Renormalizing Stochastic PDE's

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White noise driven PDE's

Space time white noise $\xi(t, x)$

$$\mathbb{E}\xi(t', x')\xi(t, x) = \delta(t' - t)\delta(x' - x)$$

- Interface growth $\phi(t, x)$ interface height (KPZ)

$$\partial_t \phi = \Delta\phi + (\nabla\phi)^2 + \xi$$

- Ginzburg-Landau (GL) model $\phi(t, x)$ magnetization

$$\partial_t \phi = \Delta\phi - \phi^3 + \xi$$

- Fluctuating hydrodynamics $\phi = (\phi_1, \phi_2, \phi_3)$

$$\partial_t \vec{\phi}_\alpha = \Delta\phi_\alpha + M_\alpha^{\beta\gamma} \partial_x \phi_\beta \partial_x \phi_\gamma + \xi_\alpha$$

ξ is very rough, are these non-linear equations well-posed?

- Given a realization ξ of noise, is there a $\phi(\xi)$ solving these equations?
- How is $\phi(\xi)$ distributed? Is there a stationary state?

In general we need to **renormalize** the equations to make them well posed.

Linear case

Linear equation $x \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$\partial_t \phi = \Delta \phi + \xi$$

$$\phi(0, x) = \phi_0(x)$$

solved by

$$\phi(t, x) = (e^{t\Delta}\phi_0)(x) + \eta(t, x)$$

with

$$\eta(t) = \int_0^t e^{(t-s)\Delta}\xi(s)ds$$

Free field

$\eta(t,x)$ is a random field with covariance

$$\mathbb{E}\eta(t,x)\eta(t,y) = C_t(x,y)$$

where $C_t(x,y)$ is the integral kernel of the operator

$$\int_0^t e^{2t\Delta} dt = -\tfrac{1}{2}\frac{1-e^{2t\Delta}}{\Delta}$$

$C_t(x,y)$ is **singular** in short scales:

$$\mathbb{E}\eta(t,x)\eta(t,y) \asymp \frac{1}{|x-y|^{d-2}}.$$

- $\eta(t,x)$ is **a.s.** not a function in $d \geq 2$
- $\nabla\eta(t,x)$ has same regularity as white noise for all d.

Integral equation

Consider nonlinear equation

$$\partial_t \phi = \Delta \phi + V(\phi) + \xi, \quad \phi(0, x) = 0.$$

Rewrite it as integral equation

$$\begin{aligned} \phi(t) &= \int_0^t e^{(t-s)\Delta}(V(\phi(s)) + \xi(s))ds \\ &= \eta(t) + \int_0^t e^{(t-s)\Delta} V(\phi(s))ds \end{aligned}$$

where $\eta(t, x)$ is the solution to the linear equation.

Fix a realization of the random field $\eta(t, x)$ and try to solve this fixed point problem in some (Banach) space of functions $\phi(t, x)$.

Perturbation theory

Study the solution iteratively:

$$\phi(t) = \eta(t) + \int_0^t e^{(t-s)\Delta} V(\eta(s)) ds + \dots .$$

This **fails**:

- For KPZ equation

  $$V(\eta(s)) = (\partial_x \eta(s,x))^2$$

  and $\partial_x \eta(s,x) =$ derivative of BM $= \infty$ almost surely.
- For GL equation

  $$V(\eta(s)) = \eta(s,x)^3 = \infty$$

  almost surely if $d \geq 2$.

Quantum Field Theory

Such divergencies are familiar from **quantum field theory**.

Formally the equation

$$\partial_t \phi = \Delta \phi - \phi^3 + \xi$$

has a **stationary measure**

$$\nu(d\phi) \propto e^{-\frac{1}{4}\int_{\mathbb{T}^d} \phi(x)^4 dx} \mu(d\phi)$$

where μ is a Gaussian measure with covariance

$$\mathbb{E}\phi(x)\phi(y) = -\frac{1}{2}\Delta^{-1}(x,y) = C|x-y|^{2-d}$$

For $d < 4$ ν can be constructed by **renormalization**.

Renormalization

Regularize

$$\phi_\epsilon(x) := (\rho_\epsilon * \phi)(x), \quad \rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$$

and **renormalize**

$$V^{(\epsilon)}(\phi_\epsilon) := \tfrac{1}{4}\phi_\epsilon^4 + r_\epsilon \phi_\epsilon^2$$

Then

$$\lim_{\epsilon\to 0} e^{-\int_\Lambda V^{(\epsilon)}(\phi_\epsilon(x))dx}\mu(d\phi)$$

exists with

$$\begin{aligned} r_\epsilon &= m\log\epsilon & d = 2 \\ r_\epsilon &= m_1\epsilon^{-1} + m_2\log\epsilon & d = 3 \end{aligned}$$

Story of 1970's.

Regularized dynamics

Consider a **regularized** equation

$$\partial_t \phi = \Delta \phi + V_\epsilon(\phi) + \xi_\epsilon$$

where

- noise $\xi_\epsilon(t) = \rho_\epsilon * \xi(t)$ is smooth on scales $\leq \epsilon$
- V_ϵ has ϵ-dependent terms added to V

Determine V_ϵ so that solutions ϕ_ϵ converge as $\epsilon \to 0$ to some distribution ϕ.

Renormalized dynamics

Renormalize:

$$
\begin{aligned}
(\partial_x \phi)^2 &\to (\partial_x \phi)^2 + a\epsilon^{-1} \\
M_\alpha^{\beta\gamma} \partial_x \phi_\beta \partial_x \phi_\gamma &\to M_\alpha^{\beta\gamma} \partial_x \phi_\beta \partial_x \phi_\gamma + a_\alpha \epsilon^{-1} + b_\alpha \log \epsilon \\
\phi^3 &\to \phi^3 + \phi \begin{cases} m \log \epsilon & d = 2 \\ m_1 \epsilon^{-1} + m_2 \log \epsilon & d = 3 \end{cases}
\end{aligned}
$$

Theorem. The following holds **almost surely in ξ**:
There exists $T > 0$ s.t. the regularized equation has a unique solution $\phi_\epsilon(t, x)$ for $t \leq T$ and

$$\phi_\epsilon \to \phi \in \mathcal{D}'([0, T] \times \mathbb{T}^d)$$

where ϕ is independent of the cutoff function ρ.

Earlier proofs: Gubinelli, Imkeller, and Perkowski, Catellier and Chouk, Hairer

Fixed point problem

Consider the fixed point problem

$$\phi(t) = \eta_\epsilon(t) + \int_0^t e^{(t-s)\Delta} V_\epsilon(\phi(s))ds$$

For $\epsilon > 0$ this has smooth solution ϕ_ϵ at least for some time.

Problem: since the limit ϕ will a distribution its not clear how to set this up as a Banach fixed point problem

Martin Hairer developed a nonlinear theory of distributions **"Regularity Structures"** allowing to formulate and solve the fixed point problem.

This can be compared to perturbative renormalization theory in QFT.

Wilson RG

We prove this result using the "Wilsonian" approach to renormalization

- Proceed scale by scale to derive **effective equation** on that scale
- No new theory of distributions needed
- Standard contraction mapping theorem
- A general method to derive counterterms for subcritical nonlinearities

Counter terms

Given a nonlinearity $V(\phi)$ how to find the counter terms ?

Why is this natural ?

Both questions can be answered by considering scale dependent **effective equations**.

Dimensionless variables

Define **space time scaling** s_μ

$$(s_\mu \phi)(t,x) := \mu^{\frac{d-2}{2}} \phi(\mu^2 t, \mu x).$$

This **preserves** the linear equation $\dot{\phi} = \Delta\phi + \xi$.

Define

$$\varphi := s_\epsilon \phi$$

Then equation

$$\dot{\phi} = \Delta\phi + \xi_\epsilon + \begin{cases} (\nabla\varphi)^2 & \text{KPZ} \\ \varphi^3 + r\varphi & \text{GL} \end{cases}$$

becomes

$$\dot{\varphi} = \Delta\varphi + \xi_1 + \begin{cases} \epsilon^{\frac{2-d}{2}}(\nabla\varphi)^2 & \text{KPZ} \\ \epsilon^{4-d}\varphi^3 + \epsilon^2 r\varphi & \text{GL} \end{cases}$$

Subcritical nonlinearity

In dimensionless variables

- Noise is smooth (UV cutoff is 1)
- Nonlinearity is **subcritical** if $d < 2$ (KPZ), $d < 4$ (GL)

However

- φ is defined on $[0, \epsilon^{-2}T] \times (\epsilon^{-1}\mathbb{T})^d$
- Need to control **arbitrary large times and volumes** as $\epsilon \to 0$

Fixed point problem

Write the PDE

$$\dot{\varphi} = \Delta\varphi + v(\varphi) + \xi_1$$

as a fixed point problem

$$\varphi = G(v(\varphi) + \xi_1)$$

with $G = (\partial_t - \Delta)^{-1}$ i.e.

$$(Gf)(t) := \int_0^t e^{(t-s)\Delta} f(s)ds$$

Note: The noise ξ_1 is a.s. smooth so this is a trivial problem for times of $\mathcal{O}(1)$.

Scale by scale

The fixed point equation

$$\varphi = G(v(\varphi) + \xi_1)$$

involves spatial scales $\in [1, \epsilon^{-1}]$ and temporal scales $\in [1, \epsilon^{-2}]$.
Fix $L > 1$ and split

$$G = G_< + G_>$$

where $G_<$ has scales $\in [1, L]$ and $G_>$ has scales $\in [L, \epsilon^{-1}]$.
Look for $\varphi = \varphi_< + \varphi_>$ so that

$$\varphi_< = G_<(v(\varphi_< + \varphi_>) + \xi_1) \quad (1)$$
$$\varphi_> = G_>(v(\varphi_< + \varphi_>) + \xi_1) \quad (2)$$

(1) is easy to solve: it has time $\mathcal{O}(L^2)$, noise is smooth and nonlinearity is small. Get $\varphi_<$ as a function of $\varphi_>$:

$$\varphi_< = \varphi_<(\varphi_>).$$

Renormalized equation

Inserting $\varphi_<(\varphi_>)$ to large scale equation (2) get

$$\varphi_> = G_>(v(\varphi_> + \varphi_<(\varphi_>)) + \xi_1)$$

Rescale $\varphi_>(t,x) = L^{\frac{2-d}{2}}\varphi'(L^{-2}t, L^{-1}x)$. Get a **renormalized equation** for φ':

$$\varphi' = G(v'(\varphi') + \xi_1)$$

This is of the same form as the original equation except that

- $\varphi'(t,x)$ has cutoff ϵ replaced by $L\epsilon$
- The nonlinearity has changed to v'.
- The map $\mathcal{R} : v \to v' := \mathcal{R}v$ is renormalization map

Iterating this we obtain a sequence of $\mathcal{R}^n v$ and equations

$$\varphi = G(\mathcal{R}^n v(\varphi) + \xi_1).$$

This φ describes solution of original PDE on scales $\geq L^n\epsilon$.

Effective equation

Upshot: solving the PDE $\Leftrightarrow$ study the iteration $\mathcal{R}^n v$.
Start with

$$v = v^\epsilon = \left\{ \begin{array}{cl} \epsilon^{\frac{1}{2}}(\nabla\varphi)^2 & \text{KPZ}_{\ d=1} \\ \epsilon^{4-d}\varphi^3 & \text{GL}_{\ d} \end{array} \right. + \text{counterterms}_\epsilon$$

Define the **effective equation for scales $\geq \mu$**

$$v_\mu^\epsilon := \mathcal{R}^{\log(\mu/\epsilon)} v^\epsilon$$

Try to fix the counter terms so that for all μ the limit

$$v_\mu := \lim_{\epsilon \to 0} v_\mu^\epsilon$$

exists.

RG map

RG map $\mathcal{R}$ acts in a space of v, the nonlinear term in the PDE.

v is a map from functions $\varphi(t,x)$ defined on space time to a functions $v(\varphi)(t,x)$ defined on space time.

$\mathcal{R}$ is a composition of translation

$$v(\varphi) \to v'(\varphi) = v(\varphi + \psi)$$

and scaling:

$$(\mathcal{S}v)(\varphi) = L^2 s^{-1} v' \circ s$$

where

$$(s\varphi)(t,x) := L^{\frac{2-d}{2}} \varphi(L^{-2}t, L^{-1}x).$$

and ψ is solved from the short time problem

$$\psi = G_<(v(\varphi + \psi) + \xi_1).$$

Linearized RG

Consider $\mathcal{R}v$ to linear order in v: $\mathcal{R}v = \mathcal{L}v + \mathcal{O}(v^2)$
Scaling operator

$$\mathcal{S}v := L^2 s^{-1} v \circ s$$

has local eigenfunctions $v(\varphi)(t,x) = \varphi(t,x)^k, (\nabla\varphi(t,x))^k \ldots$:

$$\mathcal{S}\varphi^k = L^{\alpha_k}\varphi^k, \quad \alpha_k = 2 - (k-1)\tfrac{d-2}{2}$$

$$\mathcal{S}(\nabla\varphi)^k = L^{\beta_k}(\nabla\varphi)^k, \quad \beta_k = 2 - \tfrac{k+1}{2} \quad d = 1$$

$\alpha_k > 0$ expanding (**relevant**), $\alpha_k < 0$ contracting (**irrelevant**).

To leading order in v, $\psi = G_< \xi_1$ and get

$$\mathcal{L}^n \varphi^k = L^{\alpha_k}(\varphi + \eta_n)^k$$

$$\mathcal{L}^n(\nabla\varphi)^k = L^{\beta_k}(\nabla\varphi + \nabla\eta_{L^{-n}})^k$$

$\eta_{L^{-n}}$ is the free field with UV cutoff L^{-n}

Linearized RG

For KPZ in linear approximation effective equation becomes

$$v_\mu^\epsilon = \mu^{\frac{1}{2}}(\nabla\varphi + \nabla\eta_{\epsilon/\mu})^2$$

This has no limit as $\epsilon \to 0$:

$$\mathbb{E}(\nabla\eta_{\epsilon/\mu})^2 \sim \epsilon^{-1}$$

For GL one gets

$$v_\mu^\epsilon = \mu^{4-d}(\varphi + \eta_{\epsilon/\mu})^3$$

This has no limit since

$$\mathbb{E}(\eta_{\epsilon/\mu})^2 \sim \begin{cases} \log \epsilon^{-1} & d = 2 \\ \epsilon^{-1} & d = 3 \end{cases}$$

Counterterms

Why did this happen?

- KPZ nonlinearity $(\nabla\varphi)^2$ is relevant with exponent $\frac{1}{2}$ but has size $\epsilon^{\frac{1}{2}}$ which reproduces under iteration.
- However $\mathcal{R}$ produces a **more relevant** term, constant in φ with exponent $\frac{3}{2}$ and size $\epsilon^{\frac{1}{2}}$.
- This expands under iteration to $(\frac{\mu}{\epsilon})^{\frac{3}{2}}\epsilon^{\frac{1}{2}} = \mathcal{O}(\epsilon^{-1})$.

Solution: add a constant to the original KPZ equation

$$\dot{\phi} = \Delta\phi + (\nabla\phi)^2 - \mathbb{E}(\nabla\eta_\epsilon)^2 + \xi_\epsilon.$$

Then the effective equation becomes

$$v_\mu^\epsilon = \mu^{\frac{1}{2}}[(\nabla\varphi)^2 + 2\nabla\varphi\nabla\eta_{\epsilon/\mu} + :(\nabla\eta_{\epsilon/\mu})^2:]$$

where

$$:(\nabla\eta_{\epsilon/\mu})^2 := (\nabla\eta_{\epsilon/\mu})^2 - \mathbb{E}(\nabla\eta_{\epsilon/\mu})^2)$$

Counterterms

For the GL equation $\mathcal{R}$ produces a relevant linear term in φ with exponent $= 2$.

Defining the renormalized GL equation

$$\dot{\phi} = \Delta\phi + \phi^3 - 3(\mathbb{E}\eta_\epsilon^2)\phi + \xi_\epsilon.$$

the effective equation becomes

$$v_\mu^\epsilon = \mu^{4-d}[\varphi^3 + 3\varphi^2\eta_{\epsilon/\mu} + 3\varphi : \eta_{\epsilon/\mu}^2 : + : \eta_{\epsilon/\mu}^3 :]$$

The limits

$$\lim_{\epsilon\to 0} : (\nabla\eta_{\epsilon/\mu}(t,x))^2 : \;=\; : (\nabla\eta(t,x))^2 :$$

$$\lim_{\epsilon\to 0} : \eta_{\epsilon/\mu}(t,x)^k : \;=\; : \eta(t,x)^k :$$

are distribution valued random fields.

Nonlinear corrections: $GL_{d=2}$

Denote the linear approximation by

$$u^\epsilon_\mu = \mu^2 : (\varphi + \eta_{\epsilon/\mu})^3 :$$

and write

$$v^\epsilon_\mu = u^\epsilon_\mu + w^\epsilon_\mu.$$

Since $\mathcal{L} u^\epsilon_\mu = u^\epsilon_{L\mu}$ we get

$$w^\epsilon_{L\mu} = \mathcal{L} w^\epsilon_\mu + \mathcal{O}(\mu^4).$$

In $d = 2$ $\|\mathcal{L}\| = L^2$ and so

$$\|w^\epsilon_{L\mu}\| \leq L^2 \|w^\epsilon_\mu\| + C\mu^4.$$

Since $2 < 4$ the inductive bound

$$\|w^\epsilon_\mu\| \leq \mu^{2+\delta}, \quad \delta > 0$$

iterates for $\mu \leq \mu_0$.

This becomes a proof once we work in a suitable Banach space of v's. Thus normal ordering suffices to make the PDE well posed.

Nonlinear corrections: $GL_{d=3}$

Now
$$u_\mu^\epsilon = \mu : (\varphi + \eta_{\epsilon/\mu})^3 :$$
and $\|\mathcal{L}\| = L^{5/2}$ so that
$$\|w_{L\mu}^\epsilon\| \leq L^{5/2} \|w_\mu^\epsilon\| + C\mu^2$$
$5/2 > 2 \implies$ not good! We need to compute v_μ^ϵ to second order:
$$v_\mu^\epsilon = u_\mu^\epsilon + U_\mu^\epsilon + w_\mu^\epsilon.$$
If the second order term satisfied
$$\|U_\mu^\epsilon\| \leq C\mu^2$$
we would get
$$\|w_{L\mu}^\epsilon\| \leq L^{5/2} \|w_\mu^\epsilon\| + C\mu^3$$
and since $5/2 < 3$ we would be done.

Nonlinear corrections: $GL_{d=3}$, KPZ

However, $\|U_\mu^\epsilon\|$ **diverges** as $\log \epsilon$.

U_μ^ϵ is a (nonlocal) polynomial in φ and $\eta_{\epsilon/\mu}$. and need addition $\log \epsilon$ mass renormalization to have $\epsilon \to 0$ limit.

In **KPZ** coupling constant is $\epsilon^{\frac{1}{2}}$ and $\|\mathcal{L}\| = L^{3/2} \implies$ need to go to **3rd order**.

By miracle 2nd and 3rd order terms have **vanishing relevant terms**. The random fields occurring in them have $\epsilon \to 0$ limit and no new renormalizations are needed.

This is **not true** for **multicomponent KPZ**: need a $\log \epsilon$ counter term for the random fields occurring in third order.

Noise

We assumed perturbative terms $\|u_\mu^\epsilon\|$ have the obvious bounds in powers of μ.

This can not be true since they involve the random fields $:\eta^k:$, $:(\nabla\eta)^2:$ etc.

These noise fields belong to Wiener chaos of bounded order and their covariance is in a suitable negative Sobolev space

Hypercontractivity implies good moment estimates for them.

Borel-Cantelli $\implies$ a.s. $\exists \mu_0 >$ s.t. $\|u_\mu^\epsilon\|$ has a good bound.

On that event the $\mathbb{R}$ is controlled by a simple application of contraction mapping in a suitable Banach space.

The time of existence is μ_0^2 and it is a.s. > 0.

Spaces

What is the domain and range of $v_\mu^\epsilon(\varphi)$?

The random fields in the perturbative part V_μ^ϵ are H_{loc}^{-2} in time and H_{loc}^{-4} in space. We let v_μ^ϵ take values in $H_{loc}^{-2,-4}$.

Since φ represents the large scale part of the solution we can take φ smooth:

$$\varphi \in C^{2,4}([0, \mu^{-2} T] \times \mu^{-1} \mathbb{T}^d)$$

We prove

$$v_\mu^\epsilon : C^{2,4} \to H_{loc}^{-2,-4}$$

is analytic in a ball of radius $\mu^{-\alpha}$, $\alpha > 0$.

Superrenormalizable equations

$KPZ_{d=1}$ and $GL_{d<4}$ are **superrenormalizable** (subcritical): the dimensionless strength of nonlinearity is small in short scales.

Sine-Gordon equation

$$\partial_t \phi = \Delta \phi + g \sin(\sqrt{\beta}\phi) + \xi$$

for $\beta < 16\pi$. After normal ordering dimensionless coupling is

$$\epsilon^{2-\frac{\beta}{8\pi}} g.$$

Need to expand solution to order $k-1$ where $(2-\frac{\beta}{8\pi})k > 2$. So $k \to \infty$ as $\beta \uparrow 16\pi$.

It is a challenge to carry this out for all $\beta < 16\pi$. Hairer and Shen have controlled $\beta < \frac{32\pi}{3}$.