \mathbb{R}^2$ equipped with the Lebesgue measure.)

  Frame bound $A = \|g\|^2$.

- Let $g_1, g_2 \in L^2(\mathbb{R})$ with $\langle g_1, g_2\rangle \neq 0$.

  Then $\{E_bT_ag_1\}_{a,b\in\mathbb{R}}$ and $\{\frac{1}{\langle g_1, g_2\rangle}\,E_bT_ag_2\}_{a,b\in\mathbb{R}}$ are dual continuous frames.

Example: Wavelet systems

Let $\{D_a T_b \psi\}_{a\neq 0, b\in\mathbb{R}}$ be a wavelet system for an admissible wavelet $\psi \in L^2(\mathbb{R})$ with admissibility constant C_ψ.

- For all functions $f, g \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle f, D_a T_b \psi\rangle \overline{\langle g, D_a T_b \psi\rangle} \frac{da\, db}{a^2} = C_\psi \langle f, g\rangle.$$

- $\{D_a T_b \psi\}_{a\neq 0, b\in\mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$.

  M = $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ with the Haar measure $\frac{1}{a^2} da\, db$.

- Frame bound is the admissibility constant $A = C_\psi$.

4. Decompositions in $\mathbb{R}^n$ via continuous frames

Frame generators for $\mathbb{R}^n$:

Let $\{g_k\}_{k \in M}$ be a continuous frame for $L^2(\mathbb{R})$.

$\mathcal{D}^{\frac{n-1}{2}} g_k := (\widehat{g}_k(\cdot)|\cdot|^{\frac{n-1}{2}})^\vee$ and $G_k(s) := \mathcal{D}^{\frac{n-1}{2}} g_k(s), \; s \in \mathbb{R}$.

$G_{k,u}(x) := G_k(u \cdot x)$.

Theorem

Let $\{f_k\}_{k \in M}$ and $\{g_k\}_{k \in M}$ be dual continuous frames for $L^2(\mathbb{R})$, consisting of functions in $\mathcal{S}(\mathbb{R})$ or $H^{\alpha+(n-1)/2}(\mathbb{R})$, $\alpha \geq 0$. Then, for $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$,

$$f = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_M \langle f, G_{k,u} \rangle F_{k,u} \, dk \, du.$$

[Grafakos and Sansing 2008 for Gabor systems, i.e., for generators satisfying $\langle g_1, g_2 \rangle \neq 0$.
O. Christensen, BF and P. Massopust: Bull. Aust. Math. Soc. 92 (2015), 268–281, for Sobolev spaces and general frames.]

5. Semi-discrete representations

Theorem

Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let M be a countable index set.

(i) *Let $\{g_k\}_{k\in M} \subset \mathcal{S}(\mathbb{R})$ denote a frame for $L^2(\mathbb{R})$ with bounds A, B. Then*

$$2A\,\|f\|^2 \leq \int_{\mathbb{S}^{n-1}} \sum_{k\in M} |\langle f, G_{k,u}\rangle|^2\, du \leq 2B\|f\|^2.$$

(ii) *Assuming that $\{g_k\}_{k\in M}$ and $\{f_k\}_{k\in M}$ are dual frames for $L^2(\mathbb{R})$, both consisting of functions in $\mathcal{S}(\mathbb{R})$, then*

$$f = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \sum_{k\in M} \langle f, G_{k,u}\rangle F_{k,u}\, du.$$

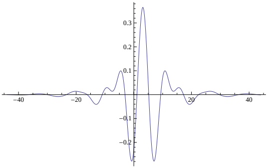
(i) and (ii) also hold for frames $\{g_k\}_{k\in M}$ and $\{f_k\}_{k\in M}$ in Sobolev space $H^\alpha(\mathbb{R})$, $\alpha > 0$.

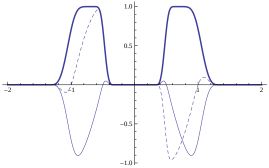
Example: The Meyer wavelet ψ

$$\widehat{\psi}(\gamma) := e^{-i\pi\gamma}(w(2\pi\gamma) + w(-2\pi\gamma))$$

with

$$w(y) := \begin{cases} \sin\left(\frac{\pi}{2}\nu(\frac{3y}{2\pi} - 1)\right), & \text{for } \frac{2\pi}{3} \leq y \leq \frac{4\pi}{3}, \\ \cos\left(\frac{\pi}{2}\nu(\frac{3y}{2\pi} - 1)\right), & \text{for } \frac{4\pi}{3} \leq y \leq \frac{8\pi}{3}, \\ 0, & \text{elsewhere,} \end{cases}$$

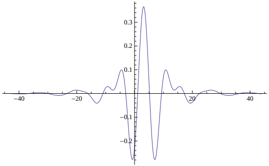


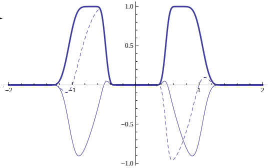


with $\nu : \mathbb{R} \to [\,0, 1\,]$ a smooth enough function of sigmoidal shape:
$\nu(y) = 0$ for $y \leq 0$,
$\nu(y) = 1$ for $y \geq 1$, and
$\nu(y) + \nu(1 - y) = 1$.

5. Semi-discrete representations

Example: The Meyer wavelet ψ





- ψ can be chosen a Schwartz function or at least $H^\alpha(\mathbb{R})$.

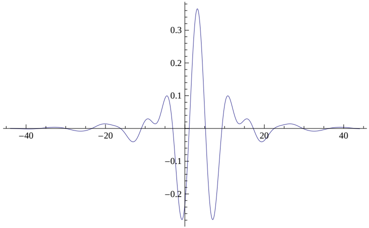
  $$\{\psi_{k,m}\}_{k,m\in\mathbb{Z}} := \{2^{-m/2}\psi(2^{-m}\cdot - k) \mid k, m \in \mathbb{Z}\}$$

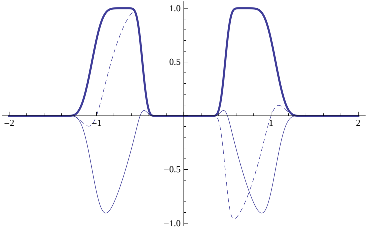
  is an orthonormal wavelet basis for $L^2(\mathbb{R})$.
- In particular, $\{\psi_{k,m}\}_{k,m\in\mathbb{Z}}$ is its own dual frame.
- Thus, we can apply the Theorem:
  The ridge frame elements $G_{k,u} = F_{k,u}$
  have the form

$$\begin{aligned}\Psi_{k,m,u}(x) &:= \Psi_{k,m}(u\cdot x) = \mathcal{D}^{\frac{n-1}{2}}\psi_{k,m}(u\cdot x)\\ &= (|\cdot|^{\frac{n-1}{2}}\widehat{\psi_{k,m}})^\vee(u\cdot x), \qquad k, m \in \mathbb{Z},\ u \in \mathbb{S}^{n-1}.\end{aligned}$$

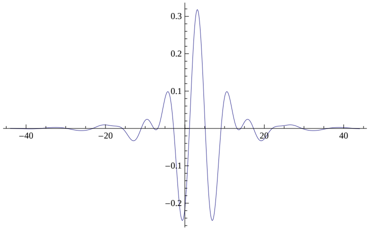
5. Semi-discrete representations

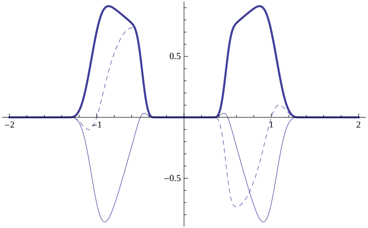
Meyer wavelet ψ





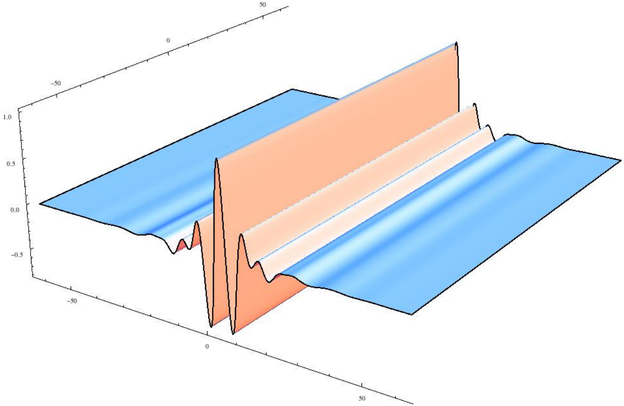
Differentiated wavelet $\mathcal{D}^{\frac{1}{2}}\psi$





5. Semi-discrete representations

2D ridge frame generator $\Psi_{0,0,u}$ with $u = (1, 0)^T$



Another Example: Complex B-splines β_z and their wavelets

Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$.

$$\widehat{\beta}_z(\gamma) := \left(\frac{1 - e^{-2\pi i \gamma}}{2\pi i \gamma} \right)^z .$$

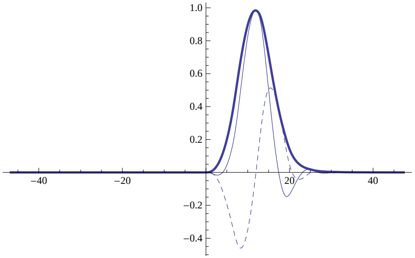
Interpretation:
Approximate single band frequency analysis

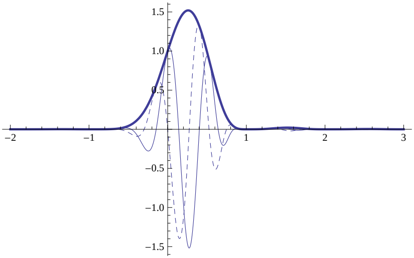
$$\widehat{\beta}_z(\gamma) = \widehat{\beta}_{\operatorname{Re} z}(\gamma)\, e^{i \operatorname{Im} z \ln |\Omega(\gamma)|}\, e^{-\operatorname{Im} z \arg \Omega(\gamma)}.$$

$\operatorname{Im} z$ enhances the positive or the negative frequency spectrum, depending on the sign of $\operatorname{Im} z$.

5. Semi-discrete representations

Complex B-spline for $z = 3.5 + i$





[B. F., T. Blu, and M. Unser. Complex B-splines. *Appl. Comp. Harmon. Anal.*, 20:281–282, 2006.]

Orthonormal complex B-splines

β_z generate a multiresolution analysis $\{V_k \mid k \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$.

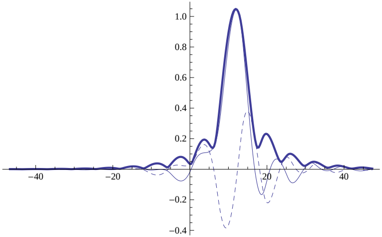
$\{\beta_z(\cdot - \ell) \mid \ell \in \mathbb{Z}\}$ is a Riesz basis for V_0.

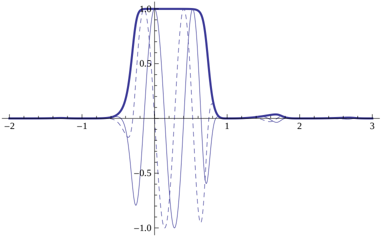
Orthonormalization
via the autocorrelation filter:

$$A_z(\gamma) = \sum_{k\in\mathbb{Z}} |\widehat{\beta}_z(\gamma + k)|^2.$$

Orthonormal complex B-spline:

$$\widehat{\beta}_{z,\perp}(\gamma) = \widehat{\beta}_z(\gamma)/\sqrt{A_z(\gamma)}$$





5. Semi-discrete representations

Complex B-spline wavelets

Scaling filter:

$$H_z(\gamma/2) = \frac{\widehat{\beta}_{z,\perp}(\gamma)}{\widehat{\beta}_{z,\perp}(\gamma/2)}.$$

Associated orthonormal wavelet $\psi_{z,\perp}$:

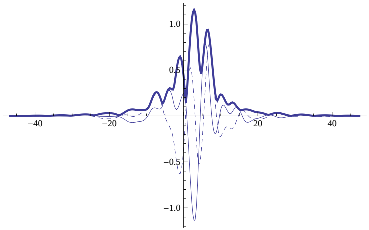
$$\widehat{\psi}_{z,\perp}(\gamma) = -e^{-i\pi\gamma}\, \overline{H_z((\gamma+1)/2)}\, \widehat{\beta}_{z,\perp}(\gamma/2).$$

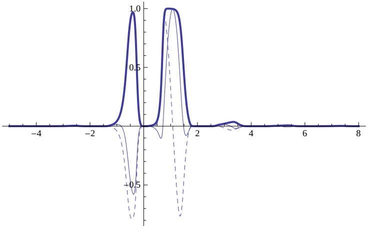
Associated ridge wavelet:

$$\Psi_{z,\perp;u}(x) := \Psi_{z,\perp}(u \cdot x) = \mathcal{D}^{\frac{n-1}{2}} \psi_{z,\perp}(u \cdot x) = (|\cdot|^{\frac{n-1}{2}} \widehat{\psi_{z,\perp}})^{\vee}(u \cdot x)$$

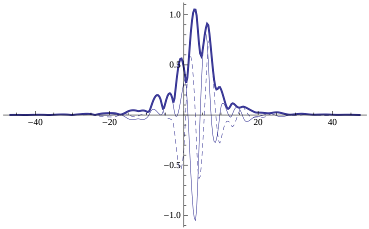
generates a ridge wavelet frame of $L^2(\mathbb{R}^n)$.

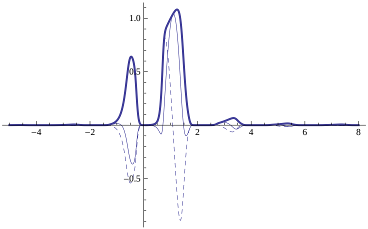
Orthonormal complex B-spline wavelet





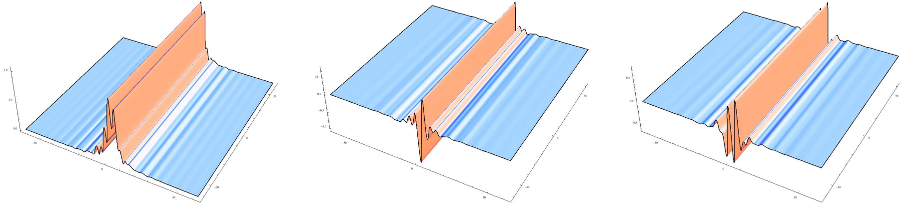
Associated complex ridge wavelet





5. Semi-discrete representations

3D complex spline ridge wavelet:



Modulus, real and imaginary part.

6. Discrete representations

Last step: Discretization of the sphere, i.e., the directions of the ridges. E.g. via ε-nets.

Definition

Let (X, d) be a metric space and a discrete set $N \subset X$. Given any $\epsilon > 0$, the set N is called an ε–net for X if

(a) $\inf\{d(y, y') \mid y \neq y' \in N\} \geq \varepsilon$;

(b) $\inf\{r \mid \bigcup_{y \in N} \overline{B}_r(y) \supseteq X\} \leq \varepsilon$, *where $\overline{B}_r(y)$ denotes the closed ball of radius $r > 0$ centered at y.*

An ε-net is called finite if N is a finite set.

We follow the construction by Candès.

[E. Candès. Harmonic analysis of neural networks. *Appl. Comp. Harm. Anal*, 6:197–218, 1999.]

General setup:

Let $g \in \mathcal{S}(\mathbb{R})$ and assume that

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{|\widehat{g}(\gamma)|^2}{|\gamma|^n} d\gamma < \infty.$

$\Rightarrow G := \mathcal{D}^{\frac{n-1}{2}} g$ satisfies the admissibility condition.

(ii) $\displaystyle\inf_{1 \le |\gamma| \le a_0} \sum_{k=0}^{\infty} \left|\widehat{g}(a_0^{-k}\gamma)\right|^2 \left|a_0^{-k}\gamma\right|^{-2(n-1)} > 0;$

(iii) $\left|\widehat{g}(\gamma)\right| \le K|\gamma|^{\alpha} (1 + |\gamma|)^{-\beta}$, for some $K > 0$, $\alpha > \frac{n-1}{2}$ and $\beta > \alpha + \frac{n+3}{2}$.

6. Discrete representations

Theorem

Let $Q := [-1, 1]^n \subset \mathbb{R}^n$.

Let $g \in \mathcal{S}(\mathbb{R})$ *be as in the general setup and let* $G := \mathcal{D}^{\frac{n-1}{2}} g$.

Then there exists $b_0 > 0$ *so that for any* $b < b_0$*, we can find constants* $A, B > 0$*, such that*

$$A\|f\|^2_{L^2(Q)} \leq \sum_{k \in I} \sum_{u \in S_k^{n-1}} \sum_{\ell \in \mathbb{Z}} |\langle f, D_{a_k} T_{\ell b} G_u \rangle|^2 \leq B\|f\|^2_{L^2(Q)}$$

for all $f \in L^2(Q)$.

I.e., the orthogonal projection of $\{D_{a_k} T_{\ell b} G_u \mid k \in I;\ \ell \in \mathbb{Z};\ u \in S_k^{n-1}\}$ *onto* $L^2(Q)$ *forms a frame for* $L^2(Q)$.

Conclusions

- Novel approach to directionally sensitive continuous frames for $L^2(\mathbb{R}^n)$ based on ridges.
- Discretization in two steps:
  - Semidiscrete representation with continuous directions;

    Frame bounds recycling up to a factor 2.
  - discrete representation via ε-nets;

    Frame bounds recycling open question.
- Examples on directional continuous wavelet frames based on the Meyer wavelet and on complex B-splines.

[O. Christensen, BF, P. Massopust: Directional time frequency analysis via continuous frames. Bull. Aust. Math. Soc. 92 (2015), 268–281.]