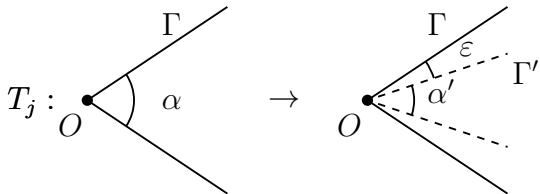


Random matrices and rarefied gases properties

Cone property: enjoyed by $\dots, T_{-1}, T_0, T_1, T_2, \dots$ of $d \times d$ real invertible matrices if



i.e. $T_j \Gamma \subset \Gamma'$ and $\Gamma'/O \subset$ of interior Γ^0 of Γ

Let $H_{n,N} v_N \stackrel{\text{def}}{=} T_n T_{n+1} \cdots T_N v_N$, $v_N \in \Gamma$ asymptotically prop. to λ^{N-n} with λ a “Lyapunov coefficient”?

λ is an “eigenvalue”. Maybe exists covariant $h(n)$? and $\Lambda(n, p)$ analytic?, s.t. ([6])

$$h(n) = \Lambda(n, p) T_n \dots T_{p-1} h(p), \quad n < p$$

Theorem 1: (1) Let $\|T_n\| < B_0$, $B_0 > 0$. Then given any sequence $\{v_j\}_{j \geq 1}$, $\|v_j\| = 1$, $v_j \in \Gamma$, $\exists \bar{\Lambda}_{v_N}(n, N)$, et $\forall \{v_j\}_{j \geq 1}$

$$h(n) = \lim_{N \rightarrow \infty} \frac{T_n T_{n+1} \cdots T_N v_N}{\bar{\Lambda}_{v_N}(n, N)}, \quad \forall 1 \leq n \leq p$$

$$\Lambda(n, p) = \lim_{N \rightarrow \infty} \frac{\bar{\Lambda}_{v_N}(p, N)}{\bar{\Lambda}_{v_N}(n, N)} > 0, \quad \forall 1 \leq n \leq p$$

(2) Λ, h is analytic for small variations of entries.

(3) The $h(n)$ are “eigenvectors” i.e. $\Lambda(n, p)$ s.t.:

$$T_{p-1}^{-1} \cdots T_n^{-1} h(n) = \Lambda(n, p) h(p)$$

(4) $\exists B$ s.t. $B^{-1} \leq \|h(n)\| \leq B$ and \limsup, \liminf of $p^{-1} \log |\Lambda(n, p)|$, as $p \rightarrow \infty$, are n independent.

$$\begin{aligned}
& (T_n T_{n+1} \cdots T_N v_N)_{i_n} \\
&= \sum_{i_{n+1}, \dots, i_N} ((T_n)_{i_n, i_{n+1}} (T_{n+1})_{i_{n+1}, i_{n+2}} \cdots (T_N)_{i_N, i_{N+1}} v_{i_{N+1}})
\end{aligned}$$

looks “partition function with boundary conditions” i_n, v .

If $(T_k)_{i,j} > 0$ then $(T_k)_{i,j} = e^{-V_k(i,j)}$ with V_k “random”.

Interpretation of “spin glass”; but $(T_k)_{i,j}$ may be $\neq > 0$

Pursue the idea: diagonalize $T_n = \sum_{\sigma=0}^{d-1} \lambda_{n,\sigma} |n, \sigma\rangle \langle n, \sigma|$

$$\begin{aligned}
H_{n,N} v_N &\stackrel{\text{def}}{=} T_n T_{n+1} \cdots T_N v_N \\
&= \sum_{\sigma_n, \dots, \sigma_N} |n, \sigma_n\rangle \cdot \left(\prod_{j=n}^N \lambda_{j, \sigma_j} \right) \prod_{j=n}^N \langle j, \sigma_j | j+1, \sigma_{j+1} \rangle
\end{aligned}$$

Here $|N + 1, \sigma_{N+1}\rangle \equiv v$ (σ_{N+1} is label, not sum index)

Call $(\sigma_n, \dots, \sigma_N)$ a “spin config”. **Isolate** the $\sigma_j = 0$.

$$\begin{array}{cccccccccccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

Fig.2

and form intervals $J = [h, h'] = (h, h + 1, \dots, h')$,
 $n \leq h \leq h' \leq N$ and, on each, *spin configurations* σ_J .

$$H(n, N)_{\sigma_{N+1}=0}$$

$$= const \sum_{\sigma_n, \dots, \sigma_N} |n, \sigma_n\rangle \cdot \left(\prod_{j=n}^N \frac{\lambda_{j, \sigma_j}}{\lambda_{j, 0}} \right) \prod_{j=n}^N \frac{\langle j, \sigma_j | j + 1, \sigma_{j+1} \rangle}{\langle j, 0 | j + 1, 0 \rangle}$$

This is (simplifying contributions of stretches with $\sigma = 0$)

$$H_{n, N} v_N = \bar{\Lambda}(n, N) \sum_{\substack{J_1 < \dots < J_s \\ s \geq 0}} \sum_{\sigma_{J_1}, \dots, \sigma_{J_s}} |n, \sigma_n\rangle \frac{\prod_{i=1}^s I(J_i, \sigma_{J_i})}{\Omega(n, N)}$$

$$\begin{aligned}
 & H(n, N)_{\sigma_{N+1}=0} \\
 &= \prod \lambda_{j,0} \sum_{\sigma_n, \dots, \sigma_N} |n, \sigma_n\rangle \cdot \left(\prod_{j=n}^N \frac{\lambda_{j, \sigma_j}}{\lambda_{j,0}} \right) \prod_{j=n}^N \langle j, \sigma_j | j+1, \sigma_{j+1} \rangle
 \end{aligned}$$

$$\begin{aligned}
 & H(n, N)_{\sigma_{N+1}=0} \\
 &= \left(\prod \lambda_{j,0} \langle j, 0 | j+1, 0 \rangle \right) \Omega(n, N) \\
 & \cdot \sum_{\sigma_n, \dots, \sigma_N} \frac{|n, \sigma_n\rangle}{\Omega(n, N)} \left(\prod_{j=n}^N \frac{\lambda_{j, \sigma_j}}{\lambda_{j,0}} \right) \prod_{j=n}^N \frac{\langle j, \sigma_j | j+1, \sigma_{j+1} \rangle}{\langle j, 0 | j+1, 0 \rangle}
 \end{aligned}$$

$$\Omega(n, N) \stackrel{\text{def}}{=} \sum_{s \geq 0} \sum_{\substack{J_1 < \dots < J_s \\ \sigma_{J_1}, \dots, \sigma_{J_s}}} \prod_{i=1}^s I(J_i, \sigma_{J_i})$$

$(H_{n,N} v_N)_\sigma =$ is therefore $\bar{\Lambda}(n, N) \sum_{\sigma_n = \sigma} P_{n, \sigma, N} |n, \sigma\rangle$

where if $J = [h, h']$:

$$I(J, \boldsymbol{\sigma}_J) \stackrel{\text{def}}{=} \left(\prod_{j=h}^{h'} \frac{\lambda_{j, \sigma_j}}{\lambda_{j, 0}} \right) \left(\prod_{j=h}^{h'-1} \frac{\langle j, \sigma_j | j+1, \sigma_{j+1} \rangle}{\langle j, 0 | j+1, 0 \rangle} \right) \\ \cdot \left(\frac{\langle h-1, 0 | h, \sigma_h \rangle}{\langle h-1, 0 | h, 0 \rangle} \right)^{\delta_{h>n}} \left(\frac{\langle h', \sigma_{h'} | h'+1, 0 \rangle}{\langle h', 0 | h'+1, 0 \rangle} \right)$$

$$\bar{\Lambda}(n, N) \stackrel{\text{def}}{=} \Omega(n, N) \prod_{j=n}^N \left(\lambda_{j, 0} \langle j, 0 | j+1, 0 \rangle \right)$$

$$\Omega(n, N) \stackrel{\text{def}}{=} \sum_{s \geq 0} \sum_{\substack{J_1 < \dots < J_s \\ \boldsymbol{\sigma}_{J_1, \dots, \boldsymbol{\sigma}_{J_s}}} } \prod_{i=1}^s I(J_i, \boldsymbol{\sigma}_{J_i})$$

The $(H_{n, N} v_N)_\sigma$ is therefore $P_{n, \sigma, N} |n, \sigma\rangle P_{n, \sigma, N}$ equal to the “probability” that at site n the “spin” is σ in the model with polymers activity $I(J, \boldsymbol{\sigma}_J)$

Bounds (*gap* $\frac{|\lambda_j|}{|\lambda_0|} \leq \gamma$, *cone property* “*inclination* α' ,
polymers number $(d - 1)$ per base site,

$$\prod \frac{|\lambda_j|}{|\lambda_0|} \leq \gamma^{h'-h+1}, \prod \frac{\langle j, \sigma_j | j+1, \sigma_{j+1} \rangle}{\langle j, 0 | j+1, 0 \rangle} \leq \frac{1}{(\cos \alpha')^{h'-h+1}}$$

$$\sum_{\sigma_J} |I(J, \sigma_J)| \leq \left(\frac{(d-1)\gamma}{\delta}\right)^{h'-h+1} \equiv \eta^{h'-h+1}, \quad \delta \stackrel{\text{def}}{=} \cos \alpha'$$

The $(H(n, N)v)$ can be written: $\sum_{\sigma=0}^{d-1} P_{n,\sigma,N} |n, \sigma\rangle$ with

$$P_{n,\sigma,N} \stackrel{\text{def}}{=} \frac{\Omega_{\sigma}(n, N)}{\Omega(n, N)} \stackrel{\text{def}}{=} \sum_{s \geq 0} \sum^{*\sigma} \frac{\prod_{i=1}^s I(J_i, \sigma_{J_i})}{\Omega(n, N)}$$

the $*\sigma$ indicates sum restricted to polymers with $\sigma_n = \sigma$.

Polymers gas on $[n, N]$ with activity $|I(S, \sigma_S)| < \eta^{|J|}$

Actually “polymers” \equiv “segments J decorated σ_J ”

Fisher droplet model (machine for (counter)examples)

Cluster expansion converges if:

$$\mu \stackrel{def}{=} \sup_{J, \sigma_J} |I(J, \sigma_J)|^{\frac{1}{2}} \exp \sum_{S, \sigma_S}^* |I(S, \sigma_S)|^{\frac{1}{2}} < 1$$

sum $*$ $\Rightarrow S \cap J \neq \emptyset$. *i.e. if $\eta = (d-1)\delta^{-1}\gamma$ small enough*

$$P_{n, \sigma, N} = \sum_{J \ni n, \sigma_J, \sigma_n = \sigma} \varphi^T(J, \sigma_J), \quad \sum_{\sigma_J} |\varphi^T(J, \sigma_J)| < C\eta^{\frac{1}{2}|J|}$$

with the sum exponentially convergent to a boundary condition independent limit.

Necessary γ =gap small enough: if not replace T_k by $T_k T_{k+1} \cdots T_{k+q}$ with q large enough! ...

Application to Dynamical systems

\mathcal{F} smooth bounded manif., $\tau : \mathcal{F} \rightarrow \mathcal{F}$ smooth, smooth inverse, map on \mathcal{F} . Let $T(x)E(x) = E(\tau x)$.

At each point $x \in \mathcal{F}$ closed convex cones $\Gamma(x) \supset \Gamma'(x)$, apex at x in a space $E(x)$ (dimens. d).

Min. angle $\Gamma(x) - \Gamma'(x) =$ “inclination” $\varepsilon(x)$

Max. angles in $\Gamma(x), \Gamma'(x) =$ “openings” be $\theta(x), \theta'(x)$.

Definition: Let $T(x)$ depend on a variable z , real analytic for $|z| < \nu$; $T(x)\Gamma(x) \subset \Gamma'(\tau x)$ for z real, $|z| < \nu$, then T “ ν' - z -analytic with cone p .”, $\nu' \leq \nu$. **Let** $T_n \stackrel{\text{def}}{=} T(\tau^{-n}x)$.

Theorem 2: $x \rightarrow v(x) \in \Gamma'(x), \|v(x)\| \equiv 1$ **measurable**

(1) \exists continuous functions $x \rightarrow b(x) \in \Gamma(x)$, and

$x \rightarrow \Lambda(x, p), p = 0, 1, \dots$, s.t. for all $p > 0, x \in \mathcal{F}, v, \exists$

$$b(x) = \lim_{N \rightarrow \infty} \frac{T(x) \cdots T(\tau^{-(N-1)}x) |v(\tau^{-N}x)\rangle}{\bar{\Lambda}(x, N)}$$

$$\Lambda(x, p) = \lim_{N \rightarrow \infty} \frac{\bar{\Lambda}(\tau^p x, N)}{\bar{\Lambda}(x, N)} > 0, \quad \forall v$$

(2) $b(x)$ are “covariant” or “eigenvectors” for products of $T(\tau^{-j}x)$ in the sense

$$T^{-1}(\tau^{-(p-1)}x) \dots T^{-1}(x)b(x) = \Lambda(x, p) b(\tau^{-p}x)$$

(3) $b(x), \Lambda(x, p)$ are v -indep., continuous and $\exists B$ s.t. $B^{-1} < \|b(x)\| < B$; the $\frac{b(x)}{\|b(x)\|}$ def. the “+ direction”, at τx .

(4) $\Lambda(x, p)$ analytic in $z, |z| < \nu$.

(5) $\limsup_{p \rightarrow \infty} \frac{1}{p} \log |\Lambda(x, p)|$ and \liminf are constants of motion (k -independent at $\tau^k x$).

Theorem 3: Let ρ be τ -invariant distribution on \mathcal{F} Then:
(1) the limits are ρ -a.e. equal (FK nor Ra not needed);
(2) ρ ergodic \Rightarrow also ρ -a.e. x -independent.

Consequence

If ρ is ergodic the logarithm $= -\lambda_+$, analytic in z .

Proof essence: understand case T_j diagonalizable, real eigenvectors and eigenvalues uniformly (in j) pairwise separated; furthermore

$$\max_{n,\sigma=1,\dots,d-1} |\lambda_{\sigma,n}|/|\lambda_{0,n}| < \varepsilon_0, \quad \varepsilon_0 \text{ small}$$

General case via $T'_n = T_{nq}T_{nq+1} \cdots T((n+1)q-1)$: then q large enough \rightarrow large gap (& no diagonalization need). Theorem holds for T'_n hence for T_n .

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