A Unified Interpolatory Subdivision Scheme for Quadrilateral Meshes

Chongyang Deng* & Weiyin Ma**

dcy@hdu.edu.cn

*Hangzhou Dianzi University, China

**City University of Hong Kong, Hong Kong

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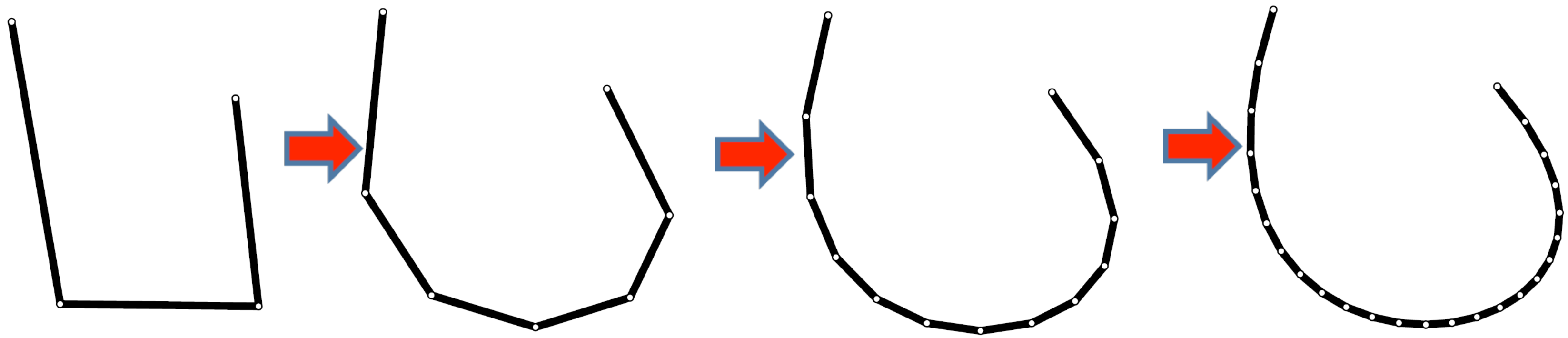
Interpolatory subdivision

- Given a point array, subdivision methods generate smooth curves or surfaces by inserting new points iteratively.
- Interpolatory subdivision:

  old points are fixed in each subdivision step.



Motivation

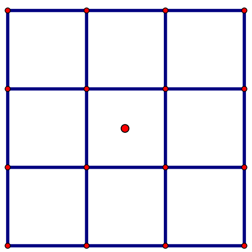
- Higher-order continuity

  ⇨ visually smoother curves and surfaces

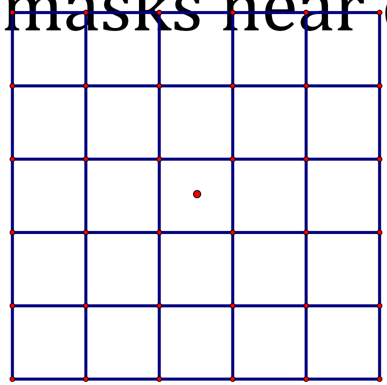
- In general, interpolatory surface schemes are only C^1
- Two difficulties with higher-order schemes

  - Low efficiency due to large local support

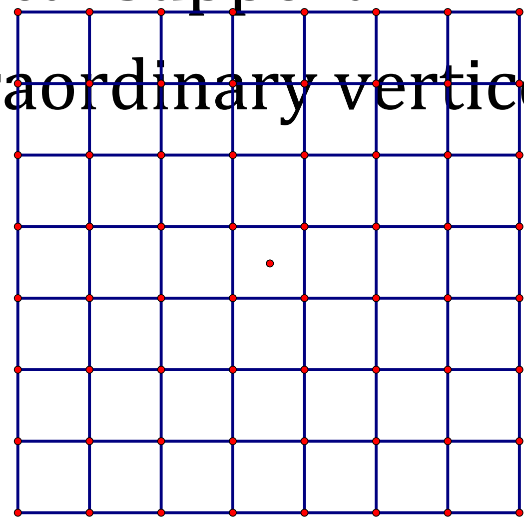
  - Complicated masks near extraordinary vertices



C^1



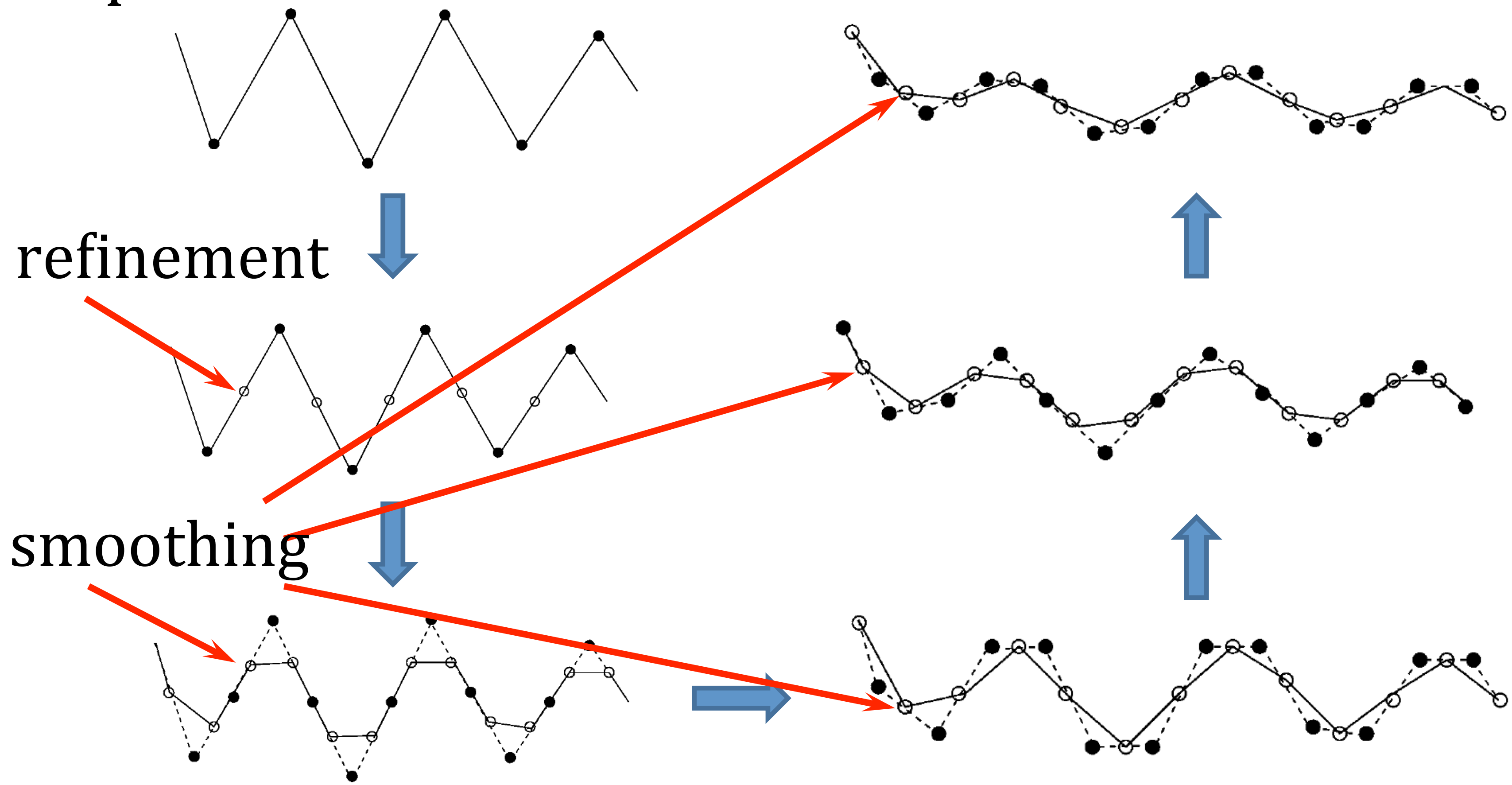
C^2



C^3

Strategy： Using Repeated Local Operations

- Example: Lane-Riesenfeld algorithm for uniform B-spline subdivision



Approximating schemes using Repeated Local Operations

- Generalized high order B-spline (Prautzsch 1998; Zorin and Schröder 2001, Warren and Weimer 2001; Stam 2001)
- Subdivision on triangular meshes (Stam 2001)
- $\sqrt{3}$ -subdivision (Oswald and Schröder 2003)
- Nonuniform B-spline curves (Cashman et al. 2009a; Schaefer and Goldman 2009)
- Nonuniform B-spline surfaces (Cashman et al 2009b)
- Generalized subdivision of B-splines, trigonometric B-splines and hyperbolic B-spline curves (Fang et al. 2010)

How about interpolatory subdivision scheme?

Interpolatory subdivision schemes

- 4-point interpolatory curve scheme (C^1) (Dubuc 1986; Dyn et al. 1987)
- $2n$-point inerpolatory scheme (C^L, L can be arbitrary integer with selected n) (Deslauriers and Dubuc 1989)
- Extensions of 4-point scheme to triangular meshes (C^1) (Dyn et al. 1990; Zorin et al. 1996; Schaefer and Warrren 2002)
- Extensions of 4-point scheme to quadrilateral meshes (C^1) (Kobbelt 1996; Li et al. 2005)
- Extension of $2n$-point scheme to surfaces?

The 2n-point interpolatory scheme

- Given vertices $\{\boldsymbol{P}_i^0\}$, the $2n$-point scheme is iteratively defined by

$$\boldsymbol{P}_{2i}^{k+1} = \boldsymbol{P}_i^k, \boldsymbol{P}_{2i+1}^{k+1} = \boldsymbol{L}_i^k(1/2).$$

  where $\boldsymbol{L}_i^k(x)$ is a $(2n-1)$-degree Lagrange polynomial interpolating adjacent $2n$ points

$$\boldsymbol{P}_{i-n+1}^k, \boldsymbol{P}_{i-n+2}^k, \cdots, \boldsymbol{P}_{i+n}^k$$

  at parameters $-n+1, -n+2, \cdots, n$.
- Continuity
  - Grows linearly with n (Daubechies 1992; Eirola 1992)
  - C^{n-1} for $n \leq 5$ (Eirola 1992)
  - $C^{\approx 0.415n}$ for large n (Eirola 1992)

The 2n-point interpolatory scheme

- Some examples

  - 2-point interpolatory scheme

$$\boldsymbol{Q}_{2i+1}^{1} = \boldsymbol{P}_{2i+1}^{k+1} = \frac{1}{2}\left(\boldsymbol{P}_{i}^{k} + \boldsymbol{P}_{i+1}^{k}\right)$$

  - 4-point interpolatory scheme

$$\boldsymbol{Q}_{2i+1}^{2} = \boldsymbol{P}_{2i+1}^{k+1} = \frac{9}{16}\left(\boldsymbol{P}_{i}^{k} + \boldsymbol{P}_{i+1}^{k}\right) - \frac{1}{16}\left(\boldsymbol{P}_{i-1}^{k} + \boldsymbol{P}_{i+2}^{k}\right)$$

  - 6-point interpolatory scheme

$$\boldsymbol{Q}_{2i+1}^{3} = \boldsymbol{P}_{2i+1}^{k+1} = \frac{150}{256}\left(\boldsymbol{P}_{i}^{k} + \boldsymbol{P}_{i+1}^{k}\right) - \frac{25}{256}\left(\boldsymbol{P}_{i-1}^{k} + \boldsymbol{P}_{i+2}^{k}\right) + \frac{3}{256}\left(\boldsymbol{P}_{i-2}^{k} + \boldsymbol{P}_{i+3}^{k}\right)$$

Recursive relations for 2n-point scheme

- Some examples

  - 2-point interpolatory scheme

$$\boldsymbol{Q}_{2i+1}^{1} = \boldsymbol{P}_{2i+1}^{k+1} = 1/2\,(\boldsymbol{P}_{i}^{k} + \boldsymbol{P}_{i+1}^{k})$$

  - 4-point interpolatory scheme

$$\boldsymbol{Q}_{2i+1}^{2} = \boldsymbol{Q}_{2i+1}^{1} + 1/8\,(2\boldsymbol{Q}_{2i+1}^{1} - \boldsymbol{Q}_{2i-1}^{1} - \boldsymbol{Q}_{2i+3}^{1})$$

$$\boldsymbol{Q}_{2i+1}^{2} = \boldsymbol{Q}_{2i+1}^{1} + 1/8\,(2\boldsymbol{D}_{2i+1}^{1} - \boldsymbol{D}_{2i-1}^{1} - \boldsymbol{D}_{2i+3}^{1}) \text{ if } \boldsymbol{Q}_{2i+1}^{0} = \mathbf{0}$$

  - 6-point interpolatory scheme

$$\boldsymbol{Q}_{2i+1}^{3} = \boldsymbol{Q}_{2i+1}^{2} + 3/16\,(2\boldsymbol{D}_{2i+1}^{2} - \boldsymbol{D}_{2i-1}^{2} - \boldsymbol{D}_{2i+3}^{2})$$

Recursive relations for 2n-point scheme

- In general, we have:

Theorem The generating function of the $2n$-point interpolatory subdivision scheme

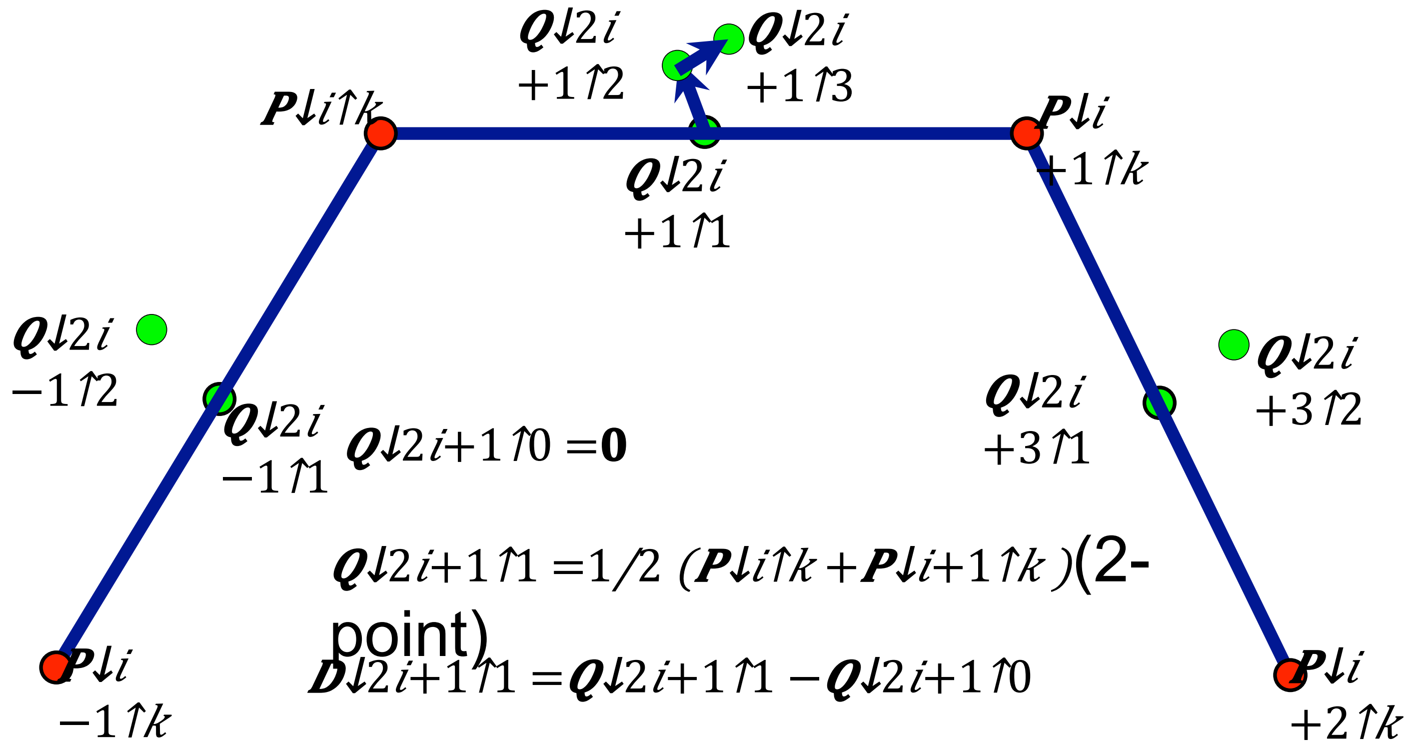
$$f_n(z)=2\sigma(z)^n\sum_{i=0}^{n-1}\binom{n-1+i}{i}\delta(z)^i,$$

where $\sigma(z)=(1+z)^2/4z$, $\delta(z)=-(1-z)^2/4z$,

satisfies the following recursive formula:

$$f_{n+1}(z)=f_n(z)+\mu_n\left[f_n(z)-f_{n-1}(z)\right]\left(2-z^2-1/z^2\right)$$

Repeated Local Operations for 2n-point scheme



$\boldsymbol{Q}_{2i+1}^0 = \boldsymbol{0}$

$\boldsymbol{Q}_{2i+1}^1 = 1/2\,(\boldsymbol{P}_i^k + \boldsymbol{P}_{i+1}^k)$ (2-point)

$\boldsymbol{D}_{2i+1}^1 = \boldsymbol{Q}_{2i+1}^1 - \boldsymbol{Q}_{2i+1}^0$

$\boldsymbol{Q}_{2i+1}^2 = \boldsymbol{Q}_{2i+1}^1 + 1/8\,(2\boldsymbol{D}_{2i+1}^1 - \boldsymbol{D}_{2i-1}^1 - \boldsymbol{D}_{2i+3}^1)$ (4-point)

$\boldsymbol{D}_{2i+1}^2 = \boldsymbol{Q}_{2i+1}^2 - \boldsymbol{Q}_{2i+1}^1$

$\boldsymbol{Q}_{2i+1}^3 = \boldsymbol{Q}_{2i+1}^2 + 3/16\,(2\boldsymbol{D}_{2i+1}^2 - \boldsymbol{D}_{2i-1}^1 - \boldsymbol{D}_{2i+3}^1)$ (6-point)

Repeated Local Operations for 2n-point scheme

- In general, we have repeated-local-operation-based algorithm for $2n$-point subdivision

Step 1 for each i,

$$\boldsymbol{Q}_{2i+1}^{0} = \mathbf{0}, \boldsymbol{Q}_{2i+1}^{1} = 1/2\,(\boldsymbol{P}_{i}^{k} + \boldsymbol{P}_{i+1}^{k})$$

Step 2 for $m=1$ to $m=n-1$

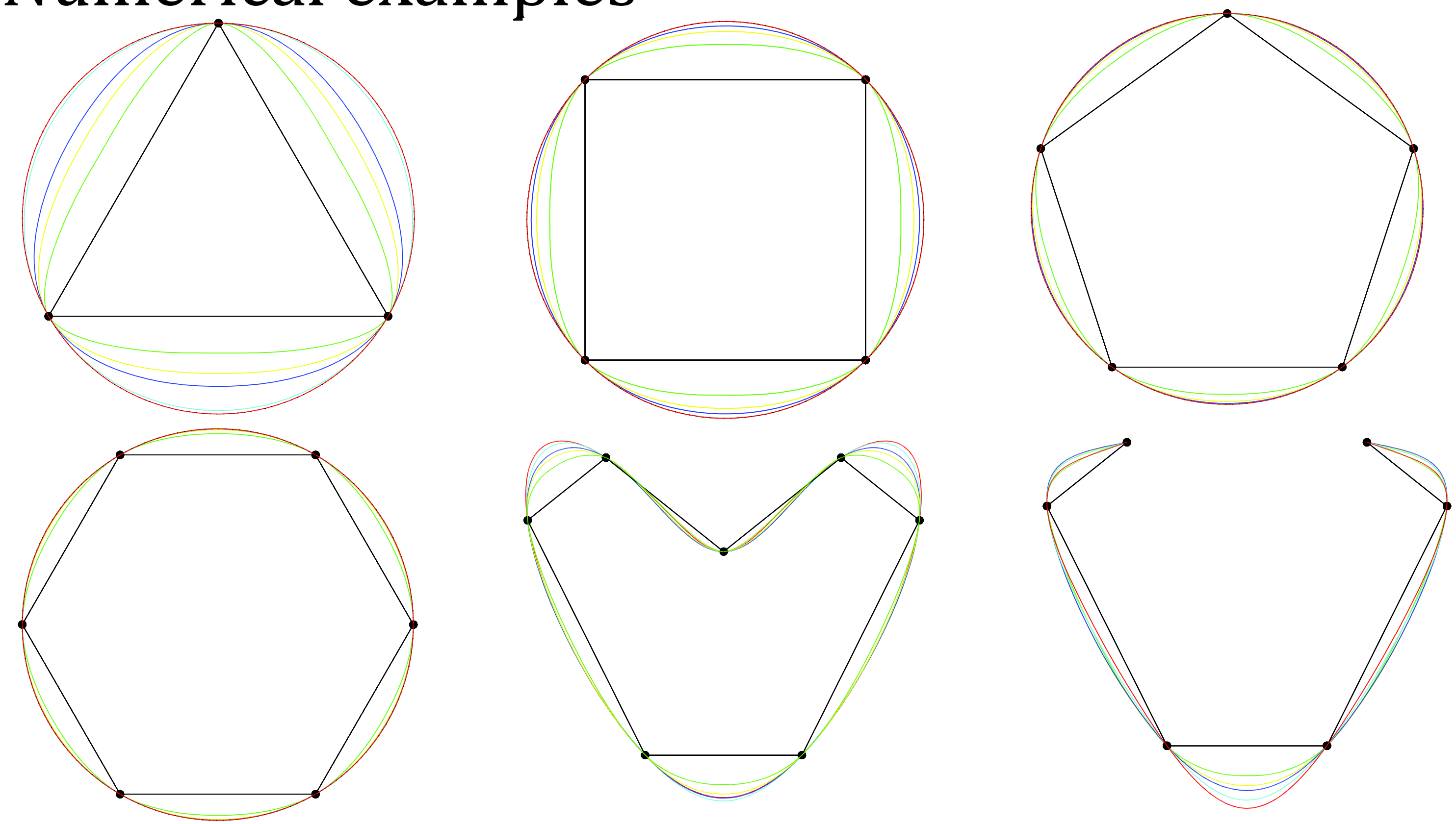
for each i,

$$\boldsymbol{D}_{2i+1}^{m} = \boldsymbol{Q}_{2i+1}^{m} - \boldsymbol{Q}_{2i+1}^{m-1}$$

$$\boldsymbol{Q}_{2i+1}^{m+1} = \boldsymbol{Q}_{2i+1}^{m} + \mu_{m}\,(2\boldsymbol{D}_{2i+1}^{m} - \boldsymbol{D}_{2i-1}^{m} - \boldsymbol{D}_{2i+3}^{m})$$

Repeated Local Operations for 2n-point scheme

- Numerical examples

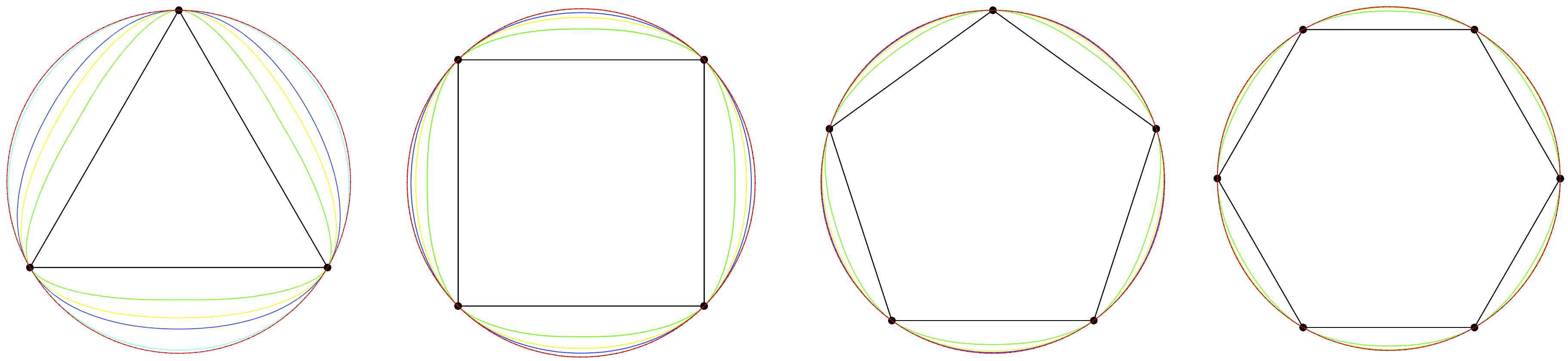


The smaller the support n, the lower the smoothness , and the closer a limit curve follows its control polygon.

Repeated Local Operations for 2n-point scheme

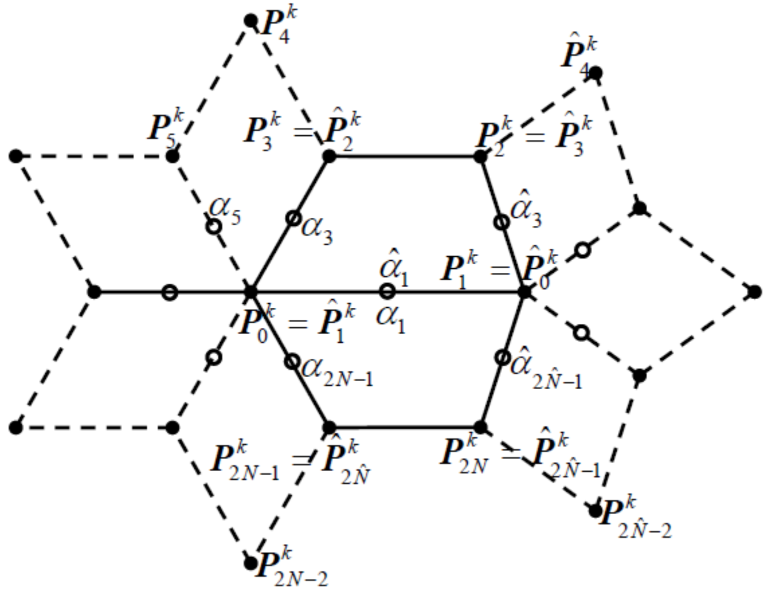
- Proposition

If the initial control vertices form an equilateral polygon, the resulting limit curve of the $2n$-point interpolatory subdivision scheme approaches a circle as n approaches infinity.



Extension to quadrilateral mesh

- Rule for new edge vertices

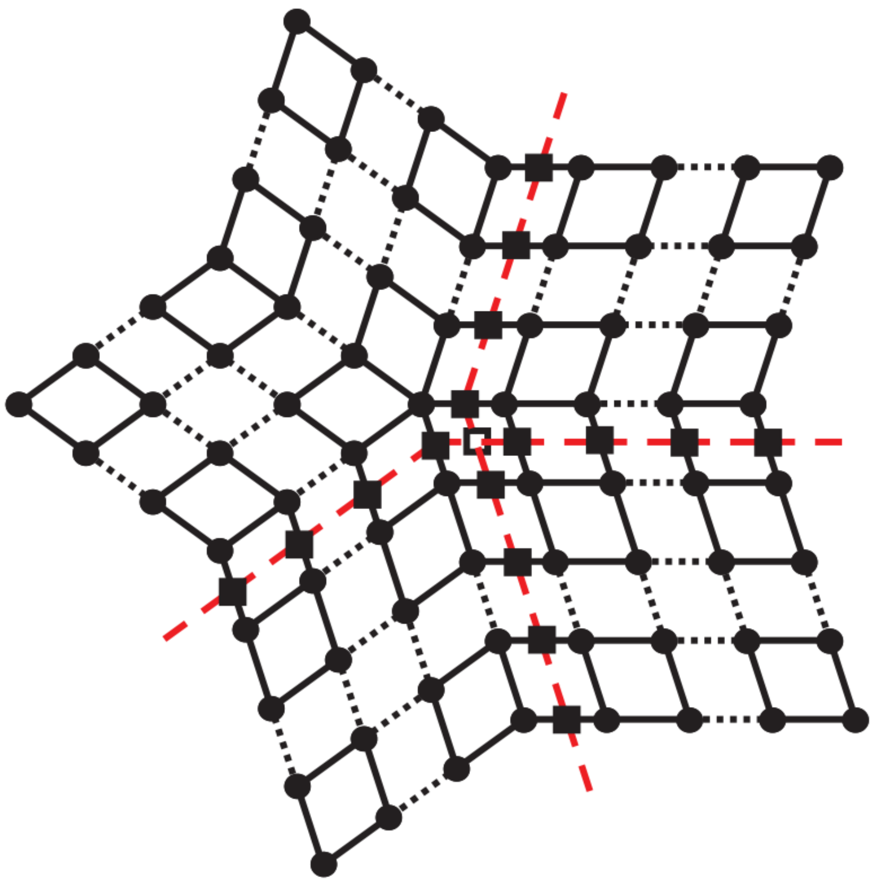


$$Q_1^{m+1} = Q_1^m + \sum_{i=0}^{N-1} \alpha_{2i+1} D_{2i+1}^m + \sum_{j=0}^{N-1} \alpha_{2j+1} D_{2j+1}^m$$

- When $n=2$, it reduces to that of (Li et al. 2005).
- When $N=4$, it is equivalent to the curve case.

Extension to quadrilateral mesh

- Rule for new face vertices



Properties of surface scheme

- The continuity of the limit surface can be of an arbitrary order $C\uparrow L$ at regular vertices.

  - follows from the continuity of $2n$-point scheme.

- The limit surface is $C\uparrow 1$ with bounded curvature at extraordinary vertices.

  - proved for $2\leq n\leq 5,\ 3\leq N\leq 50$.

- The implementation is efficient due to the Repeated Local Operations.

Numerical examples

- For more examples, see [**Chongyang Deng**, Weiyin Ma, A unified interpolatory subdivision scheme for quadrilateral meshes, **ACM Transactions on Graphics**, 2013, 32(3):23:1-11.]

Conclusions and future work

- Repeated Local Operations for $2n$-point interpolatory subdivision scheme
- Extension of $2n$-point scheme to quadrilateral meshes
- Nature rules for surface point?
- Extension of $2n$-point scheme to triangular meshes?
- Extension to other subdivision?

Thanks for your attention!