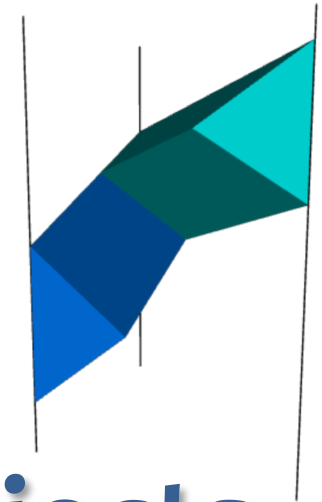
Tel-Aviv University
School of Mathematical Sciences



Reconstruction of 3D Objects from 1D Cross-Sections by Piecewise Linear Interpolation

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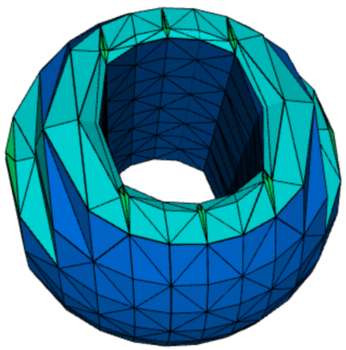

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The Reconstruction Problem

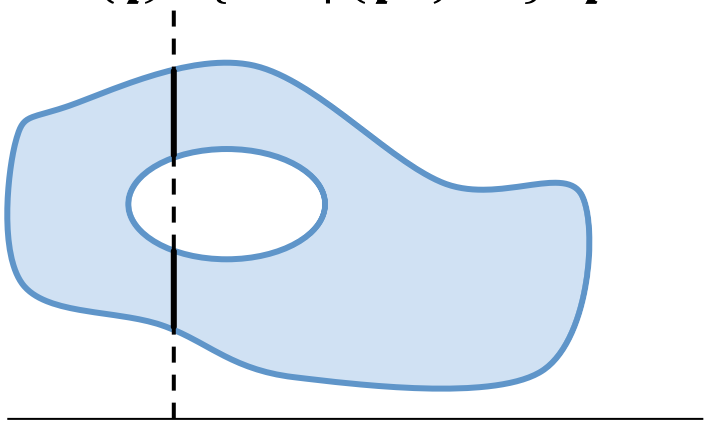
Reconstruction of a 3D object $\boldsymbol{M}$ from a collection of its parallel 1D cross-sections.

Representation of a 3D Object as a Set-Valued Function (SVF)

The sampling plane: $\{(q;0)\in\mathbb{R}^3 : q\in\mathbb{R}^2\}$

Ω – The orthogonal projection of M on the sampling plane

$\boldsymbol{F(q)} := \{\boldsymbol{z}\in\mathbb{R} \mid (\boldsymbol{q};\boldsymbol{z})\in\boldsymbol{M}\},\ \boldsymbol{q}\in\boldsymbol{\Omega}$



The Reconstruction Problem

Assumptions on M

M is bounded inside: $\Omega\times[M_{min}, M_{max}]$

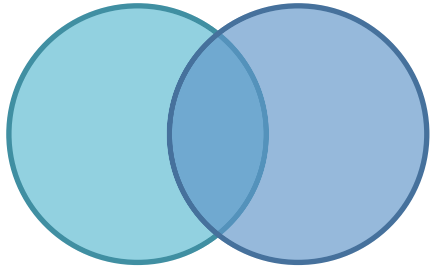
The values of F can be written as:

$F(q) := \bigcup_{i=1}^{I(q)} [a_{2i-1}(q), a_{2i}(q)],\ \ a_{2i} < a_{2i+1},\ \ \sup_{q\in\Omega} \{I(q)\} < \infty$

The Problem: Reconstruct the object M from a collection $\{F(V_l)\}$, $l=1,\ldots,L$ of its samples at the vertices of a triangulation $\mathcal{T}$, $diam(\mathcal{T})\leq h$.

The symmetric difference metric:

$$dist(A,B) = \mu(A \Delta B)$$



$$A \Delta B = (A \setminus \mathrm{B}) \cup (B \setminus \mathrm{A})$$

$\mu(C)$ - measure of C

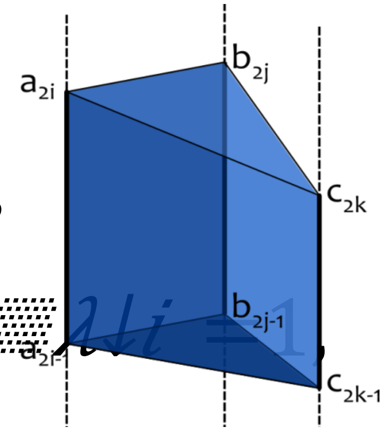
1. Stable – not affected by single points.
2. Every function that is continuous in the symmetric difference metric is continuous in the Hausdorff metric.
3. Approximation results can be obtained for a larger class of functions then when using the Hausdorff metric.

Earlier Works

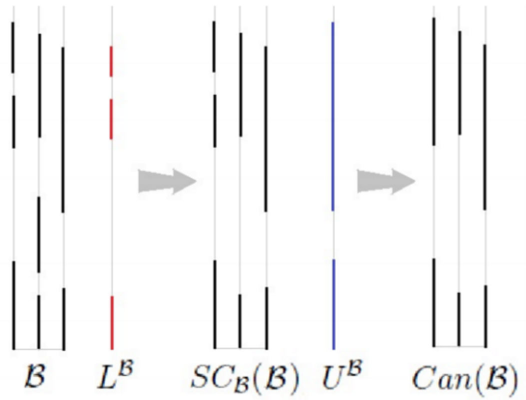
The CPP Method

Minkowski convex combination of intervals

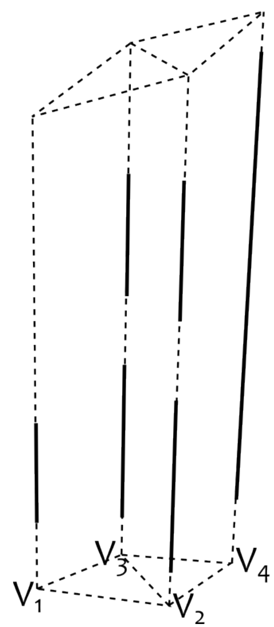
$$\{\lambda_1 I_1 + \lambda_2 I_2 + \lambda_3 I_3 \mid \sum_{i=1}^{3} \lambda_i = 1, \lambda_i \geq 0\}$$

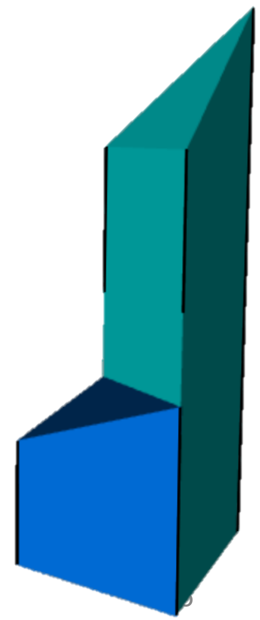


Canonization



Approximation order O(h)





The PLI Methods

Piecewise Linear Interpolant

- The 2D-PLI
- The 3D-PLI

The 2D-PLI

Piecewise Linear Interpolant

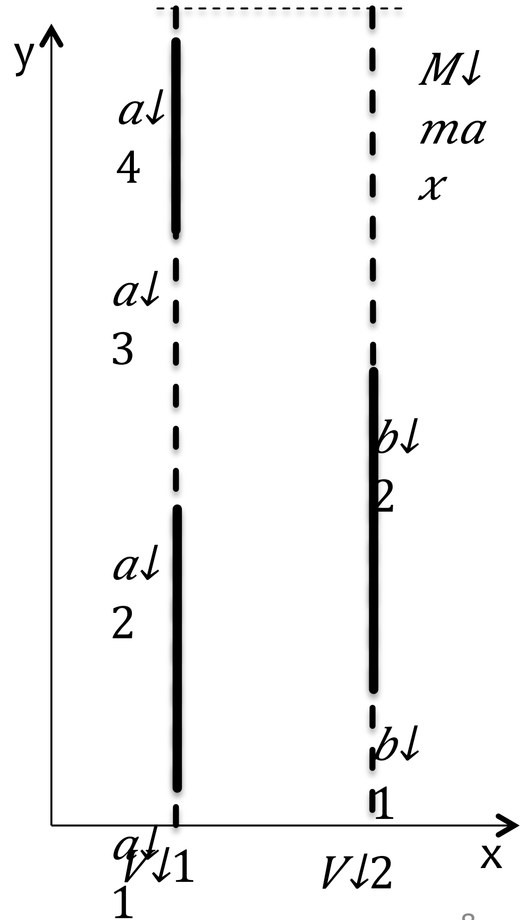
The 2D-Basic Elements

Biconnected 2D-Basic element:

$$[a_{2i-1}, a_{2i}] \cap [b_{2j-1}, b_{2j}] \neq \emptyset$$

Singular 2D-Basic element:

$$[a_{2i-1}, a_{2i}] \subset (b_{2j}, b_{2j+1})$$



The 2D-PLI

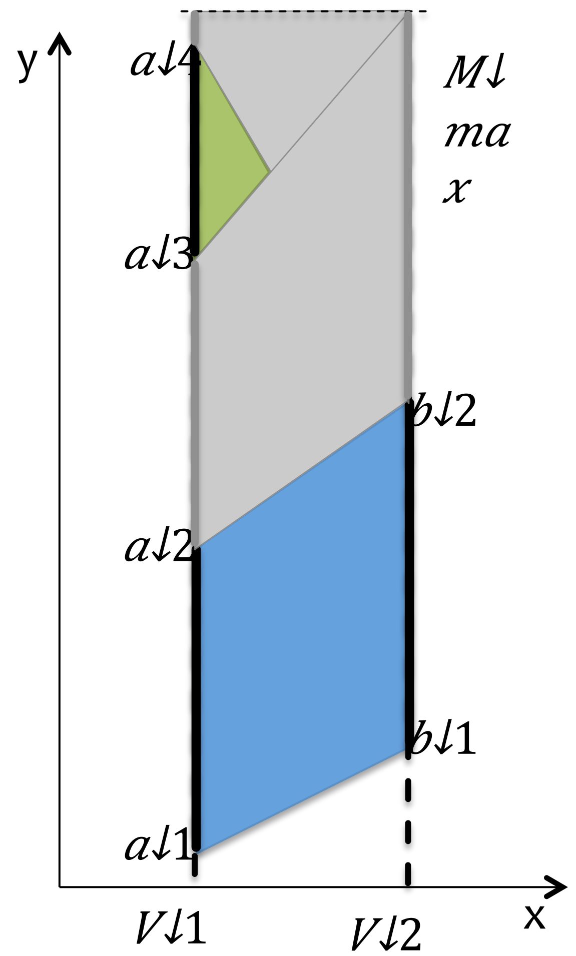
The Construction

For a bicconected 2D-basic element:

$$\mathbf{Tra}([a_{2i-1}, a_{2i}], [b_{2j-1}, b_{2j}])$$

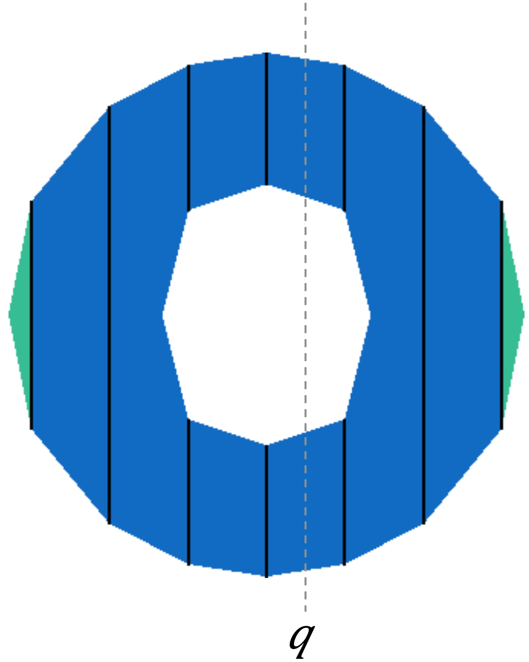
For a singular 2D-basic element:

$$\mathbf{Tri}([a_{2i-1}, a_{2i}], (b_{2j}, b_{2j+1}))$$



The 2D-PLI

Properties

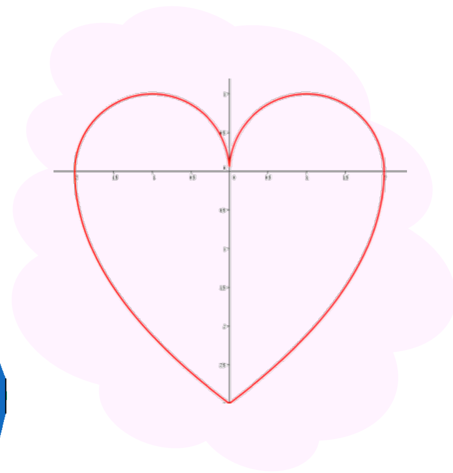


1. Interpolation
2. Duality of the construction
3. Continuity
4. Inclusion $q \in co\{V_1, V_2\}$

   $F(V_1) \cap F(V_2) \subset F(q) \subset F(V_1) \cup F(V_2)$
5. Connectivity preservation
6. Topology preservation

Examples

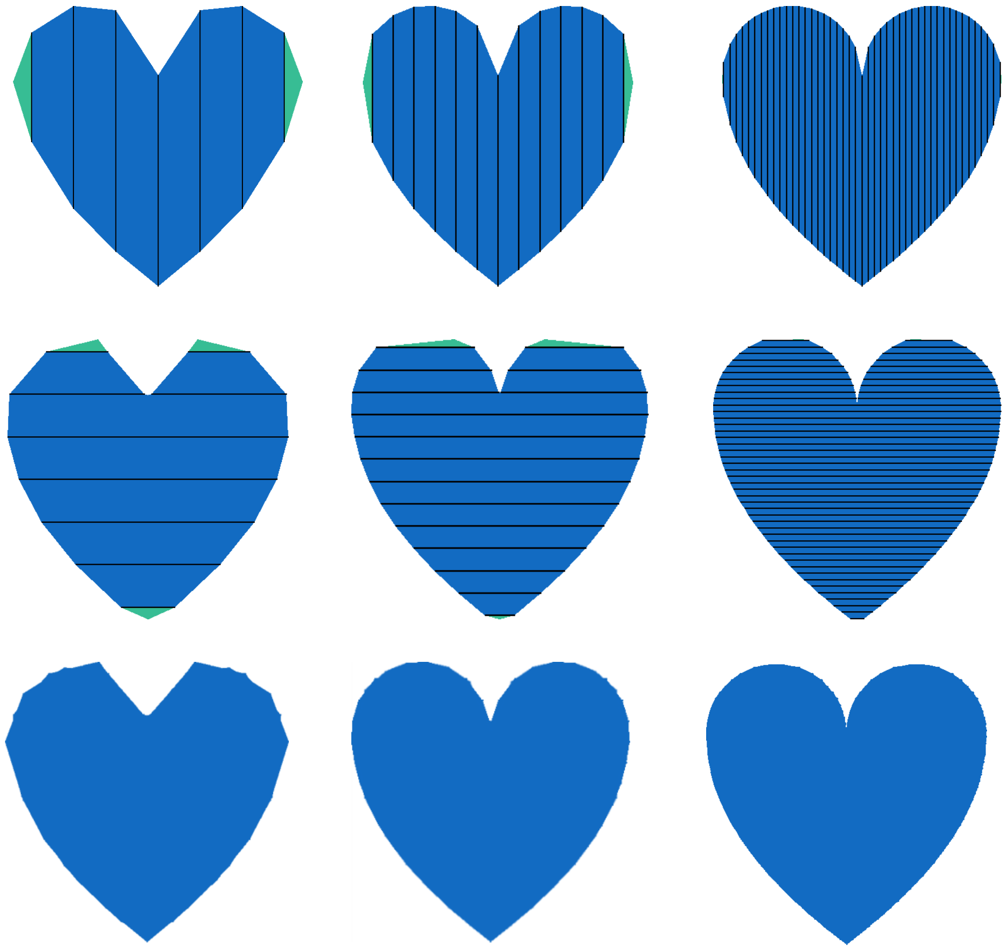
The 2D-PLI



Vertical sampling

Horizontal sampling

Union of vertical & horizontal



h=0.6 h=0.3 h=0.09

The 3D-PLI

The 3D-basic Elements

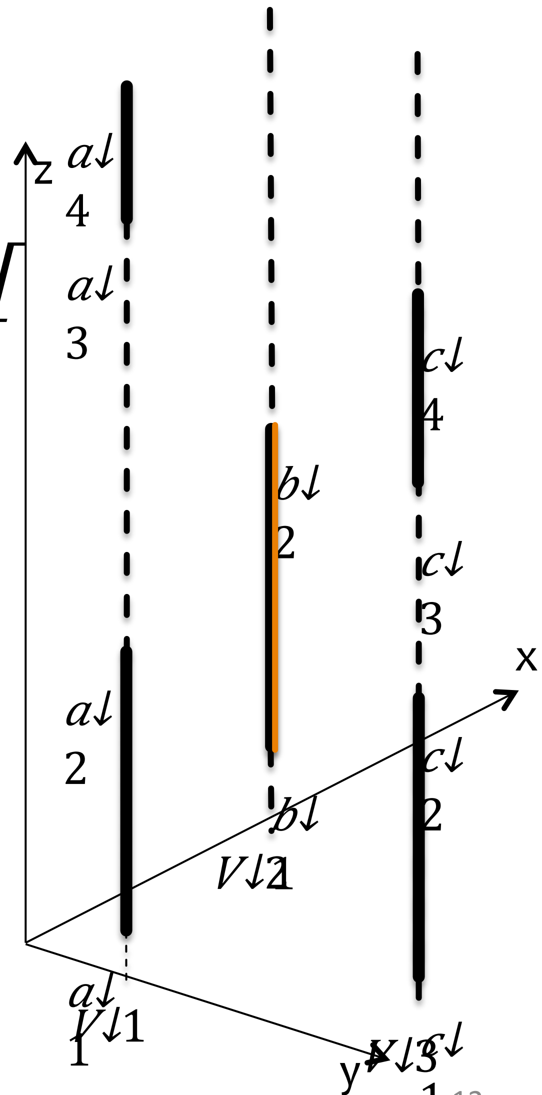
Canonical 3D-basic element:

$$[a_{2i-1}, a_{2i}] \cap [b_{2j-1}, b_{2j}] \cap [c_{2k-1}, c_{2k}] \neq \emptyset$$

Two-fold singular 3D-basic element:

$$[a_{2i-1}, a_{2i}] \subset (b_{2j}, b_{2j+1})$$

$$[a_{2i-1}, a_{2i}] \subset (c_{2k}, c_{2k+1})$$



Pair singular 3D-basic element:

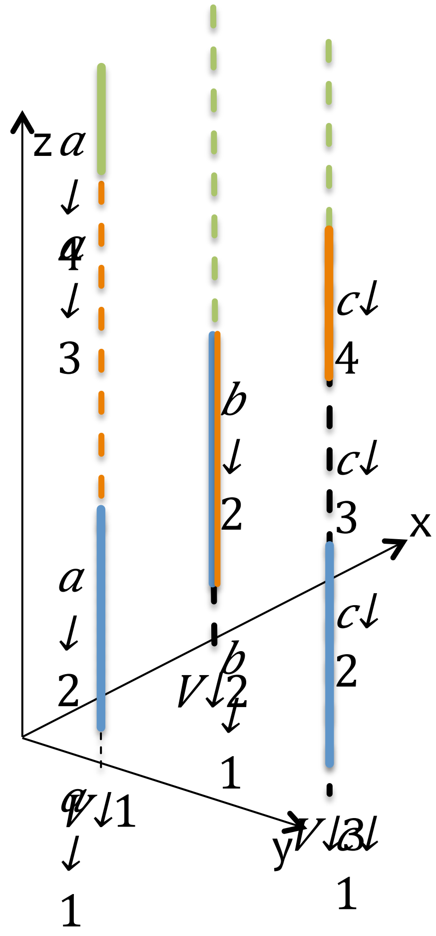
The 3D-PLI

Reconstruction

We apply the appropriate reconstruction rule to each 3D-basic element of every triangle in $\mathcal{T}$. The union of these polyhedrons over the triangles of $\mathcal{T}$ is the 3D-PLI object corresponding to the given samples of M.

We denote by $\mathbf{F}$ the 3D-PLI of F:

$$\mathrm{F}\ (q)=\{z\in\mathbb{R}\,|\,(q;z)\in M\ \}.$$

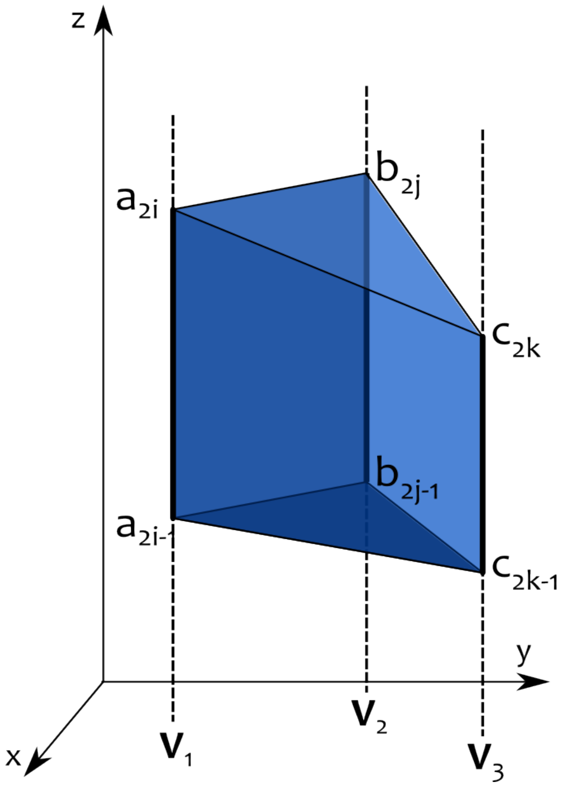


The 3D-PLI

Reconstruction for the 3D-basic elements

A. Canonical 3D-basic element

$$\mathbf{PLI}((V_1;[a_{2i-1},a_{2i}]),(V_2;[b_{2j-1},b_{2j}]),(V_3;[c_{2k-1},c_{2k}]))$$



Inclusion property:

$$F(q)\subset[a_{2i-1},a_{2i}]\cup[b_{2j-1},b_{2j}]\cup[c_{2k-1},c_{2k}]$$

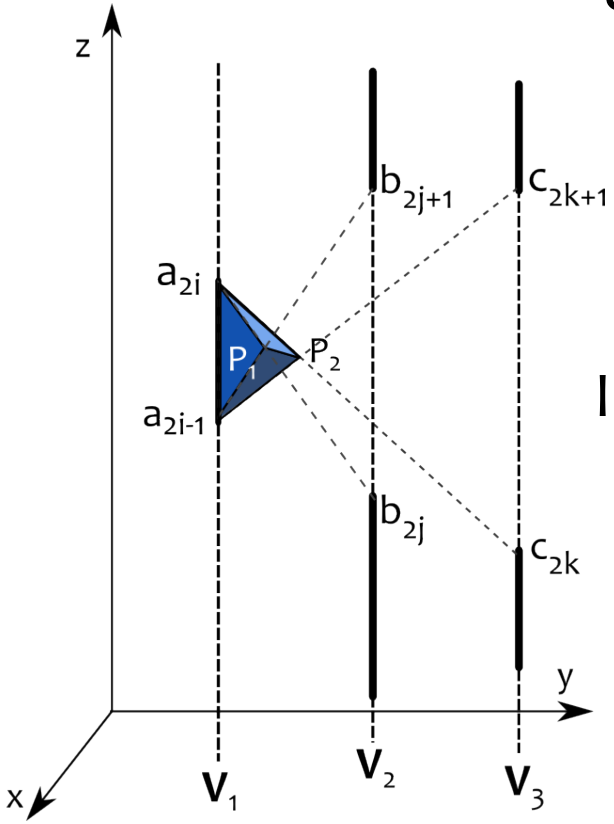
$$F(q)\supset[a_{2i-1},a_{2i}]\cap[b_{2j-1},b_{2j}]\cap[c_{2k-1},c_{2k}]$$

$$\forall\ q\in co\{V_1,V_2,V_3\}$$

The 3D-PLI

Reconstruction for the 3D-basic elements

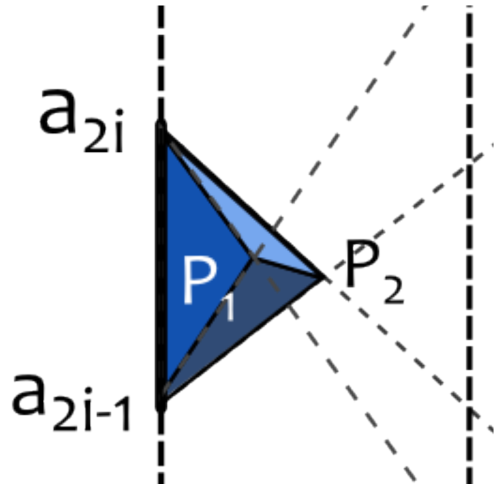
B. Two-fold singular 3D-basic element



Inclusion property:

$$F(q) \subset [a_{2i-1}, a_{2i}]$$

$$\forall\, q \in co\{V_1, V_2, V_3\}$$

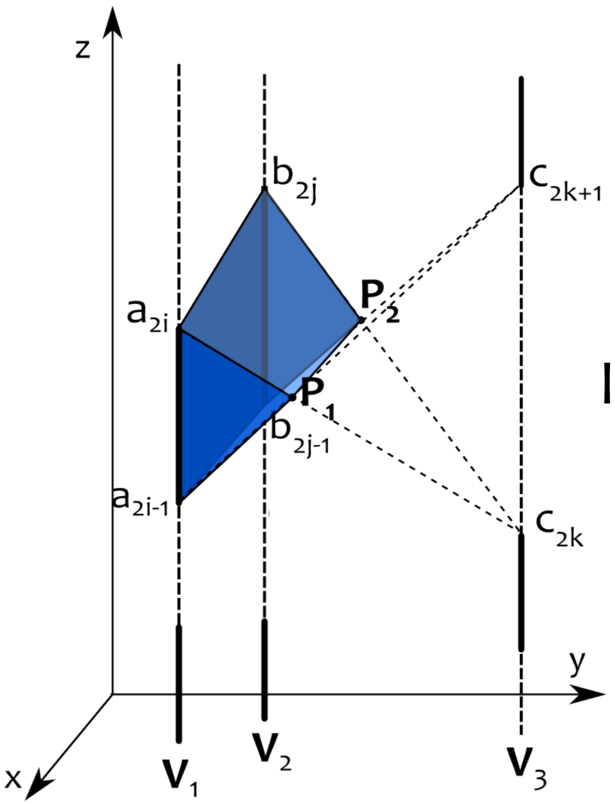


$$\mathbf{PLI}((V_1; [a_{2i-1}, a_{2i}]), (V_2; (b_{2j}, b_{2j+1})), (V_3; (c_{2k}, c_{2k+1})))$$

The 3D-PLI

Reconstruction for the 3D-basic elements

C. Pair-singular 3D-basic element



Inclusion property:

$$F(q) \subset [a_{2i-1}, a_{2i}] \cup [b_{2j-1}, b_{2j}]$$

$$\forall\, q \in co\{V_1, V_2, V_3\}$$

$$\mathbf{PLI}((V_1; [a_{2i-1}, a_{2i}]), (V_2; [b_{2j-1}, b_{2j}]), (V_3; (c_{2k}, c_{2k+1})))$$

The 3D-PLI

Properties

1. Interpolation

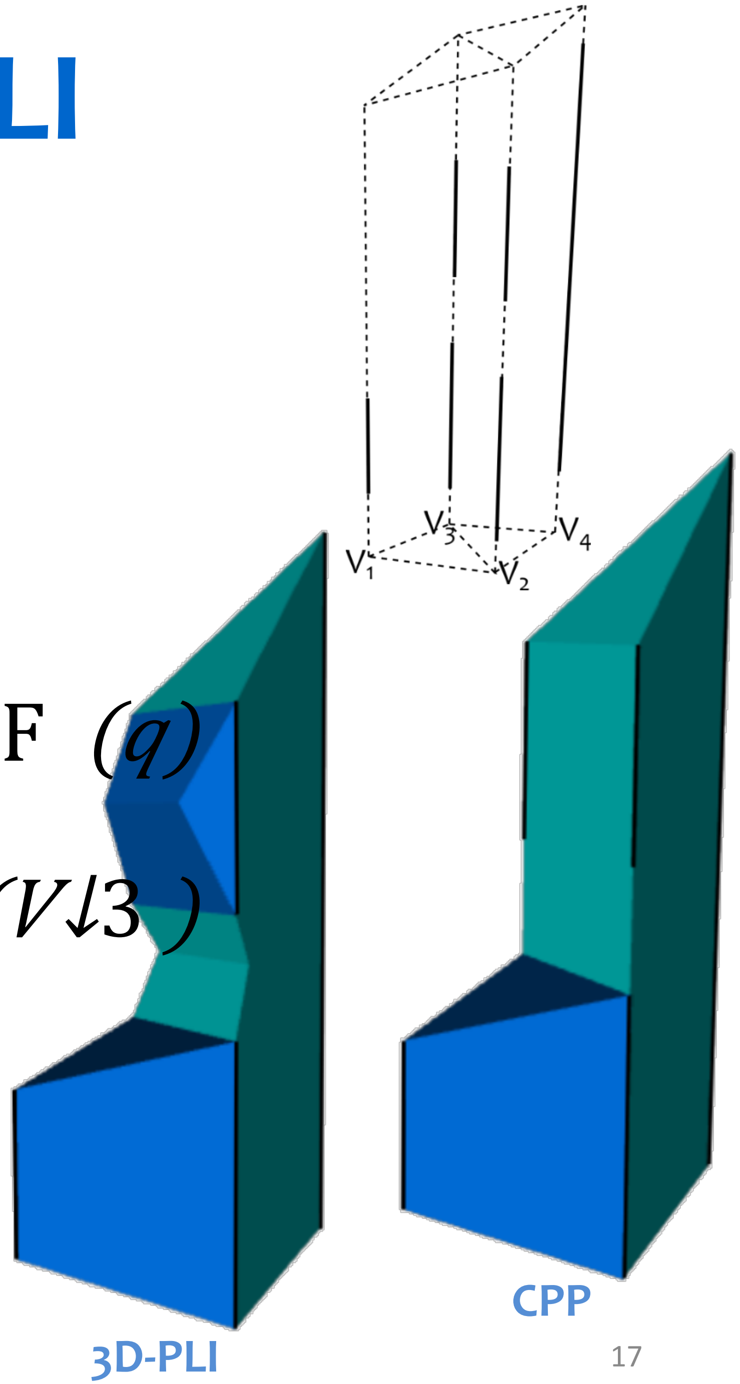
2. Inclusion

$$F(V_1) \cap F(V_2) \cap F(V_3) \subset F(q)$$

$$F(q) \subset F(V_1) \cup F(V_2) \cup F(V_3)$$

$$\forall\, q \in co\{V_1, V_2, V_3\}$$

3. Continuity



4. Connectivity preservation

Approximation order of the 3D-PLI

Approximation Order of the 3D-PLI SVF

$dist(F\ (q),F(q))$

Definition: A multifunction F is **Hölder continuous** if there exist

$L \geq 0$ and $\alpha \in (0,1]$ such that for every $p,q \in \Omega$:

$$dist(F(p),F(q)) \leq L||p-q||\uparrow\alpha\ .$$

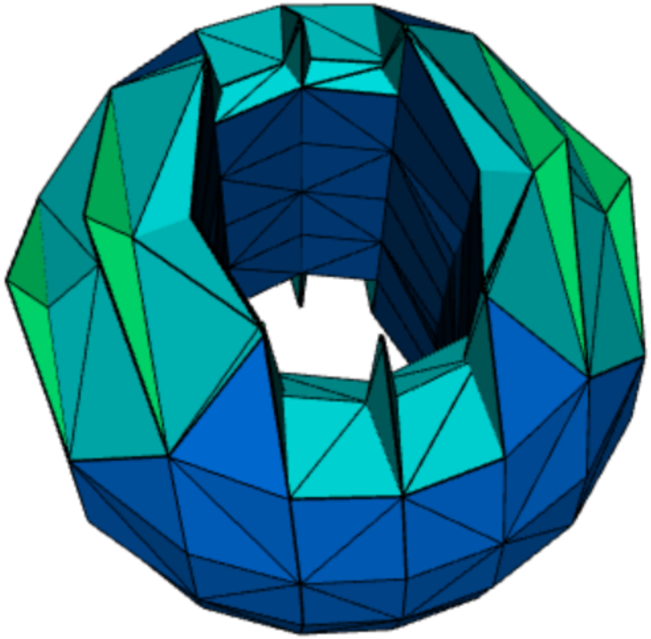
Lemma: If F is Hölder continuous with constants $L,\ \alpha$ in Ω, then $h=diam(T)$

for every p,q in a triangle T in $\mathcal{T}$:

$$dist(F\ (p),F\ (q)) \leq 3Lh\uparrow\alpha\ .$$

Examples

The 3D-PLI



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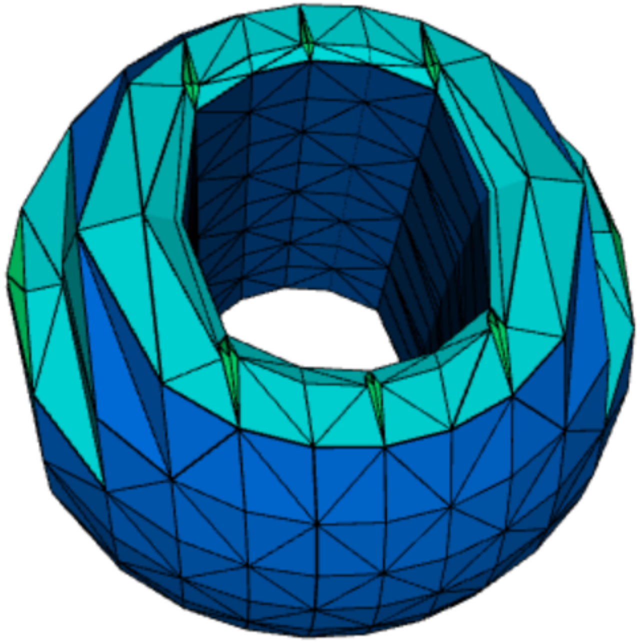


h=0.5

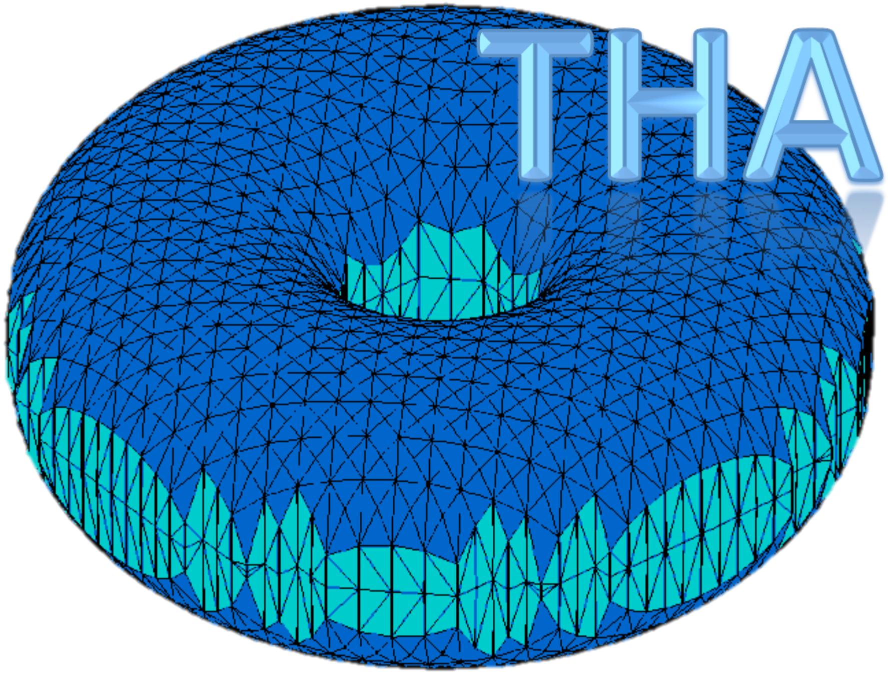
Conclusions

Properties of the Reconstruction Method

1. The reconstructed object **interpolates** the given samples.
2. The reconstructed object is **piecewise linear,** in the sense that on each triangle the object is a union of polyhedrons.
3. The reconstructed SVF is **continuous** in the symmetric difference metric.
4. The method provides an **approximation rate higher than $O(h)$** for some types of objects / SVFs.

Future Work

1. Dealing with **noisy data**.
2. Efficient **computation of the surface** of the approximating object and of the **values of the corresponding SVF**.
3. Improve the visual **appearance** and the **approximation order** of the basic interpolating polyhedrons.
4. Obtain a **subdivision scheme** of 1D sets yielding a smooth approximant.
5. Choosing **optimal sampling direction**.
6. Enhanced method based on **sampling in multiple directions**.



THANK YOU