The unitary extension principle on LCA groups

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HATA: Harmonic Analysis - Theory and Applications
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Plan for the talk

- Frames: If a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert spaces $\mathcal{H}$ is a tight frame with frame bound $A = 1$, then

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \; f \in \mathcal{H}. \qquad \text{General dual frames: } f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k$$

- Wavelet frames $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$
  - The unitary extension principle by Ron & Shen;
- The unitary extension principle on locally compact abelian (LCA) groups, e.g., $\mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_N$.
  - Explicit constructions, typically based on B-splines.

Key point: The unitary extension principle can be generalized to LCA groups, as well on the theoretical level as on the level of concrete constructions.

Frames

Definition: A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a *frame* if

$$\exists A, B > 0: \; A\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2, \; \forall f \in \mathcal{H}.$$

- If $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator $S: \mathcal{H} \to \mathcal{H}$, $Sf = \sum \langle f, f_k\rangle f_k$,

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k\rangle f_k, \;\; \forall f \in \mathcal{H}.$$

- If the frame $\{f_k\}_{k=1}^{\infty}$ is tight, $A = B$, then $S = A\,I$ and

$$f = \frac{1}{A} \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, \;\; \forall f \in \mathcal{H}.$$

- If $\{f_k\}_{k=1}^{\infty}$ is overcomplete, there exist frames $\{g_k\}_{k=1}^{\infty} \neq \{S^{-1} f_k\}_{k=1}^{\infty}$ s.t.

$$f = \sum \langle f, g_k\rangle f_k = \sum \langle f, S^{-1} f_k\rangle f_k, \; \forall f \in \mathcal{H}.$$

Key tracks in frame theory:

- Frames in finite-dimensional spaces;
- Frames in general separable Hilbert spaces
- Concrete frames in concrete Hilbert spaces:
  - Gabor frames in $L^2(\mathbb{R}), L^2(\mathbb{R}^d)$;
  - Wavelet frames;
  - Shift-invariant systems, generalized shift-invariant (GSI) systems;
  - Shearlets, etc.
- Frames in Banach spaces;
- (GSI) Frames on LCA groups
- Frames via integrable group representations, coorbit theory.

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An Introduction to Frames and Riesz bases, 2.edition, Birkhäuser 2016

Operators on $L^2(\mathbb{R})$

Translation by $a \in \mathbb{R}$: $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (T_a f)(x) = f(x-a).$

Modulation by $b \in \mathbb{R}$: $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (E_b f)(x) = e^{2\pi i b x} f(x).$

Dyadic scaling: $D : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (Df)(x) = 2^{1/2} f(2x).$

All these operators are unitary on $L^2(\mathbb{R})$, and

$$T_a E_b = e^{-2\pi i b a} E_b T_a,\ \ T_{b/2} D = D T_b,\ \ D E_{b/2} = E_b D$$

For $f \in L^1(\mathbb{R})$, the *Fourier transform* is defined by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma}\, dx,\ \gamma \in \mathbb{R}.$$

The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R})$, and

$$\begin{aligned} \mathcal{F}T_a &= E_{-a}\mathcal{F},\ \ \mathcal{F}E_a = T_a\mathcal{F}, \\ \mathcal{F}D^{-1} &= D\mathcal{F},\ \ \mathcal{F}D = D^{-1}\mathcal{F}. \end{aligned}$$

Construction of wavelet ONB via MRA

Theorem: Let $\phi \in L^2(\mathbb{R})$, and assume that the following conditions hold:

(i) $\inf_{\gamma \in]-\epsilon,\epsilon[} |\hat{\phi}(\gamma)| > 0$ for some $\epsilon > 0$;

(ii) The scaling equation

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma),$$

is satisfied for a bounded 1-periodic function H_0;

(iii) $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system.

Then ϕ generates a multiresolution analysis, and the function ψ given by

$$\widehat{\psi}(2\gamma) = H_1(\gamma)\widehat{\phi}(\gamma)$$

(with $H_1(\gamma) = \overline{H_0(\gamma + 1/2)}e^{-2\pi i\gamma}$) generates an orthonormal basis $\{D^j T_k\psi\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$. Alternatively, for any $j_0 \in \mathbb{Z}$,

$$\{D^{j_0}T_k\phi\}_{k\in\mathbb{Z}} \cup \{D^j T_k\psi\}_{k\in\mathbb{Z}, j\geq j_0}$$

is an ONB.

Spline wavelets B_N

- The B-splines $B_N,\ N \in \mathbb{N}$, are given by

$$B_1 = \chi_{[-1/2,1/2]},\ \ B_{N+1} = B_N * B_1.$$

- One can consider even order splines B_N and define associated multiresolution analyses, which leads to wavelets of the type

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k B_N(2x+k).$$

- These wavelets are called *Battle–Lemarié wavelets.*
- Only shortcoming: all coefficients c_k are non-zero, which implies that the wavelet ψ has support equal to $\mathbb{R}$.
- Chui & He & Stöckler: There does not exists an ONB or even a tight frame $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ generated by a finite linear combination

$$\psi(x) = \sum c_k B_N(2x+k).$$

The unitary extension principle by Ron & Shen (1997)

Solution: consider systems of the wavelet-type, but generated by more than one function.

Setup for construction of tight wavelet frames by Ron & Shen:

Let $\psi_0 \in L^2(\mathbb{R})$ and assume that

(i) There exists a function $H_0 \in L^\infty(\mathbb{T})$ such that

$$\widehat{\psi}_0(2\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma).$$

(ii) $\lim_{\gamma\to 0} \widehat{\psi}_0(\gamma) = 1.$

Further, let $H_1, \dots, H_n \in L^\infty(\mathbb{T})$, and define $\psi_1, \dots, \psi_n \in L^2(\mathbb{R})$ by

$$\widehat{\psi_\ell}(2\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma),\ \ell = 1, \dots, n.$$

The unitary extension principle

- $\widehat{\psi}_0(2\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma)$.
- $\widehat{\psi_\ell}(2\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma),\ \ell = 1, \ldots, n$.
- We want to find conditions on the functions $H_1, \ldots, H_n$ such that $\psi_1, \ldots, \psi_n$ generate a tight multiwavelet frame for $L^2(\mathbb{R})$.
- Then

$$f = \sum_{\ell=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f, D^j T_k \psi_\ell \rangle D^j T_k \psi_\ell,\ \forall f \in L^2(\mathbb{R}).$$

- Let H denote the $(n+1) \times 2$ matrix-valued function defined by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & H_0(\gamma + 1/2) \\ H_1(\gamma) & H_1(\gamma + 1/2) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & H_n(\gamma + 1/2) \end{pmatrix},\ \gamma \in \mathbb{R}.$$

The unitary extension principle

Theorem (Ron and Shen, 1997): *Let* $\{\psi_\ell, H_\ell\}_{\ell=0}^{n}$ *be as in the general setup, and assume that* $H(\gamma)^* H(\gamma) = I$ *for a.e.* $\gamma \in \mathbb{T}$*. Then the multiwavelet system* $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *constitutes a tight frame for* $L^2(\mathbb{R})$ *with frame bound equal to* 1*. Alternatively, for any* $j_0 \in \mathbb{Z}$,

$$\{D^{j_0} T_k \psi_0\}_{k\in\mathbb{Z}} \cup \{D^j T_k \psi_\ell\}_{k\in\mathbb{Z},\ell=1,\ldots,n,j\geq j_0}$$

is a tight frame with frame bound 1.

Oblique extension principle (2001): equivalent to the UEP, but provides more natural constructions of frames with high approximation orders and optimal number of vanishing moments. Developed by

Daubechies & Han & Ron & Shen, and Chui & He & Stöckler

The unitary extension principle and B-splines

Exmple: For any $m = 1, 2, \ldots,$ we consider the (centered) B-spline

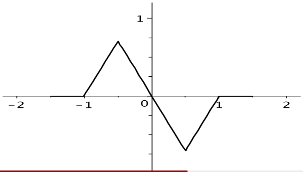
$$\psi_0 := B_{2m}$$

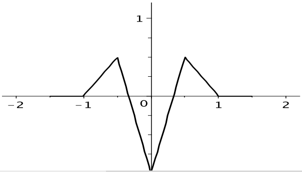
of order $2m$. Then

$$\widehat{\psi}_0(\gamma) = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^{2m}, \ \lim_{\gamma\to 0} \widehat{\psi}_0(\gamma) = 1, \ \widehat{\psi}_0(2\gamma) = \cos^{2m}(\pi\gamma)\widehat{\psi}_0(\gamma).$$

The condition $H(\gamma)^*H(\gamma) = I$ is satisfied with

$$H_\ell(\gamma) = \sqrt{\binom{2m}{\ell}} \sin^\ell(\pi\gamma)\cos^{2m-\ell}(\pi\gamma), \ell = 1, \ldots, 2m.$$





Wavelets and B-splines

Applications to image analysis (restoring, deblurring, inpainting) by Cai, Osher & Shen (2009-2015).

Cai, J. F., Osher, S., and Shen, Z.: *Split Bregman methods and frame based image restoration.* Multiscale Model. Simul., **8** (2009), 337–369.

Cai, J. F., Dong, B., Osher, S., and Shen, Z.: *Image restoration: Total variation, wavelet frames, and beyond.* J. Amer. Math. Soc. **25** (2012), 1033–1089.

Pseudosplines

Pseudosplines (Daubechies & Han & Ron & Shen): based on the filter

$$H_0(\gamma) := (\cos^2)^m \pi\gamma \sum_{k=0}^{\ell} \binom{m+\ell}{k} \sin^{2k}\pi\gamma \, \cos^{2(\ell-k)}\pi\gamma, \; \gamma \in \mathbb{R},$$

where $\ell < m$ are nonnegative integers. and the associated refinable function ψ_0 such that

$$\widehat{\psi_0}(2\gamma) = H_0(\gamma)\widehat{\psi_0}(\gamma).$$

Generalization to Complex pseudosplines (Massopust & Forster & C., 2015), by replacing $m \in \mathbb{N}$ by $z \in \mathbb{C}$ with $\alpha := Re(z) \geq 1$ and $0 \leq \ell \leq \lfloor \alpha \rfloor - 1$. Wavelet frames can be obtained in a similar fashion via the UEP.

Motivation for the generalization (B. Forster): Real-valued transforms can only provide a symmetric spectrum and are therefore unable to separate positive and negative frequency bands. Moreover, real-valued transforms are not applicable in the context of phase retrieval. Here, complex-valued transforms and frames are indispensably needed.

Towards a generalization of the unitary extension principle

- The unitary extension principle provides conditions for a set of functions

$$\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n} = \{2^{j/2}\psi_\ell(2^jx-k)\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$$

  to form a tight frame for $L^2(\mathbb{R})$.
- Let G be a locally compact abelian (LCA) group with Haar measure μ.
  - Typical examples: $\mathbb{R}, \mathbb{R}^s, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_N$;
  - The operator T_a immediately generalizes to $L^2(G)$; $T_af(x)=f(x-a)$
  - The operator E_b has a generalization to $L^2(G)$;
  - The operator D^j is not well defined for $j<0$: $D^{-1}f(x)=2^{-1/2}f(x/2)$???

How can the unitary extension principle be generalized to LCA groups?

Frames on LCA groups

Advantages of the LCA approach:

- Applying various groups $(\mathbb{R}, \mathbb{T}, \mathbb{Z}, \mathbb{Z}_N)$, frames in $L^2(\mathbb{R}), \ell^2(\mathbb{Z}), L^2(0,1)$ and $\mathbb{C}^N$ are obtained as manifestations of a single theory.
- Wavelet frames on $L^2(\mathbb{R})$ and periodic wavelet frames are covered by the same approach
- The group $\mathbb{Z}$ is covered, which leads to frames in $\ell^2(\mathbb{Z})$.
- Generalizations to higher dimensions are provided without any additional notational complication.
- [Gabor case: uniform treatment of various cases treated separatly in the literature]

Towards a generalization of the unitary extension principle

- Assume that $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ is a frame. Applying the Fourier transform we obtain the frame

$$\{\mathcal{F}D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n} = \{E_{k/2^j}\mathcal{F}D^j\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}.$$

  Letting $\Lambda_j := 2^{-j}\mathbb{Z}$, $\Psi_j^\ell := \mathcal{F}D^j\psi_\ell$, we arrive at the frame

$$\begin{aligned}\{\mathcal{F}D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n} &= \{E_\lambda \Psi_j^\ell\}_{\lambda\in\Lambda_j, j\in\mathbb{Z},\ell=1,\dots,n} \\ &= \{E_\lambda \Psi_k^\ell\}_{\lambda\in\Lambda_k, k\in\mathbb{Z},\ell=1,\dots,n}.\end{aligned}$$

- This form can be generalized to LCA groups: indeed, the sets $\Lambda_k = 2^{-k}\mathbb{Z}$ are lattices in the LCA group $\mathbb{R}$, and multiplication with E_λ is a special case of multiplication with a character.

LCA groups

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- A *character* on G is a function $\gamma : G \to \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, for which $\gamma(x+y) = \gamma(x)\gamma(y), \ \forall x, y \in G$.

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- The set of continuous characters is denoted by $\widehat{G}$, and also forms a LCA group, the *dual group* of G, when equipped with an appropriate topology and the composition

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Example: $\widehat{\mathbb{R}} = \mathbb{R}$, and $\widehat{\mathbb{Z}} = \mathbb{T}$.

LCA groups

- Can prove: $\widehat{\widehat{G}} = G$.
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- The *annihilator* $\Lambda^\perp$ of Λ is defined by

$$\Lambda^\perp := \{\gamma \in \widehat{G} \mid (x, \gamma) = 1, \ \forall x \in \Lambda\}.$$

  The annihilator $\Lambda^\perp$ is a closed subgroup of $\widehat{G}$.

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  Example: for $G = \mathbb{R}$, and $\Lambda = b\mathbb{Z}$, we have $\Lambda^\perp = b^{-1}\mathbb{Z}$

Towards a generalization of the unitary extension principle

Recall: Via the Fourier transform, a frame $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ was turned into the frame

$$\{\mathcal{F}D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n} = \{E_\lambda \Psi_k^\ell\}_{\lambda\in\Lambda_k,k\in\mathbb{Z},\ell=1,\dots,n},$$

where

$$\Lambda_k = 2^{-k}\mathbb{Z}, \Psi_k^\ell = \mathcal{F}D^k\psi_\ell.$$

Interpretation: The operators E_λ are multiplications with characters in the LCA group $\mathbb{R}$, and the sets Λ_k are lattices!

More generally: exactly the same procedure turns a frame

$$\{D^{j_0} T_k \psi_0\}_{k\in\mathbb{Z}} \cup \{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n,j\geq j_0}$$

into a frame $\{E_\lambda \Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \cup \{E_\lambda \Psi_k^\ell\}_{\lambda\in\Lambda_k,k\geq k_0,\ell=1,\dots,n}$, where

$$\Phi_k = \mathcal{F}D^k\psi_0.$$

Towards a generalization of the unitary extension principle

Note: by the scaling equation $D\widehat{\psi_0}(\gamma) = 2^{1/2} H_0(\gamma)\widehat{\psi_0}(\gamma)$, so

$$\begin{aligned} \Phi_k(\gamma) = D^{-k}\mathcal{F}\psi_0(\gamma) = D^{-k-1}D\widehat{\psi_0}(\gamma) &= 2^{1/2}D^{-k-1}\left(H_0\widehat{\psi_0}\right)(\gamma) \\ &= H_{k+1}(\gamma)\Phi_{k+1}(\gamma), \end{aligned}$$

where $H_{k+1}(\gamma) := 2^{1/2}H_0(\gamma/2^{k+1})$ satisfies that

$$H_{k+1}(\gamma+\omega) = H_{k+1}(\gamma),\ \omega \in 2^{k+1}\mathbb{Z}.$$

Interpretation: The function H_k is periodic with respect to the lattice $2^k\mathbb{Z} = \Lambda_k^{\perp}$, the annihilator of the lattice $\Lambda_k = 2^{-k}\mathbb{Z}$ indexing the frame

$$\{E_\lambda \Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \cup \{E_\lambda \Psi_k^{\ell}\}_{\lambda\in\Lambda_k, k\geq k_0, \ell=1,\dots,n},$$

i.e.,

$$H_{k+1}(\gamma+\omega) = H_{k+1}(\gamma),\ \omega \in \Lambda_{k+1}^{\perp}.$$

Towards a generalization of the unitary extension principle

- Consider the space $L^2(\widehat{G})$, where the integration is with respect to the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$.
- For $\lambda \in G$, consider the unitary operator

$$\mathcal{M}_\lambda : L^2(\widehat{G}) \to L^2(\widehat{G}),\ (\mathcal{M}_\lambda f)(\gamma) := (\lambda, \gamma) f(\gamma).$$

The operator $\mathcal{M}_\lambda$ generalizes the modulation operator

$$E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ E_b f(x) = e^{2\pi ibx} f(x).$$

The unitary extension principle on LCA groups

General setup:

- Let $\{\Lambda_k\}_{k=k_0}^{\infty}$ be a nested sequence of lattices in G, i.e.,

$$\Lambda_k \subset \Lambda_{k+1}, \ \forall k \geq k_0.$$

- Let V_k denote a fundamental domain associated with the lattice $\Lambda_k^{\perp}$ in $\widehat{G}$, i.e., we have

$$\widehat{G} = \bigcup_{\omega \in \Lambda_k^{\perp}} (\omega + V_k), \quad (\omega + V_k) \cap (\omega' + V_k) = \emptyset \text{ for } \omega \neq \omega', \ \omega, \omega' \in \Lambda_k^{\perp}.$$

- Let $\{\Phi_k\}_{k=k_0}^{\infty}$ be a sequence of functions in $L^2(\widehat{G})$ (the "scaling functions"). For the UEP on $\mathbb{R}$ we had $\Phi_k = \mathcal{F}D^k\psi_0$, but now the functions Φ_k might not be related, i.e., the nonstationary case is included.

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Assume that for some periodic functions $H_{k+1} \in L^\infty(V_{k+1})$ (with $H_{k+1}(\gamma+\omega) = H(\gamma)$ for $\gamma \in \widehat{G}$, $\omega \in \Lambda_{k+1}^\perp$),

$$\Phi_k(\gamma) = H_{k+1}(\gamma)\, \Phi_{k+1}(\gamma),\ \gamma \in \widehat{G}.$$

Given periodic functions $G_{k+1}^{(m)} \in L^\infty(V_{k+1})$, $m = 1, \dots \rho_k$, define the functions $\Psi_k^{(m)} \in L^2(\widehat{G})$, $m = 1, \dots, \rho_k$, by

$$\Psi_k^{(m)}(\gamma) := G_{k+1}^{(m)}(\gamma)\, \Phi_{k+1}(\gamma),\ \gamma \in \widehat{G}. \tag{1}$$

Our goal is to identify conditions on the filters H_k and $G_k^{(m)}$ such that the functions

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0,\, \lambda \in \Lambda_k, m=1,\dots,\rho_k} \tag{2}$$

form a tight frame for $L^2(\widehat{G})$ with frame bound 1.

The unitary extension principle on LCA groups

Technical conditions: For every compact set $S \subset \widehat{G}$ and any $\epsilon > 0$ there exists K such that for all $k \geq K$,

$$|\mu(V_k)\,|\Phi_k(\gamma)|^2 - 1| \leq \epsilon,\ \forall \gamma \in S.$$

and

$$\text{card}\{(\Lambda_k^{\perp} + \gamma) \cap S\} \leq 1,\ \forall\, \gamma \in V_k.$$

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Note:

- The assumption

$$\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots$$

  implies that

$$\cdots \Lambda_2^\perp \subset \Lambda_1^\perp \subset \Lambda_0^\perp.$$

- For each $k \geq k_0$ we can choose a sequence $\{\nu_{k,\ell}\}_{\ell=1,\dots,d_k} \subset \widehat{G}$ such that $\nu_{k,1} = 0$ and

$$\Lambda_k^\perp = \bigcup_{\ell=1}^{d_k} (\nu_{k,\ell} + \Lambda_{k+1}^\perp), \;\; (\nu_{k,\ell} + \Lambda_{k+1}^\perp) \cap (\nu_{k,\ell'} + \Lambda_{k+1}^\perp) = \emptyset \text{ for } \ell \neq \ell'.$$

The unitary extension principle on LCA groups

For $k \geq k_0$, consider the $(\rho_k + 1) \times d_k$ matrix-valued function P_k defined by

$$P_k(\gamma) := \begin{pmatrix} H_{k+1}(\gamma + \nu_{k,1}) & \cdot & \cdot & \cdot & H_{k+1}(\gamma + \nu_{k,d_k}) \\ G^{(1)}_{k+1}(\gamma + \nu_{k,1}) & \cdot & \cdot & \cdot & G^{(1)}_{k+1}(\gamma + \nu_{k,d_k}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G^{(\rho_k)}_{k+1}(\gamma + \nu_{k,1}) & \cdot & \cdot & \cdot & G^{(\rho_k)}_{k+1}(\gamma + \nu_{k,d_k}) \end{pmatrix}, \ \gamma \in V_k.$$

Theorem: (C. & Goh, 2014–2016) In addition to the general setup, assume that for $k \geq k_0$, the matrix-valued function P_k satisfies that

$$P_k(\gamma)^* P_k(\gamma) = \frac{\mu(V_{k+1})}{\mu(V_k)} \, I_{d_k}, a.e. \, \gamma \in V_k.$$

Then the collection

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\ldots,\rho_k}$$

form a tight frame for $L^2(\widehat{G})$ with frame bound 1.

The unitary extension principle on LCA groups

Alternatively, the generalized shift-invariant system

$$\{T_\lambda \mathcal{F}^{-1}\Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \bigcup \{T_\lambda \mathcal{F}^{-1}\Psi_k^{(m)}\}_{k\geq k_0,\lambda\in\Lambda_k,m=1,\dots,\rho_k}$$

forms a tight frame for $L^2(G)$ with frame bound 1.

Key steps in the proof of the UEP

Lemma For any $F \in C_c(\widehat{G})$ and any $\epsilon > 0$, there is a $K \in \mathbb{N}$ such that for $k \geq K$,

$$(1-\epsilon)\|F\|^2 \leq \sum_{\lambda\in\Lambda_{k+1}} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 \leq (1+\epsilon)\|F\|^2.$$

Lemma In addition to the general setup, assume that for some $k \geq k_0$, the matrix-valued function P_k satisfies that

$$P_k(\gamma)^* P_k(\gamma) = \frac{\mu(V_{k+1})}{\mu(V_k)} I_{d_k},\ a.e.\, \gamma \in V_k.$$

Then for all $F \in C_c(\widehat{G})$,

$$\sum_{\lambda\in\Lambda_{k+1}} |\langle F, \mathcal{M}_\lambda \Phi_{k+1}\rangle|^2 = \sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 + \sum_{m=1}^{\rho_k}\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Psi_k^{(m)}\rangle|^2.$$

B-splines on LCA groups

- Dahlke, Tikhomirov, 1994: definition of B-splines on LCA-groups.
- Extension to a definition of weighted splines (C. & Goh, 2014)

Definition Let Λ denote a lattice in the LCA group G, with associated fundamental domain Q, i.e.,

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + Q) \text{ and } (\lambda + Q) \cap (\lambda' + Q) = \emptyset,\ \lambda \neq \lambda'.$$

Let $r \in \mathbb{N}$. Given functions $g_1, \ldots, g_r \in L^2(Q)$ the function defined by the r-fold convolution

$$W_r := g_1 \chi_Q * g_2 \chi_Q * \cdots * g_r \chi_Q$$

is called a *weighted B-spline of order r*.

B-splines on LCA groups

Lemma (C. & Goh, 2014) Let Λ denote a lattice in the LCA group G, with associated fundamental domain Q. Given functions $g_1, \ldots, g_r \in L^2(Q)$, the weighted B-spline

$$W_r := g_1 \chi_Q * g_2 \chi_Q * \cdots * g_r \chi_Q$$

has the following properties:

(i) $\{T_\lambda W_r\}_{\lambda \in \Lambda}$ is a Bessel sequence with bound $\prod_{j=1}^r ||g_j||^2_{L^2(Q)}$.

(ii) $\operatorname{supp} W_r \subseteq \overline{rQ}$

(iii) If $r \geq 2$, then $W_r \in C_c(G)$; in particular, $W_r \in L^p(G)$ for all $p \geq 1$.

(iv) If $g_j > 0$ on int(Q) for $j = 1, \ldots, r$, then $W_r > 0$ on int(rQ);

(v) If $g_j = C$ for some $j = 1, \ldots, r$, then W_r satisfies the partition of unity condition up to a constant, i.e.,

$$\sum_{\lambda \in \Lambda} W_r(x - \lambda) = \frac{1}{\mu_G(Q)} \prod_{j=1}^r \int_Q g_j(x)\, dx.$$

Extra information for the "Atoll of spline lovers"

Theorem (C. & Goh, 2014) Given a lattice Γ in $\widehat{G}$, let $\Omega \subset \widehat{G}$ denote a fundamental domain, i.e.,

$$\widehat{G} = \bigcup_{\gamma \in \Gamma} (\gamma + \Omega). \qquad [\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (nb + [0, b[)]$$

- For a fixed $r \in \mathbb{N}$, consider the function

$$W_r := g_1 \chi_\Omega * g_2 \chi_\Omega * \cdots * g_r \chi_\Omega,$$

  with the assumption that $g_j > 0$ and $g_j = C$ for at least one index $j = 1, \ldots, r$.
- Given a lattice Λ in G, and assume that the fundamental domain V associated with $\Lambda^\perp$ satisfies that $r\Omega \subseteq V$.

Then $\{\mathcal{M}_\lambda T_k W_r\}_{\lambda \in \Lambda, k \in \Gamma}$ is a frame for $L^2(\widehat{G})$.

Extra information for the "Atoll of spline lovers"

Example Consider a Gabor system $\{E_{mb}T_nB_N\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, which corresponds to $\{\mathcal{M}_\lambda T_k W_r\}_{\lambda\in\Lambda,k\in\Gamma}$ with $\Lambda = b\mathbb{Z}, \Gamma = \mathbb{Z}$. Then

- $\Lambda^\perp = \frac{1}{b}\mathbb{Z}, V = [0, 1/b[$;
- $\Omega = [0, 1[$;
- The condition $r\Omega \subseteq V$ means that $[0, r[\subseteq [0, 1/b[$, i.e., $r \leq 1/b$; this is exactly the classical Gabor condition:

Corollary: $\{E_{mb}T_nB_N\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if $b \leq 1/N$.

The UEP on LCA groups and B-splines

Given the fundamental domain Q_k associated with the lattice Λ_k, define the B-spline of Nth order on level k by the N-fold convolution

$$\phi_k := \mu(Q_k)^{-N+1/2} \chi_{Q_k} * \cdots * \chi_{Q_k}.$$

Consider the functions Φ_k defined by

$$\Phi_k(\gamma) := \widehat{\phi}_k(\gamma) = \mu(Q_k)^{-N+1/2} \left(\int_{Q_k} (-x, \gamma)\, dx \right)^N .$$

Lemma The function Φ_k satisfies the scaling equation

$$\Phi_k(\gamma) = H_{k+1}(\gamma) \Phi_{k+1}(\gamma),$$

where $H_{k+1} \in L^\infty(V_{k+1})$ is given by

$$H_{k+1}(\gamma) = \frac{1}{2^{N-1/2}} \left(1 + (-\eta_k, \gamma)\right)^N$$

for some $\eta_k \in G$.

The UEP on LCA groups and B-splines

For the B-spline case:

- Provided that the group G has "enough lattices," there is a canonical way of choosing the filters $G_k^{(m)}$ such that

$$P_k(\gamma)^* P_k(\gamma) = \frac{\mu(V_{k+1})}{\mu(V_k)} I_{d_k}, \ a.e. \gamma \in V_k.$$

- For $G = \mathbb{R}$, the classical UEP is obtained and leads to a tight frame with wavelet structure.
- All the technical conditions are satisfied for $G = \mathbb{Z}$, leading to a tight frame for $\ell^2(\mathbb{Z})$ consisting of modulates of a finite collection of functions.

The UEP on LCA groups and B-splines

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Alternative construction:

- Shannon–type constructions, i.e., $\Phi_k = \chi_{\Omega_k}$ for some sets Ω_k in $\widehat{G}$;
- Concrete applications to all the elementary LCA groups $\mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_N$.

Conclusion

The UEP can be generalized to LCA groups, as well as the level of deriving the theorem as on the level of applications to B-splines and characteristic functions.

LCA groups

Lemma Let G be a LCA group and Λ a lattice in G. Then the following hold:

(i) There exists a relatively compact set $Q \subseteq G$ such that

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + Q), \qquad (\lambda + Q) \cap (\lambda' + Q) = \emptyset \text{ for } \lambda \neq \lambda'.$$

The set Q is called a *fundamental domain* for the lattice Λ.
Example: $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (nb + [0, b[)$

(ii) The set $\Lambda^{\perp}$ is a lattice in $\widehat{G}$, and there exists a relatively compact set $V \subseteq \widehat{G}$ such that

$$\widehat{G} = \bigcup_{\omega \in \Lambda^{\perp}} (\omega + V), \qquad (\omega + V) \cap (\omega' + V) = \emptyset \text{ for } \omega \neq \omega'.$$

Example: $\widehat{\mathbb{R}} = \mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n/b + [0, 1/b[)$