Berrut's Rational Interpolants

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Good Polynomial Interpolation Points

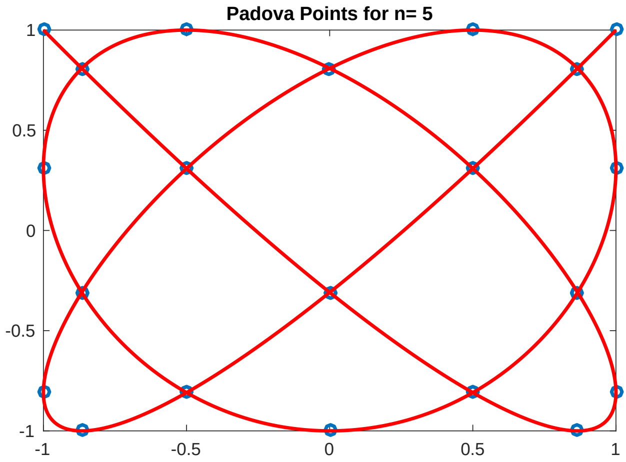
Theorem

(Berman-Boucksom) For $K \subset \mathbb{R}^d$ compact (non pluripolar), arrays of "good" points for total degree polynomial interpolation converge weak- to the equilibrium measure of Pluripotential Theory.*

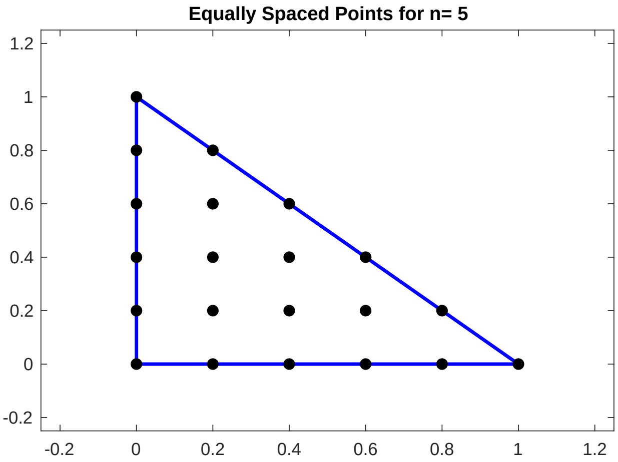
Good Interpolation Points

CHALLENGE: Find explicit arrays of points with minimal growth of the Lebesgue Constant.
Univariate Example: Chebyshev Points.
Only few multivariate examples known.

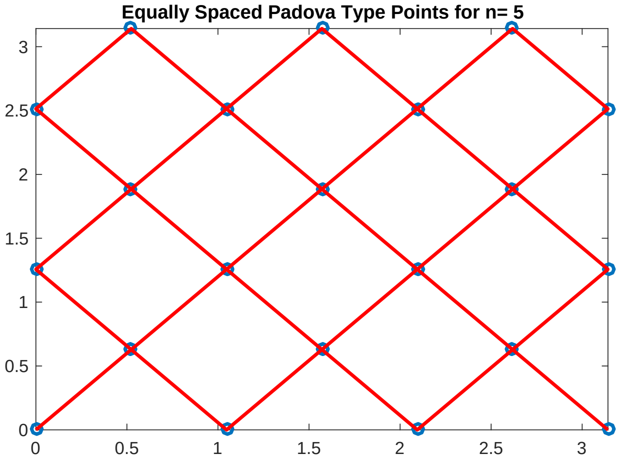
Padova Points



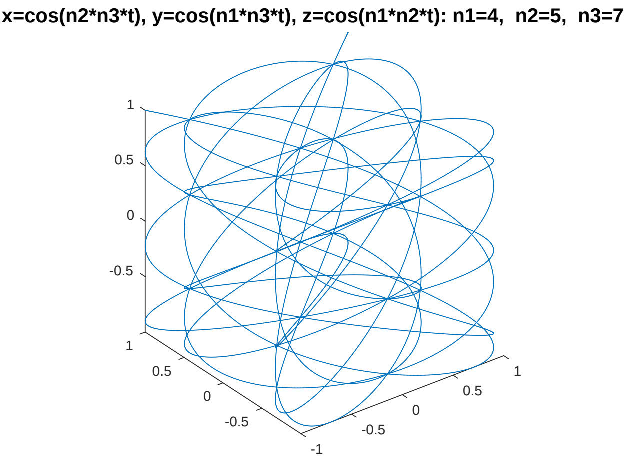
Equally Spaced Triangle Points



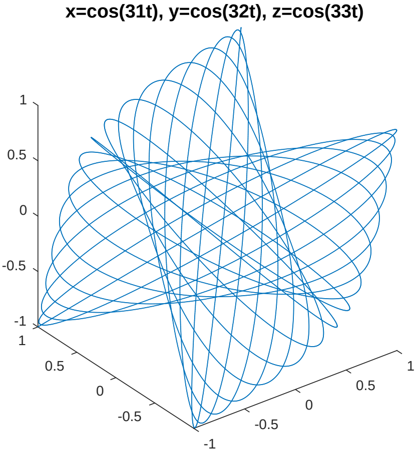
Equally Spaced Padova Type Points



A 3d Lissajous Curve



Another 3d Lissajous Curve



Whittaker-Shannon Sampling

Theorem

Suppose that $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ *and that* $\widehat{f}(\omega) = 0$ *for* $|\omega| \geq h/2$. *Then*

$$f(x) = \sum_{k=-\infty}^{\infty} f(kh)\mathrm{sinc}\left(\frac{1}{h}(x - kh)\right). \tag{1}$$

$$\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$$

is the sinc function and we define the Fourier transform

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} e^{-2\pi i \omega x} f(x)dx.$$

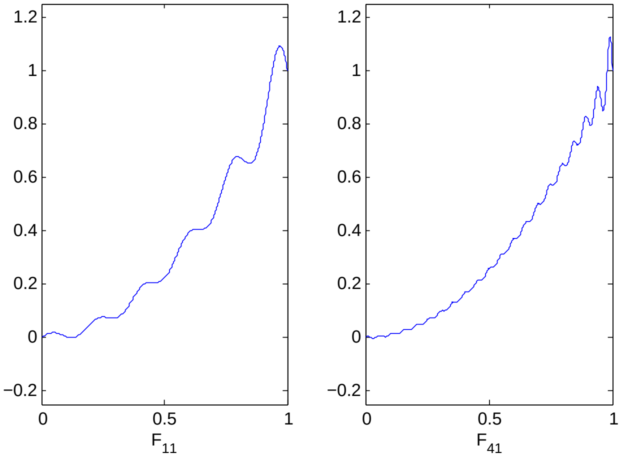
Truncated Whittaker-Shannon Sampling

In the case of f with domain restricted to some compact subinterval of $\mathbb{R}$, say to $[0,1]$, the formula (1) of course no longer makes sense. However, taking $h = 1/n$, we may consider the partial sum

$$f(x) \approx F_n(x) := \sum_{k=0}^{n} f(k/n)\mathrm{sinc}\,(n(x-k/n)) = \sum_{k=0}^{n} f(x_k)\mathrm{sinc}\,(n(x-x_k)) \quad (2)$$

where we have set $x_k := k/n$, $0 \le k \le n$.

Truncated Whittaker-Shannon Sampling Example Plots for $f(x) = x^2$



Truncated Whittaker-Shannon Sampling 2

Although (2) no longer reproduces $f(x)$ for all $x \in [0,1]$, it is an *interpolant* in that

$$F_n(x_j) = f(x_j),\ 0 \le j \le n, \qquad (3)$$

as easily follows from the cardinality property of the translated sinc functions, i.e.,

$$\mathrm{sinc}(n(x_j - x_k)) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

Truncated Whittaker-Shannon Sampling 3

This interpolant F_n was already studied by de la Vallée Poussin (1908) who showed that under some weak regularity conditions on $f(x)$,

$$\lim_{n\to\infty} F_n(x) = f(x),\ \ x \in [0,1],$$

with error essentially of $O(1/n)$. The reader interested in further details may find them in the excellent survey by Butzer and Stens (1992).

Berrut's Improvement

In order to alleviate the poor approximation quality of F_n Berrut (1989) suggested normalizing the formula (2) for F_n to obtain

$$B_n(x) := \frac{\sum_{k=0}^{n} f(x_k)\mathrm{sinc}\,(n(x-x_k))}{\sum_{k=0}^{n} \mathrm{sinc}\,(n(x-x_k))}. \tag{4}$$

As is easily seen, B_n remains an interpolant of f at the nodes x_k, $k = 0, \cdots, n$ but has the advantage of reproducing constants, i.e., if $f(x) = 1$ then $B_n(x) = 1$.

Simplified Formula - Berrut's First Interpolant

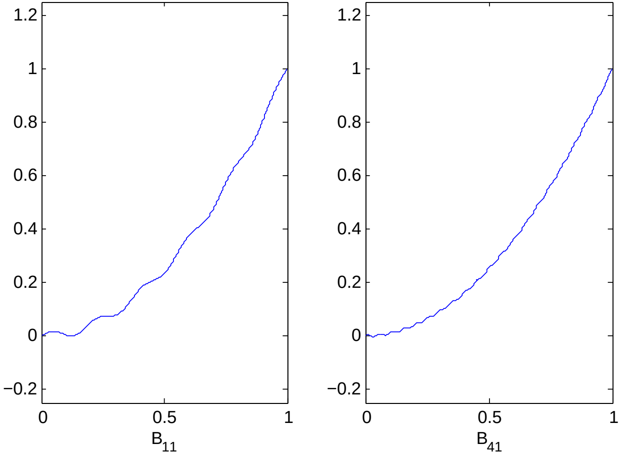
Notice that

$$\mathrm{sinc}(n(x-x_k)) = (-1)^k \frac{\sin(n\pi x)}{n\pi(x-x_k)}.$$

Hence,

$$\begin{aligned} B_n(x) &= \frac{\sum_{k=0}^n f(x_k)\mathrm{sinc}\left(n(x-x_k)\right)}{\sum_{k=0}^n \mathrm{sinc}\left(n(x-x_k)\right)} \\ &= \frac{\sin(n\pi x)\sum_{k=0}^n (-1)^k \frac{f(x_k)}{x-x_k}}{\sin(n\pi x)\sum_{k=0}^n (-1)^k \frac{1}{x-x_k}} \\ &= \frac{\sum_{k=0}^n (-1)^k f(x_k)/(x-x_k)}{\sum_{k=0}^n (-1)^k/(x-x_k)}. \qquad (5) \end{aligned}$$

Berrut's First Interpolant Sample Plots for $f(x) = x^2$



Berrut's First Interpolant – Lebesgue Constants

Besides being an improved approximant, B_n is also numerically stable as its associated Lebesgue constant has $O(\log(n))$ growth, as was shown by B., De Marchi and Hormann (2011)

Floater-Hormann Extension

$$FH_n(x) := \frac{\sum_{k=0}^{n} (-1)^k \beta_k^{(d)} f(x_k)/(x-x_k)}{\sum_{k=0}^{n} (-1)^k \beta_k^{(d)}/(x-x_k)}. \quad (6)$$

- Weights $\beta_k^{(d)}$ are chosen so that FH_n reproduces polynomials of degree at most d

Floater-Hormann Extension 2

In the specific case of equally spaced nodes their formula for the $\beta_k^{(d)}$ reduces to

$$\beta_k^{(d)} := \begin{cases} \sum_{j=0}^{k} \binom{d}{k} & 0 \leq k \leq d \\ 2^d & d \leq k \leq n-d \\ \beta_{n-k} & n-d \leq k \leq n \end{cases} \tag{7}$$

where $n \geq 2d$, by assumption.

For $d = 1$ this is Berrut's second interpolant.

Floater-Hormann Polynomial Reproduction

Theorem

[Floater and Hormann (2007)] Consider the Floater-Hormann inerpolant FH_n *with weights* $\beta_k^{(d)}$ *given by* (7), $n \geq 2d$, *and equally spaced nodes* $x_k = k/n$. *Then if* $f(x)$ *is a polynomial of degree at most* d,

$$FH_n(x) = f(x).$$

Lebesgue Constants

Besides having improved approximation properties, the Floater-Hormann remains numerically stable as its associated Lebesgue constant is also of logarithmic growth in n, as is shown in the recent paper by B., De Marchi, Hormann and Klein.

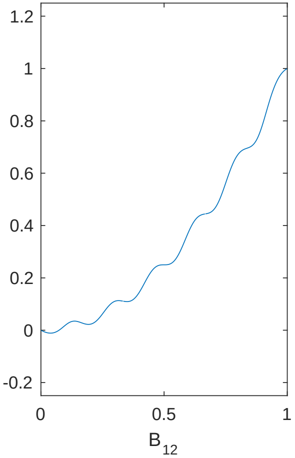
Back to Sampling

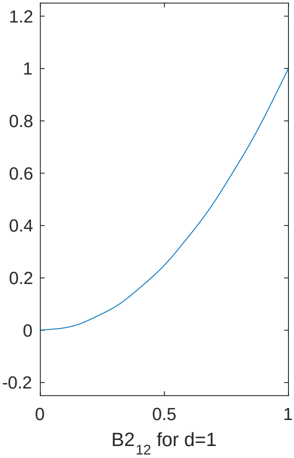
The Berrut-Floater-Hormann interpolant for equally spaced is a simple improvement on the sampling operator.
Indeed,

$$FH_n(x) = \frac{\sum_{k=0}^{n} \beta_k^{(d)} f(x_k)\mathrm{sinc}(n(x-x_k))}{\sum_{k=0}^{n} \beta_k^{(d)}\mathrm{sinc}(n(x-x_k))}.$$

The weights $\beta_k^{(d)}$ are constant except for the first d and last d and hence (for small d) are only a small modification of the normalized sampling operator $B_n(x)$, but FH_n reproduces polynomials of degree d and yet enjoys a Lebesgue constant of minimal growth.

Berrut Examples for $f(x) = x^2$





Lebesgue Function 1

The Lebesgue function is defined to be:

$$\Lambda_n(x) := \sum_{k=0}^{n} |b_k(x)|$$

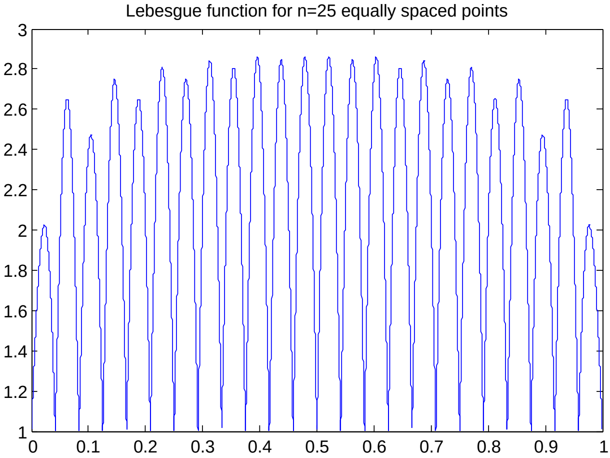
The Lebesgue *constant* is

$$\lambda_n = \max_{0\le x\le 1} \Lambda_n(x)$$

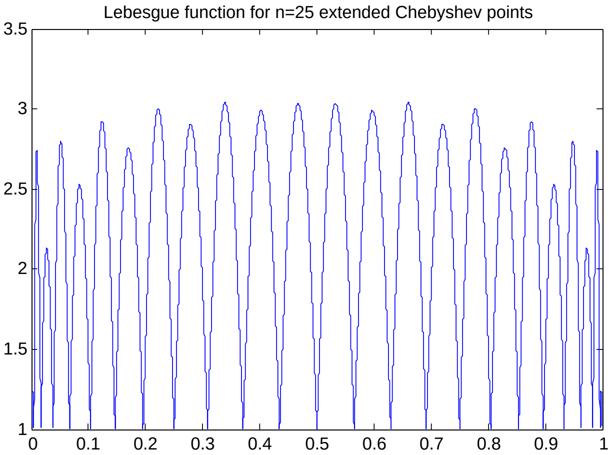
It is the norm of the interpolation operator

$$f(x) \longrightarrow \sum_{k=0}^{n} f(x_k) b_k(x)$$

Lebesgue Function for Equally Spaced Points



Lebesgue Function for Extended Chebyshev Points



Lebesgue Function 2

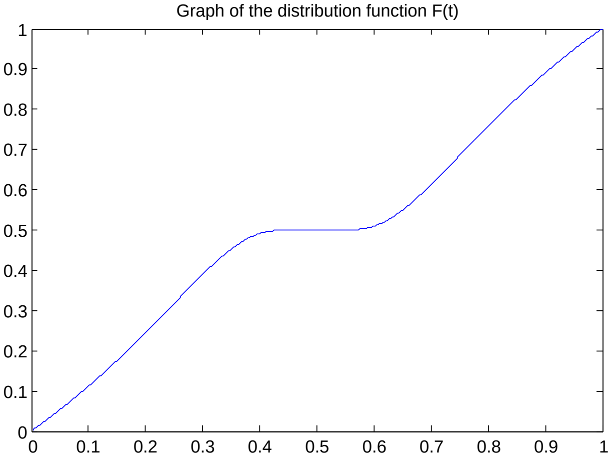
Theorem (B., Hormann and De Marchi)

Suppose that the nodes are generated as $x_k = F(k/n)$ where $F : [0,1] \to [0,1]$ is a regular distribution function. Then the Lebesgue constants for B_n, Berrut's first interpolant, have logarithmic growth in n.

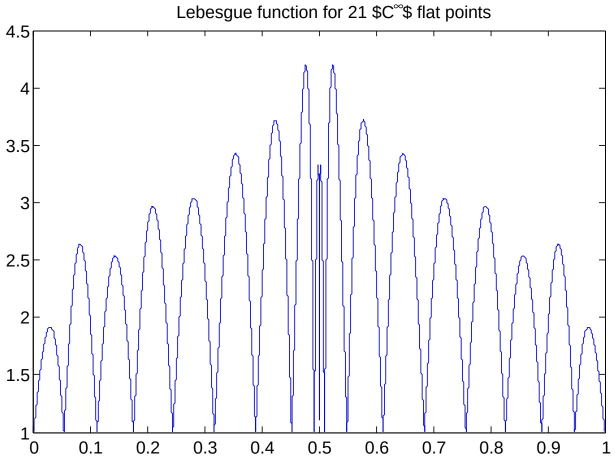
Definition

An increasing function $F : [0,1] \to [0,1]$ is said to be a regular distribution function if $F \in C^1[0,1]$ and F' has a finite number of zeros in $[0,1]$ of at most algebraic order.

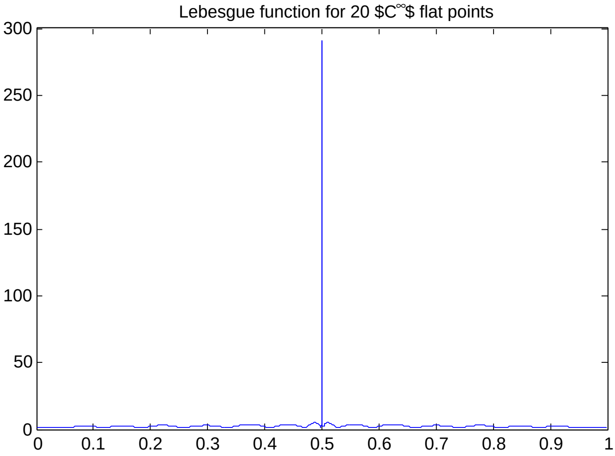
Function $F(t)$ C^∞ flat at $t = 1/2$



Lebesgue Function $F(t)$ C^∞ flat at $t = 1/2$ ODD number of points



Lebesgue Function $F(t)$ C^∞ flat at $t = 1/2$ EVEN number of points



Bivariate Whittaker-Shannon Sampling

Set $x_i := i/n$ and $y_j := j/n$

$$f(x,y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f(x_i, y_j) \mathrm{sinc}(n(x - x_i)) \mathrm{sinc}(n(y - y_j))$$

Truncate triangularly

$$F_n(x,y) := \sum_{0 \le i+j \le n} f(x_i, y_j)\, \mathrm{sinc}(n(x - x_i))\, \mathrm{sinc}(n(y - y_j))$$

Bivariate Whittaker-Shannon Sampling 2

Normalize:

$$B_n(x,y) := \frac{\sum_{0\le i+j\le n} f(x_i,y_j)\,\mathrm{sinc}(n(x-x_i))\,\mathrm{sinc}(n(y-y_j))}{\sum_{0\le i+j\le n}\mathrm{sinc}(n(x-x_i))\,\mathrm{sinc}(n(y-y_j))}$$

Berrut One in Two Dimensions

Simplify:

$$B_n(x,y) := \frac{\sum_{0\le i+j\le n}(-1)^{i+j}\frac{f(x_i,y_j)}{(x-x_i)(y-y_j)}}{\sum_{0\le i+j\le n}(-1)^{i+j}\frac{1}{(x-x_i)(y-y_j)}}$$

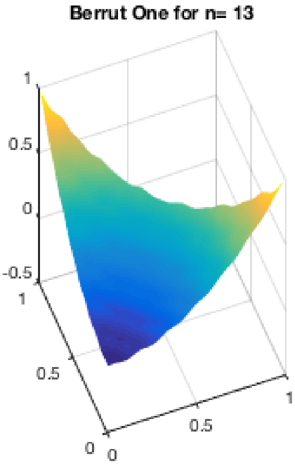
Berrut One in Two Dimensions

Problem: The Denominator has real zeros!!

$$D_n(x,y) := w_n(x)w_n(y) \sum_{0\le i+j\le n} (-1)^{i+j} \frac{1}{(x-x_i)(y-y_j)}$$

is zero at $\left(\frac{\alpha}{n}, \frac{\beta}{n}\right)$ for $0 \le \alpha, \beta \le n$ and $\alpha + \beta > n$

Berrut One 2D for $f(x, y) = x^2 + y^2$



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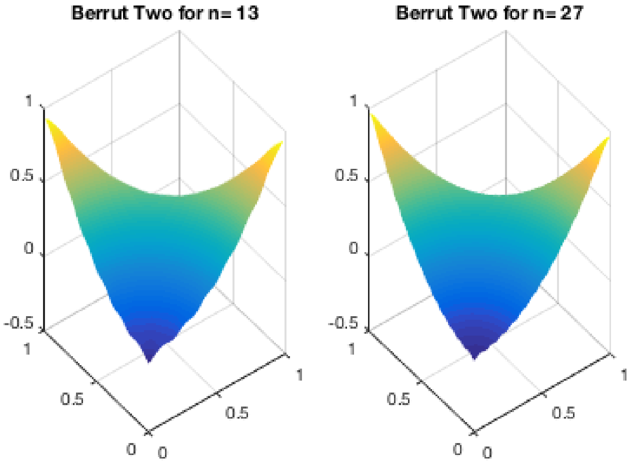

Merci e au revoir!!