# MACROSCOPIC FLUCTUATION THEORY

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# Joint with

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# The starting point

# Framework

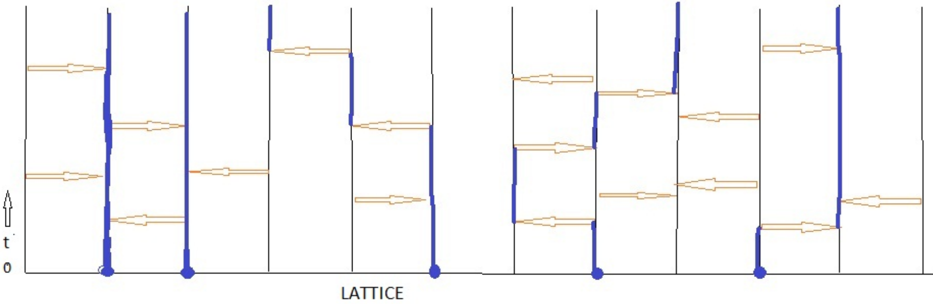
- $\Lambda_N \subseteq \frac{1}{N}\mathbb{Z}^d$, lattice with mesh $\frac{1}{N}$
- $\eta =$ configuration of particles
- $\eta_t(i) =$ number of particles at site $i \in \Lambda_N$, at time $t$
- $\eta \in S^{\Lambda_N} =$ configuration space

$$S = \begin{cases} \{0, 1\} \\ \mathbb{N} \\ \mathbb{R} \\ \dots \end{cases}$$

# Stochastic dynamics

- $\eta_t \to$ stochastic Markovian evolution
- Many degrees of freedom interacting
- Harris graphical construction: $\eta_t$ as a function of independent Poisson processes

# Simple exclusion process



# Generator

- The dynamics is encoded in the generator

$$L_N f(\eta) = \sum_{\eta'} c(\eta, \eta') \left[ f(\eta') - f(\eta) \right]$$

- $c(\eta, \eta')$ = rate of transition from $\eta$ to $\eta'$
- $\eta'$ local modification of $\eta$

# Zero Range

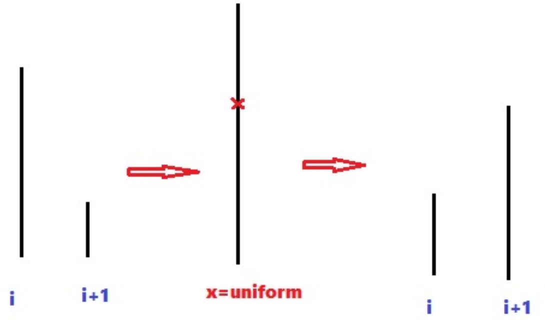
- $\eta \in \mathbb{N}^{\Lambda_N}$ = configuration space
- If one particle jumps from $i$ to $j$ we write $\eta \to \eta^{i,j}$

$$\eta^{i,j}(k) = \begin{cases} \eta(i) - 1 & \text{if } k = i \\ \eta(j) + 1 & \text{if } k = j \\ \eta(k) & \text{if } k \neq i, j \end{cases}$$

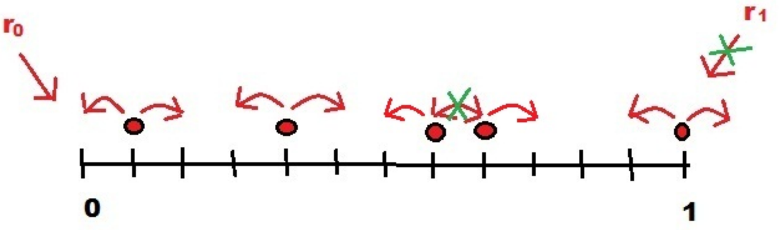
- Rate of jump $c(\eta, \eta^{i,j}) = g(\eta(i))p(i,j)$
- $g : \mathbb{N} \to \mathbb{R}^+$, $g(0) = 0$

# KMP model

- $\eta \in (\mathbb{R}^+)^{\Lambda_N}$; $\eta(i)$ = energy of an oscillator at $i \in \Lambda_N$



# Boundary driven models



# Invariant measure

- $\mu_N$ probability measure on the configuration space
- $\mathbb{P}_{\mu_N}$ = Markovian probability measure on paths with initial condition $\mu_N$
- $\mu_N$ is invariant if

$$\mathbb{P}_{\mu_N}\left(\eta_t = \eta'\right) = \mu_N(\eta') \qquad \forall \eta'$$

- Detailed balance $\Longleftrightarrow$ reversibility

$$\mu_N(\eta)c(\eta, \eta') = \mu_N(\eta')c(\eta', \eta)$$

- Detailed balance $\Longrightarrow$ $\mu_N$ is invariant

# Large deviations

- Law of large numbers

$$\frac{1}{N}\sum_{i=1}^{N} X_i \to \mathbb{E}(X_1)$$

- Central limit Theorem

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} [X_i - \mathbb{E}(X_i)] \to \mathcal{N}(0,\sigma^2)$$

- Large deviations (Cramer Theorem)

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} X_i \in A\right) \simeq e^{-N \inf_{x\in A} I(x)}$$

- $I(x) =$ Rate functional, $I(x) \geq 0$ and $I\left(\mathbb{E}(X_1)\right) = 0$

# LDP formal statement

Sequence of random variables $X_N$ taking values on a Polish space $M$ satisfies LDP with speed $\alpha(N)$ and rate functional $I : M \to \mathbb{R}^+ \cap \{+\infty\}$ if

- Upper bound

$$\limsup_{N\to+\infty} \frac{1}{\alpha(N)} \log \mathbb{P}\left(X_N \in C\right) \leq -\inf_{x\in C} I(x) \qquad \forall C \text{ closed}$$

- Lower bound

$$\liminf_{N\to+\infty} \frac{1}{\alpha(N)} \log \mathbb{P}\left(X_N \in O\right) \geq -\inf_{x\in O} I(x) \qquad \forall O \text{ open}$$

We write

$$\mathbb{P}\left(X_N \sim x\right) \simeq e^{-\alpha(N) I(x)}$$

# Empirical measure

- Coarse graining of a configuration of particles
- $\Lambda \subseteq \mathbb{R}^d$ bounded domain
- $\Lambda_N = \Lambda \cup \frac{1}{N}\mathbb{Z}^d$
- $\eta \to \pi_N(\eta) \in \mathcal{M}^+(\Lambda)$ positive measures on $\Lambda$

$$\pi_N(\eta) = \frac{1}{N^d} \sum_{i \in \Lambda_N} \eta(i)\delta_i$$

$$\int_\Lambda f \, d\pi_N(\eta) = \frac{1}{N^d} \sum_{i \in \Lambda_N} f(i)\eta(i)$$

- A sequence of configurations is associated to a density profile $\rho(x)$ if

$$\pi_N(\eta) \stackrel{N \to +\infty}{\to} \rho(x)dx$$

# LDP and coarse graining

- Exact microscopic structure of a stationary non equilibrium state is difficult
- Law of large numbers

$$\lim_{N\to+\infty} P_{\mu_N}\left(\left|\int_{[0,1]} f\, d\,\pi_N(\eta) - \int_{[0,1]} f(x)\bar{\rho}(x)\, d\,x\right| \geq \epsilon\right) = 0$$

- We are satisfied with the LDP asymptotics

$$P_{\mu_N}\left(\pi_N(\eta) \sim \rho(x)dx\right) \simeq e^{-N^d V(\rho)}$$

- We have $V(\rho) \geq 0$ and $V(\bar{\rho}) = 0$

# LDP and relative entropy

- Relative entropy

$$H(\nu_N|\mu_N) = \sum_\eta \nu_N(\eta) \log \frac{\nu_N(\eta)}{\mu_N(\eta)}$$

- Rate functional of large deviations is computed as a density of relative entropy

$$h = \lim_{N\to+\infty} \frac{1}{\alpha(N)} H(\nu_N|\mu_N)$$

- You have to find a suitable class of perturbations $\nu_N^x$ and $I(x) = h$

# Contraction principle

- $X_N$ satisfies a LDP on a metric space $M$ with a rate functional $I$
- $f : M \to M'$ continuous
- $f(X_N)$ satisfies a LDP on $M'$ with rate function

$$I'(y) = \inf_{x : f(x)=y} I(x)$$

# LDP for product measures

- 1 dimensional SEP with equal sources at the boundaries is reversible and has product invariant measure $\mu_N^\alpha$
- To compute LD the perturbations are still product $\Longrightarrow$ direct computation
- Law of large numbers $\pi_N(\eta) \to \alpha dx$ in $\mu_N^\alpha$ probability
- LDP rate functional

$$V_{\mathrm{SEP}}^{\alpha}(\rho) = \int_{[0,1]} dx \big[ f(\rho(x)) - f(\alpha) - f'(\alpha)(\rho(x) - \alpha) \big]$$

# Density of free energy

- For SEP the density of free energy

$$f(\rho) = \rho \log \rho + (1-\rho)\log(1-\rho)$$

- Up to an affine term coincides with

$$f(\rho) = \sup_\lambda \left[\rho\lambda - p(\lambda)\right]$$

where

$$p(\lambda) = \lim_{N\to+\infty} \frac{1}{N} \log \mathbb{E}_{\mu_N^\alpha}\left(e^{\lambda \sum_{i\in\Lambda_N} \eta(i)}\right)$$

(Gärtner–Ellis, pressure)

# LDP for zero range

- Invariant measure always product even if not homogeneous
- law of large numbers $\pi_N(\eta) \to \bar{\rho}(x)dx$ in general not constant
- Direct explicit computation of LD rate functional

$$V_{\mathrm{ZR}}(\rho) = \int_\Lambda dx \left[ f(\rho(x)) - f(\bar{\rho}(x)) - f'(\bar{\rho}(x))(\rho(x) - \bar{\rho}(x)) \right]$$

- $f$ and $\bar{\rho}(x)$ depend on the function $g$

# Non equilibrium SEP

- When in contact with different reservoirs the 1-d boundary driven SEP is not reversible
- The invariant measure has combinatorial representations
- The LD rate functional is not local

$$V(\rho) = \sup_f \mathcal{G}(\rho, f)$$

- The supremum is over functions satisfying $f(0) = \rho_-$ and $f(1) = \rho_+$ determined by the sources

# LDP for non equilibrium KMP

- 1 dimensional non equilibrium boundary driven KMP is macroscopically exactly solvable
- LD rate functional for the invariant measure is

$$V(\rho) = \inf_f \mathcal{G}(\rho, f)$$

  where

$$\mathcal{G}(\rho, f) = \int_0^1 dx \left[ \frac{\rho}{f} - 1 - \log \frac{\rho}{f} - \log \frac{\nabla f}{\rho_+ - \rho_-} \right]$$

- The optimal $f$ solves

$$f^2 \frac{\Delta f}{(\nabla f)^2} - f = -\rho$$

- $\mathcal{G}(\rho, f)$ is a joint LDP rate functional $\Longrightarrow$ natural interpretation as contraction over an hidden temperature profile $f$

# Current

- $(i, j)$ edge of the lattice
- $\mathcal{N}_{i,j}(t) =$ number of particles jumped from $i$ to $j$ up to time $t$
- Net current across the edge $(i, j)$

$$Q_{i,j}(t) = \mathcal{N}_{i,j}(t) - \mathcal{N}_{j,i}(t)$$

- antisymmetric

$$Q_{i,j}(t) = -Q_{j,i}(t)$$

# Empirical current

- Empirical current is a vector valued measure on $\Lambda \times [0, t]$
- It is a function of a trajectory $(\eta_s)_{s\in[0,t]}$

$$\mathcal{J}_N(\eta, s) = \frac{1}{N^d} \sum_{\{i,j\}} (j - i)\delta_{\frac{i+j}{2}} \frac{d\, Q_{i,j}(s)}{ds}$$

- satisfies a discrete continuity equation

$$\partial_t \pi_N(\eta_t) + \nabla \cdot \mathcal{J}_N(\eta, t) = 0$$

# Empirical current integrated

- Let $V(x, s)$ a smooth vector field
- We have

$$\int_0^t \int_\Lambda \mathcal{J}_N(\eta, s) \cdot V(s)\, dx\, ds$$

$$= \frac{1}{N^d} \sum_\ell V\left(\frac{i(\ell) + j(\ell)}{2}, \tau_\ell\right) \cdot \big(j(\ell) - i(\ell)\big)$$

- The sum is over all jumps in $[0, t]$
- At time $\tau_\ell$ one particle jumps from $i(\ell)$ to $j(\ell)$

# LDP current 1-d open boundary ZR

- $\frac{Q_{i,i+1}(t)}{t}$ satisfies LDP when $t \to +\infty$
- Particles created at 0 contribute with $+$ when die at 1
- Particles created at 1 contribute with $-$ when die at 0
- $0 \Rightarrow 1$ with probability $\frac{1}{N+1}$
- $1 \Leftarrow 0$ with probability $\frac{1}{N+1}$
- At the left boundary effective Poisson of parameter $\frac{\alpha}{N+1}$
- At the right boundary effective Poisson of parameter $\frac{\beta}{N+1}$

# Large deviations

- LDP for a Poisson process $\Gamma_t^\lambda$ of parameter $\lambda$

$$\mathbb{P}\left(\frac{\Gamma_t^\lambda}{t} \sim x\right) \simeq e^{-t\Psi(x,\lambda)}$$

where

$$\Psi(x,\lambda) = x \log \frac{x}{\lambda} + \lambda - x$$

- The current is exponentially close to the difference of two effective independent Poisson

$$Q_{i,i+1}(t) \simeq \Gamma_t^{\frac{\alpha}{N+1}} - \Gamma_t^{\frac{\beta}{N+1}}$$

# Contraction

$$\mathbb{P}\left(\frac{Q_{i,i+1}(t)}{t} \sim J\right) \simeq e^{-t\varphi_N(J)}$$

where by contraction

$$\varphi_N(J) := \inf_{\{x^+ - x^- = J\}}\left(\Psi\left(x^+, \frac{\alpha}{N+1}\right) + \Psi\left(x^-, \frac{\beta}{N+1}\right)\right) =$$

$$J\left[\sinh^{-1}\left(\frac{J(N+1)}{2\sqrt{\alpha\beta}}\right) + \log\sqrt{\frac{\beta}{\alpha}}\right] + \frac{\alpha+\beta}{(N+1)} - \sqrt{J^2 + \frac{4\alpha\beta}{(N+1)^2}}$$

# Scaling

Accelerating the rates by $N^2$ ($\alpha_N = \alpha N^2$ and $\beta_N = \beta N^2$) we have

$$\frac{1}{N}\varphi_N(NJ) \to \Phi(J)$$

$$\Phi(J) = J\left[\sinh^{-1}\left(\frac{J}{2\sqrt{\alpha\beta}}\right) + \log\sqrt{\frac{\beta}{\alpha}}\right] + \alpha + \beta - \sqrt{J^2 + 4\alpha\beta}\,.$$

# Freidlin–Wentzell theory

- $b$ Lipshitz vector field with an unique equilibrium point $b(\bar{x}) = 0$, globally attractive
- diffusion process

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \sqrt{\epsilon} d W_t$$

- Invariant measure $\mu^\epsilon(dx)$ solves the corresponding partial differential equation

# Large deviations

- If $b(x) = -\nabla U(x)$ the process is reversible
- The invariant measure is

$$\mu^{\epsilon}(dx) = \frac{1}{Z_{\epsilon}} e^{-2\epsilon^{-1}U(x)}\, dx$$

- By Laplace Theorem we deduce a LDP when $\epsilon \to 0$ with rate function coinciding with $2U(x)$ up to a constant.
- If the process is not reversible explicit computations are difficult

# Dynamic LDP

- Trajectories $(X^\epsilon(s))_{s\in[0,t]}$ satisfy LDP when $\epsilon \to 0$ with rate functional (action of a Lagrangian)

$$I_{[0,t]}(x) = \frac{1}{2}\int_0^t |\dot{x}(s) - b(x(s)|^2 \, d\, s$$

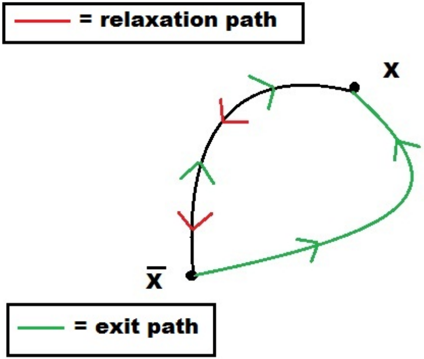
- Quasipotential

$$V(x) = \inf_{\{y: y(-t)=\bar{x}, y(0)=x\}} I_{[-t,0]}(y)$$

- The quasipotential coincides with the rate functional of the invariant measure

# Minimization

- Reversible case: simple minimizer, time reversal
- Non reversible case: difficult problem, no time reversal symmetry



# Hamilton–Jacobi equation

- The quasipotential $V(x)$ solves the Hamilton–Jacobi equation

$$\nabla V(x) \cdot \left[ \nabla V(x) + b(x) \right] = 0$$

- Orthogonality condition

# Scaling limits

- Diffusive rescaling $L_N \to N^2 L_N$
- The empirical measure satisfies law of large numbers $\pi_N(\eta_t) \to \rho(x,t)dx$

$$\lim_{N\to+\infty} \mathbb{P}_{\nu_N}\left(\left|\int_{[0,1]} f\, d\,\pi_N(\eta_t) - \int_{[0,1]} f(x)\rho(x,t)\, d\, x\right| \geq \epsilon\right) = 0$$

- $\rho(x,t)$ solves

$$\begin{cases} \partial_t \rho = \nabla \cdot (D(\rho)\nabla\rho) \\ \rho(x,0) = \rho_0(x) \\ \rho(x,t) = \psi(x) & \qquad x \in \partial\Lambda \end{cases}$$

# Examples

- $D(\rho) =$ Diffusion matrix, density dependent positive defined symmetric matrix
- SEP $D(\rho) = \mathbb{I}$
- Zero range $D(\rho) = \phi'(\rho)\mathbb{I}$, the function $\phi$ depends on the function $g$
- KMP model $D(\rho) = \mathbb{I}$

# Dynamic large deviations

- Deterministic initial condition associated to a profile $\pi_N(\eta) \to \rho_0(x)$
- Dynamic large deviations

$$\mathbb{P}_{\rho_0}\big(\pi_N(\eta_s) \sim \rho(s), \mathcal{J}_N(\eta, s) \sim j(s); s \in [0, t]\big) \simeq e^{-N^d \mathcal{I}_{[0,t]}(\rho, j)}$$

- If $\rho(x, t) \neq \psi(x), x \in \partial\Lambda$ then $I_{[0,t]}(\rho, j) = +\infty$
- If $\partial_t \rho + \nabla \cdot j \neq 0$ then $I_{[0,t]}(\rho, j) = +\infty$
- The rate functional is computed from relative entropy of the paths measure $\mathbb{P}_{\rho_0}\big|_{[0,t]}$ and a suitable weak asymmetric perturbation $\mathbb{P}^F_{\rho_0}\big|_{[0,t]}$

# Weakly asymmetric models

- $E : \Lambda \to \mathbb{R}^d$ smooth vector field
- When 1 particle jumps from $i$ to $j$ then $\eta \to \eta^{i,j}$ and the work done by the field is

$$\int_{(i,j)} E \cdot dl = O\left(\frac{1}{N}\right)$$

- Perturbed rates

$$c^E(\eta, \eta^{i,j}) = c(\eta, \eta^{i,j}) e^{\int_{(i,j)} E \cdot dl} = c(\eta, \eta^{i,j}) + O\left(\frac{1}{N}\right)$$

- Weakly asymmetric model with generator $L^E$ having rates $c^E$

# Scaling limits WA

- Diffusive rescaling $L_N^E \to N^2 L_N^E$
- The empirical measure satisfies law of large numbers $\pi_N(\eta_t) \to \rho(x,t)dx$
- $\rho(x,t)$ solves

$$\begin{cases} \partial_t \rho = \nabla \cdot (D(\rho)\nabla\rho) - \nabla \cdot (\chi(\rho)E) \\ \rho(x,0) = \rho_0(x) \\ \rho(x,t) = \psi(x) & x \in \partial\Lambda \end{cases}$$

- $\chi(\rho)$ positive definite symmetric and density dependent mobility matrix
- Einstein relation

$$D(\rho) = \chi(\rho) f''(\rho)$$

# Examples

- SEP $\chi(\rho) = \rho(1-\rho)\mathbb{I}$
- Zero range $\chi(\rho) = \phi(\rho)\mathbb{I}$, the function $\phi$ depends on the function $g$
- KMP model $\chi(\rho) = \rho^2\mathbb{I}$

# Scaling limit empirical current

- Empirical current satisfies law of large numbers

$$\mathcal{J}_N(\eta, t) \to j(x,t)dxdt$$

$$j(x,t) = -D(\rho(x,t))\nabla\rho(x,t) + \chi(\rho(x,t))E(x,t) = J(\rho)$$

and $\rho(x,t)$ is the solution of the hydrodynamic equation

# The principal formula

- The suitable external field for computing LD is given by

$$j(x,t) = -D(\rho(x,t))\nabla\rho(x,t) + \chi(\rho(x,t))F(x,t)$$

- The rate function suitably weights the external field

$$\mathcal{I}_{[0,t]}(\rho,j) = \frac{1}{4}\int_0^t \int_\Lambda F \cdot \chi(\rho) F \, dx \, dt$$

- Principal formula

$$\mathcal{I}_{[0,t]}(\rho,j) = \frac{1}{4}\int_0^t \int_\Lambda (j - J(\rho)) \cdot \chi^{-1}(\rho)\,(j - J(\rho))\, dx\, dt$$

# Lagrangian structure

- Introduce the vector field

$$A(x,t) = A_0(x) - \int_0^t j(x,s)ds$$

where $\nabla \cdot A_0(x) = \rho_0(x)$

- We have

$$j(x,t) = -\partial_t A(x,t) \qquad \rho(x,t) = \nabla \cdot A(x,t)$$

- The rate function becomes

$$\mathcal{I}_{[0,t]}(\rho, j) = \int_0^t \mathbb{L}(A, \partial_s A)\, ds$$

- The Lagrangian is

$$\mathbb{L}(A, \partial_t A) = \frac{1}{4}\int_\Lambda (\partial_t A + J(\nabla \cdot A)) \cdot \chi^{-1}(\nabla \cdot A)\,(\partial_t A + J(\nabla \cdot A))\ dx$$

- The constraint of the continuity equation disappears since it is automatically satisfied (Schwartz theorem)

# Hamiltonian structure

- The corresponding Hamiltonian is

$$\mathbb{H}(A,B) = \sup_{\xi}\left\{\int_\Lambda B(x)\cdot\xi(x)\,dx - \mathbb{L}(A,\xi)\right\}$$
$$= \int_\Lambda \left[B\cdot\chi^{-1}(\nabla\cdot A)B - B\cdot J(\nabla\cdot A)\right]\,dx$$

- the Hamilton equations

$$\begin{cases} \partial_t A = 2\chi(\nabla\cdot A)B - J(\nabla\cdot A) \\ \partial_t B = -\nabla\left[\mathrm{Tr}\big(D(\nabla\cdot A)\nabla^T B\big) - B\cdot\chi'(\nabla\cdot A)B\right] \end{cases}$$

# Time reversal

- Stationary process $\mathbb{P}_{\rm st}(\eta_t = \eta') = \mu_N(\eta')\ \forall t$
- $\theta =$ time reversal $(\theta\eta)_t = \eta_{-t}$
- $\theta\eta$ is still Markovian with generator $L_N^*$ with rates

$$c^*(\eta, \eta') = \frac{\mu_N(\eta')c(\eta', \eta)}{\mu_N(\eta)}$$

- $\mu_N$ is still invariant for $L_N^*$

# Time reversal again

- By definition we have

$$\mathbb{P}_{\mathrm{st}}\left(\pi_N(\eta_s) \simeq \rho(s)\,,\mathcal{J}_N(\eta,s) \simeq j(s)\,; s \in [t_1,t_2]\right) =$$
$$\mathbb{P}^*_{\mathrm{st}}\left(\pi_N(\theta\eta_s) \simeq (\theta\rho)(s)\,,\mathcal{J}_N(\theta\eta,s) \simeq (\theta j)(s)\,; s \in [-t_2,-t_1]\right)$$

- where $(\theta\rho)(x,s) = \rho(x,-s)$ and $(\theta j)(x,s) = -j(x,-s)$
- By Markov property

$$\mathbb{P}_{\mathrm{st}}\left(\pi_N(\eta_s) \simeq \rho(x,s)dx\,,\mathcal{J}_N(\eta,s) \simeq j(x,s)dxds\,; s \in [t_1,t_2]\right)$$
$$\simeq e^{-N^d\left[V(\rho(t_1))+\mathcal{I}_{[t_1,t_2]}(\rho,j)\right]}$$

# Time reversal and large deviations

- Time reversal symmetry at large deviations scale reads

$$V(\rho(t_1)) + \mathcal{I}_{[t_1,t_2]}(\rho, j) = V(\rho(t_2)) + \mathcal{I}^*_{[-t_2,-t_1](\theta\rho,\theta j)}$$

- $\mathcal{I}^*$ is the dynamic rate functional for the time reversed process
- Assume $\mathcal{I}^*$ has the same structure of the direct process

$$\mathcal{I}^*_{[0,t]}(\rho, j) = \int_0^t \int_\Lambda (j - J^*(\rho)) \cdot \chi^{-1}(\rho)\,(j - J^*(\rho))\,dx\,dt$$

- $J^*(\rho)$ is the typical current observed in the reversed process associated to the density profile $\rho$

# Fluctuation dissipation and Hamilton–Jacobi

- Take the time reversal relationship for LD in $[-t, t]$, divide by $2t$, take the limit $t \to 0 \implies$ instantaneous relationships
- Fluctuation dissipation

$$J(\rho) + J^*(\rho) = -2\chi(\rho)\nabla \frac{\delta V}{\delta \rho}$$

- Hamilton–Jacobi equation

$$\int_\Lambda J_S(\rho) \cdot \chi^{-1}(\rho) J_A(\rho)\, dx$$

- Symmetric and antisymmetric part of the current

$$\begin{cases} J_S(\rho) = \frac{J(\rho)+J^*(\rho)}{2} \\ J_A(\rho) = \frac{J(\rho)-J^*(\rho)}{2} \end{cases}$$

# Projection to the density

- Large deviations for fluctuations of the density alone

$$\mathbb{P}_{\rho_0}\left(\pi_N(\eta_s) \sim \rho(x,s)dx\,,\, s \in [0,t]\right) \sim e^{-N^d I_{[0,t]}(\rho)}$$

- By contraction

$$I_{[0,t]}(\rho) = \inf_{\{j\,:\nabla j = -\partial_s \rho\}} \mathcal{I}_{[0,t]}(\rho, j)$$

# Dynamic LDP for the density

- The minimum is obtained for gradient vector fields

$$I_{[0,t]}(\rho) = \int_0^t \int_\Lambda \nabla H \cdot \chi(\rho) \nabla H \, dx \, ds$$

- where $H$ solves

$$\begin{cases} -\partial_s \rho + \nabla \cdot (D(\rho) \nabla \rho) = \nabla \cdot (\chi(\rho) \nabla H) \\ H(x,s) = 0 \qquad x \in \partial\Lambda \end{cases}$$

# Quasipotential

- Quasipotential
$$V(\rho) = \inf_t \inf_{\hat{\rho}} I_{[-t,0]}(\hat{\rho})$$
- The infimum is over time dependent density trajectories that satisfies $\hat{\rho}(x,-t) = \bar{\rho}(x)$ and $\hat{\rho}(x,0) = \rho(x)$
- $\bar{\rho}(x)$ is the stationary solution of the hydrodynamic equation
$$\begin{cases} \nabla \cdot (D(\bar{\rho})\nabla\bar{\rho}) = 0 & \\ \bar{\rho}(x) = \psi(x) & x \in \partial\Lambda \end{cases}$$
- The quasipotential coincides with the LDP rate functional for the invariant measure

# Time reversal again

- Time reversal symmetry for densities

$$V(\rho(t_1)) + I_{[t_1,t_2]}(\rho) = V(\rho(t_2)) + I^*_{[-t_2,-t_1]}(\theta\rho)$$

- For a time dependent density trajectory $\hat{\rho}$ satisfying $\hat{\rho}(-t) = \bar{\rho}$ and $\hat{\rho}(0) = \rho$ we have

$$I_{[-t,0]}(\hat{\rho}) = V(\rho) - V(\bar{\rho}) + I^*_{[0,t]}(\theta\hat{\rho}) \geq V(\rho)$$

- The minimizer $\hat{\rho}_m$ solves

$$I^*(\theta\hat{\rho}_m) = 0$$

- This means $\hat{\rho}_m = \theta\hat{\rho}^*$ where $\hat{\rho}^*$ is the solution of the hydrodynamic equation of the time reversed process with initial condition $\rho(x)$

# Examples

- Equilibrium 1-d boundary driven SEP (Reversibility)
- Reversibility: $L_N = L_N^*$, $I = I^*$
- Minimizer time reversal of hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta \rho \\ \rho(0,t) = \rho(1,t) = \alpha \end{cases}$$

- $J(\rho) = J^*(\rho) = -\nabla \rho$
- $J_S(\rho) = J(\rho)$, $J_A(\rho) = 0$

# Equilibrium (non homogeneous)

- Boundary sources and external field
- Stationary solution hydrodynamics $\nabla \cdot J(\bar{\rho}) = 0$
- Macroscopic equilibrium condition: $J(\bar{\rho}) = 0$
- Local rate functional

$$V(\rho) = \int_{[0,1]} dx \left[ f(\rho(x)) - f(\bar{\rho}(x)) - f'(\bar{\rho}(x))(\rho(x) - \bar{\rho}(x)) \right]$$

- This happens if $J(\rho) = -D(\rho)\nabla\rho + \chi(\rho)\nabla G$ for a suitable function $G$
- $J(\rho) = J^*(\rho)$

# Zero Range

- Always local rate functional but in general $J(\bar{\rho}) \neq 0$, not reversible
- We have

$$\begin{cases} J(\rho) = -\phi'(\rho)\nabla\rho \\ J^*(\rho) = -\phi'(\rho)\nabla\rho + \phi(\rho)E \end{cases}$$

- The external field associated to the hydrodynamics of the time reversed process is

$$E(x) = 2f''(\bar{\rho}(x))\nabla\bar{\rho}(x)$$

# Conditions for locality

- The large deviations rate functional is local for any boundary conditions and for any external field when $D$ and $\chi$ are diagonal and

$$D(\rho)\chi''(\rho) = D'(\rho)\chi'(\rho)$$

- On the $d-$dimensional torus when $D(\rho) = D(\rho\mathbb{I})$ and $\chi(\rho) = \chi(\rho)\mathbb{I}$ and there is an external field

$$E(x) = -\nabla U(x) + \tilde{E}(x)$$

where $\nabla\tilde{E}(x) = 0$ and $\nabla U(x) \cdot \tilde{E}(x) = 0$ for any $x$.

- In this case in general no reversibility and

$$\begin{cases} J(\rho) = -D(\rho)\nabla\rho + \chi(\rho)(-\nabla U + \tilde{E}) \\ J^*(\rho) = -D(\rho)\nabla\rho + \chi(\rho)(-\nabla U - \tilde{E}) \end{cases}$$

# Non equilibrium 1-d SEP

- Different boundary reservoirs, non reversible, non local rate functional

$$V(\rho) = \sup_f \int_0^1 dx \left[\rho \log \frac{\rho}{f} + (1-\rho) \log \frac{(1-\rho)}{(1-f)} + \log \frac{\nabla f}{\rho_+ - \rho_-}\right]$$

- The sup is over increasing functions such that $f(0) = \rho_-$ and $f(1) = \rho_+$
- The optimal $f$ solves

$$f(1-f)\frac{\Delta f}{(\nabla f)^2} + f = \rho$$

- We have $J(\rho) = -\nabla \rho$ and $J^*(\rho) = -\nabla \rho + \chi(\rho) E$ where

$$E(x) = \frac{2\nabla f(x)}{f(x)(1-f(x))}$$

# Hamilton Jacobi

- Hamilton–Jacobi equation can be rewritten as

$$\int_\Lambda \left[ \nabla \frac{\delta V}{\delta \rho} \cdot \chi(\rho) \nabla \frac{\delta V}{\delta \rho} - \frac{\delta V}{\delta \rho} \nabla \cdot J(\rho) \right] dx = 0$$

- 1-d SEP; we search for a solution of the form

$$\frac{\delta V}{\delta \rho} = \log \frac{\rho}{1-\rho} - \log \frac{f}{1-f}$$

- After some tricky integrations by parts H-J equation becomes

$$\int_0^1 \left( \cdots \right) \frac{\delta \mathcal{G}}{\delta f} \, dx = 0$$

# The solution

- We use the functional

$$\mathcal{G}(\rho, f) = \int_0^1 dx \left[\rho \log \frac{\rho}{f} + (1-\rho) \log \frac{(1-\rho)}{(1-f)} + \log \frac{\nabla f}{\rho_+ - \rho_-}\right]$$

- Note that $\frac{\delta \mathcal{G}}{\delta \rho} = \log \frac{\rho}{1-\rho} - \log \frac{f}{1-f}$
- We have that $\mathcal{G}(\rho, f[\rho])$ solves the Hamilton–Jacobi equation when $f[\rho]$ is critical since

$$\frac{\delta \mathcal{G}(\rho, f[\rho])}{\delta \rho} = \frac{\delta \mathcal{G}}{\delta \rho} = \log \frac{\rho}{1-\rho} - \log \frac{f[\rho]}{1-f[\rho]}$$

- Same computation if $D(\rho) = 1$ and $\chi(\rho)$ a second order polynomial

# Time averaged current

- Average current on $[0, t]$

$$\frac{1}{t}\int_0^t \mathcal{J}_N(\eta, s)\, ds$$

- LD for averaged current

$$\mathbb{P}_{\rho_0}\left(\frac{1}{t}\int_0^t \mathcal{J}_N(\eta, s)\, ds \sim J(x)dx\right) \simeq e^{-N^d t\Phi_t(J)}$$

- By contraction

$$\Phi_t(J) = \frac{1}{t}\inf_{(\rho,j)\in\mathcal{A}_t} \mathcal{I}_{[0,t]}(\rho, j)$$

- The infimum is over

$$\mathcal{A}_t = \left\{(\rho, j)\ : \partial_s\rho + \nabla\cdot j = 0\,,\ \frac{1}{t}\int_0^t j(s)ds = J\right\}$$

# Long time behavior

- When $t \to +\infty$ only divergence free $J$ are relevant
- For $J$ divergence free $t\Phi_t(J)$ is subadditive

$$(t+s)\Phi_{t+s}(J) \leq t\Phi_t(J) + s\Phi_s(J)$$

- Indeed if $(\rho_1, j_1) \in \mathcal{A}_t$ and $(\rho_2, j_2) \in \mathcal{A}_s$ then $\rho_1(0) = \rho_1(t) = \rho_2(0)$. We can concatenate the trajectories getting an element of $\mathcal{A}_{t+s}$
- There exists

$$\Phi(J) = \lim_{t \to +\infty} \Phi_t(J) = \inf_t \Phi_t(J)$$

# Independence from the initial condition

- In principle $\Phi(J)$ depends on the initial condition $\rho_0$
- This dependence is irrelevant
- Starting from a different initial condition $\rho_0'$ you go in finite time to $\rho_0$ then at the end you come back to $\rho_0'$ with an inverted current. The finite transient is irrelevant on long times

# Convexity

- If $J = pJ_1 + (1-p)J_2$ then

$$\Phi(J) \leq p\Phi(J_1) + (1-p)\Phi(J_2)$$

- Let $(\rho_1, j_1) \in \mathcal{A}_{pt}(J_1)$ and $(\rho_2, j_2) \in \mathcal{A}_{(1-p)t}(J_2)$.
- Since $J_1$ and $J_2$ are divergence free we can concatenate them into $(\rho, J) \in \mathcal{A}_t(J)$
- Since

$$\frac{1}{t}\mathcal{I}_{[0,t]}(\rho, j) = p\frac{1}{pt}\mathcal{I}_{[0,pt]}(\rho_1, j_1) + (1-p)\frac{1}{(1-p)t}\mathcal{I}_{[pt,t]}(\rho_2, j_2)$$

- Optimizing over $(\rho_1, j_1)$ and $(\rho_2, j_2)$ we get

$$\Phi_t(J) \leq p\Phi_{pt}(J_1) + (1-p)\Phi_{(1-p)t}(J_2)$$

- Take now the limit $t \to +\infty$

# Special paths

- Since $J$ is divergence free then a time independent path $(\rho, j) = (\rho(x), J(x)) \in \mathcal{A}_t$ and

$$\mathcal{I}_{[0,t]}(\rho, j) = t \int_\Lambda (J - J(\rho)) \cdot \chi^{-1}(\rho) \, (J - J(\rho)) \, dx$$

- Since we have independence from the initial condition we deduce

$$\Phi_t(J) \leq \inf_\rho \int_\Lambda (J - J(\rho)) \cdot \chi^{-1}(\rho) \, (J - J(\rho)) \, dx = U(J)$$

- $U(J)$ is the prediction for current fluctuations of the additivity principle; not necessarily convex
- We deduce $\Phi(J) \leq U(J)$. When does equality hold?

# Dynamic phase transitions

- When $\Phi(J) < U(J)$ we say that there is a dynamic phase transitions
- Open systems, no external field $D(\rho) = D(\rho)\mathbb{I}$ and $\chi(\rho) = \chi(\rho)\mathbb{I}$ with

$$D(\rho)\chi''(\rho) \leq D'(\rho)\chi'(\rho)$$

- In this case $\Phi(J) = U(J)$, no phase transition
- Follows by a joint convexity argument

$$\frac{1}{t}\mathcal{I}_{[0,t]}(\rho, j) \geq \int_\Lambda (J - J(\rho^*)) \cdot \chi^{-1}(\rho^*)\,(J - J(\rho^*))\,dx$$

where $J = \frac{1}{t}\int_0^t j(s)ds$ and $\rho^* = \frac{1}{t}\int_0^t \rho(s)ds$

- This implies $\Phi(J) \geq U(J)$ and the equality follows

# Periodic systems

- A system in the $d$ dimensional torus of side length 1 with constant external field $E$
- If $D(\rho) = D(\rho)\mathbb{I}$ and $\chi(\rho) = \chi(\rho)\mathbb{I}$ and

$$\frac{|J|^2}{\chi(\rho)} + |E|^2\chi(\rho)$$

  is convex in $\rho$
- The minimizer for computing $U(J)$ is constant in space $\bar{\rho}$

$$U(J) = \frac{1}{4}\frac{|J - \chi(\bar{\rho})E|^2}{\chi(\bar{\rho})}$$

# Traveling waves

- For special models on the torus it is possible to construct a periodic traveling wave $(\rho, j) = (\rho(x - vt), j(x - vt))$ of period $T$ such that

$$\frac{1}{T}\mathcal{I}_{[0,T]}(\rho, j) < U(J)$$

- Dynamic phase transition
- WASEP for special values of the external field and current
- KMP without external field for large enough currents

# An example

- 1 dimensional open boundary systems without external field, $U(J)$ is explicitely computed
- Divergence free currents are constant
- Zero range we known $\Phi(J) = U(J)$ and

$$U(J) = \inf_{\rho} \int_0^1 \frac{\left(J + \phi'(\rho(x))\nabla\rho(x)\right)^2}{\phi(\rho(x))} dx$$

- Change of variables $\alpha(x) = \phi(\rho(x)) \Longrightarrow$ independence on $g$

$$U(J) = \inf_{\alpha} \int_0^1 \frac{\left(J + \nabla\alpha(x)\right)^2}{\alpha(x)} dx$$

- Euler-Lagrange equations $\Longrightarrow$ explicit solution

# Gallavotti–Cohen type symmetries

- By a symmetry that holds at finite time we deduce

$$\Phi(J) - \Phi(-J) = -\int_\Lambda J \cdot E \, dx + \int_{\partial\Lambda} d\sigma \lambda(x) J(x) \cdot n(x)$$

- When $\Phi = U$ this can be generalized.
- $J$ and $J'$ divergence free and $|J(x)|^2 = |J'(x)|^2$ for any $x$ then

$$U(J) - U(J') = \frac{1}{2}\left[\int_\Lambda (J' - J) \cdot E \, dx + \int_{\partial\Lambda} d\sigma \lambda (J - J') \cdot n\right]$$