Proportionate growth and pattern formation in growng sandpiles

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Work done with

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- S. B. Singha
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- Rajeev Kapri
- R. Dandekar
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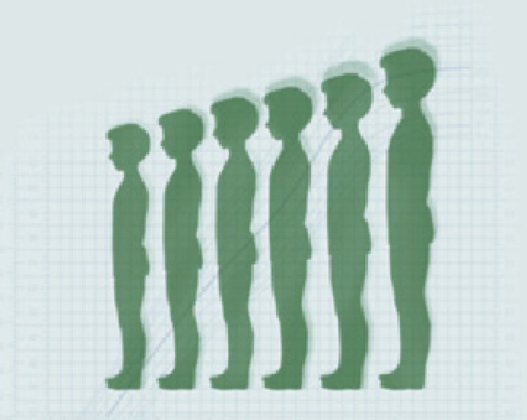
Outline

- Why is this problem interesting?
  - Proportionate growth in Animals
  - Pattern formation in Physics
  - Some interesting Mathematics: Discrete harmonic functions and tropical polynomials
- Definition of the model
- Examples of patterns
- Exact characterization
- Summary

Growth in biological systems



Different body parts in animals grow roughly at the same rate.

Proportionate growth in biology

- Proportionate growth is typical in animal kingdom.
  Also some parts of plants, say leaves.
- Easier problem than development of animal from a single cell.
- Qualitatively different for previously studied models of growth in physics.

Proportionate growth in biology

- Proportionate growth requires regulation, and/or communication between different parts.
- Same food becomes different tissues in different parts of the body.
- The standard biological answer only identifies different chemical agents that achieve this. Variously called growth factors, inhibitors, harmones.... "the knife did it!"
- This growth is orchestrated by the genetic program encoded in the animal's DNA, which turns on and off the production of different proteins, and controls which cells divide, and when.

This is undoubtedly correct, but a bit reminiscent of organized placard displays.

A 1940's picture of an organized placard display in China.

Can we find a simpler physical/ mathematical model that achieves this function, ignoring chemical detail?

In the spirit of
d' Archy Wentworth Thomson, " on Growth and form", (1917).
Von Neumann, " Theory of self-reproducing automata", (1966)

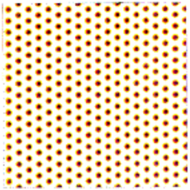
Growth Phenomena in Physics

Figure: Earlier sudied models of growth in physics: (a) A DLA cluster, (b) Epsom salt crystals grown from solution, (c) An invasion percolation cluster

In all these cases, inner parts do not grow further. Proportionate growth is qualitatively different.
In fact, non-biological physical systems showing proportionate growth are hard to find.

Pattern formation in Physics

Time-evolving complex spatio-temporal patterns.



Stationary Turing Pattern

Stationary Meander

Tiger skin pattern

Belousov-Zhabotinsky reaction

Characterizing complex patterns

Simple dynamical rules can generate complex patterns.
A long string of binary digits 001001101001..., with update rule $x_i' = [x_{i-1} + x_{i+1}](mod 2)$, generates the Sierpinski gasket.

More complex patterns like the Mandelbrot set are specified by simple rules, but are difficult to characterize in detail.

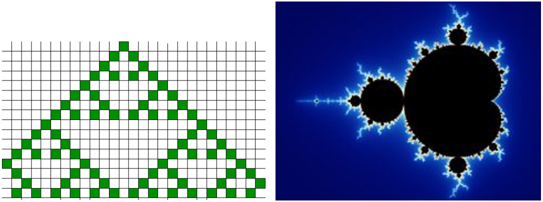


Figure: (a) The Sierpinski gasket (b) Mandelbrot set

Self-organization and sandpiles

In 1970's, Haken, Prigogine introduced the idea of living systems being 'self-organized'.

In 1987, Bak et al realized that many natural systems are self-organized to be at the edge of stability, and called these Self-Organized Critical.

They proposed a sandpile model as prototype model of SOC. Many earlier studies about the power-laws in distribution of avalanche sizes.

Pattern formation and proportionate growth in growing sandpiles

- A simple cellular automaton model showing proportionate growth
- Analytically tractable, non-trivial beautiful patterns
- Robustness with respect to noise

Proportionate growth in sandpile patterns

Basic facts from biology:

- Food required for growth. Reaches different body parts.
- Cell-division occurs only if the cell has enough nutrients.

A well-studied model of threshold dynamics is the Abelian Sandpile Model

Definition of ASM:

- Non-negative integer height z_i at sites i of a square lattice
- Add rule: $z_i \to z_i + 1$
- Relaxation rule : if $z_i > z_c = 3$, topple, and move one grain to each neighbor.

Rule for forming patterns:

Add N particles at one site on a periodic background, and relax.

Generalization to other lattices, higher dimensions

Sandpile Model: toppling rules

Start with a stable configuration, and add a particle :

0	0	0	0	0
0	0	0	0	0
0	0	4	0	0
0	0	0	0	0
0	0	0	0	0

$\longrightarrow$

0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	0	1	0	0
0	0	0	0	0

$\longrightarrow$

0	0	0	0	0
0	0	3	0	0
0	3	4	3	0
0	0	3	0	0
0	0	0	0	0

Finally, we get stable configuration:

0	0	1	0	0
0	2	1	2	0
1	1	0	1	1
0	2	1	2	0
0	0	1	0	0

.

Proportionate growth

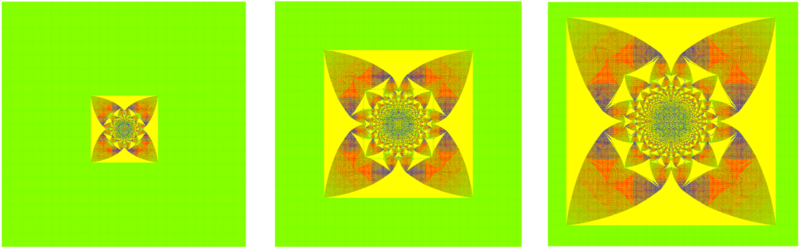


Figure: Patterns formed on a square lattice with initial height 2 at all sites. $N = (a) 4 \times 10^4$ (b) 2×10^5 (c) 4×10^5. Color code $0, 1, 2, 3 =$ R,B,G,Y

Diameter $\sim \sqrt{N}$.

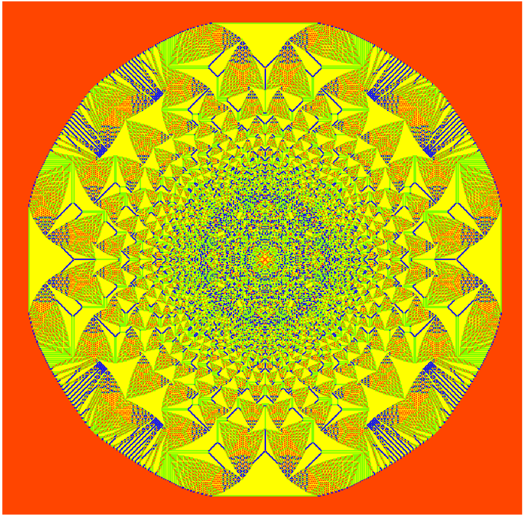


Figure: Patterns produced by adding 400000 particles at the origin, on a square lattice ASM, with initial state (a) all 0. Color code 0, 1, 2, 3 = R,B,G,Y

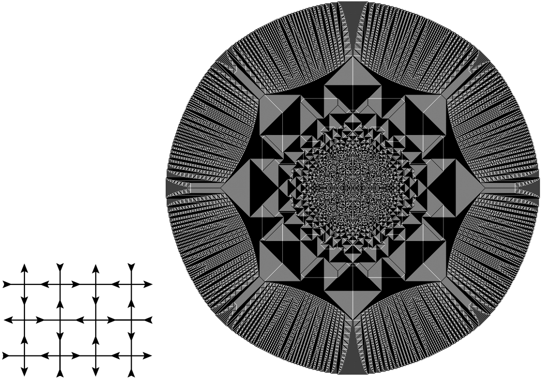


Figure: (a) the F lattice (b) Pattern produced by adding 10^5 particles at the origin, with initial state alternating columns of 1's and 0's.
Color code: $B = 0, W = 1$

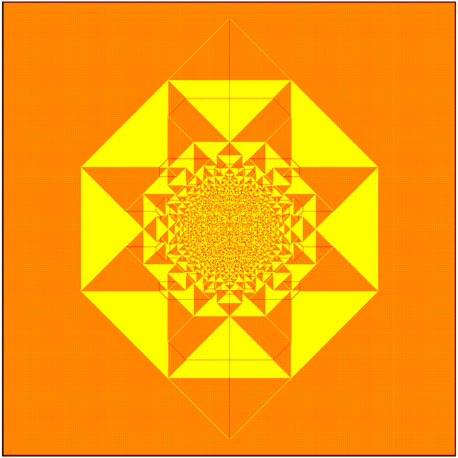


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard. Color code: 0 =R, 1=Y

Precise definition of Proportionate Growth

Let $T_N(\vec{R})$ = the number of topplings at point $\vec{R}$.

Define reduced coordinate $\vec{r} = \vec{R}/\Lambda$, Λ = diameter

Proportionate growth if scaling for large N:

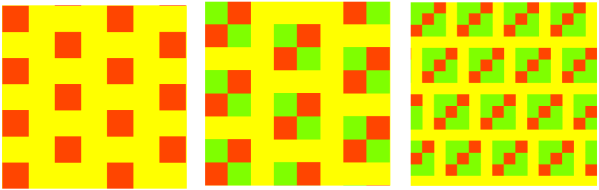
For large N, $T_N(\vec{R}) \sim \Lambda^a \phi(\vec{r})$, with $\Lambda \sim N^{1/b}$
A non-trivial $\phi(\vec{r})$ defines proportionate growth, and the asymptotic pattern.

The excess density of grains $\nabla^2\phi(\vec{r})$ is bounded, for $\vec{r} \neq \vec{0}$ implies
$a \leq 2$.

The Key Observation
S. Ostojic (2003).

- Proportionate growth.
- Periodic height pattern in each patch. [ignoring Transients]

Examples of periodic patterns in patches



The Main Result

In each patch with a periodic height pattern, we can only have

$a = 2$, and $\phi(x, y)$ is a quadratic function of x and y,
Or
$a = 1$, and $\phi(x, y)$ is a linear function of x and y.

Proof:
Expand $\phi(x_0 + \Delta x, y_0 + \Delta y)$ in a Taylor series:
$\phi(x_0 + \Delta x, y_0 + \Delta y) = \phi(x_0, y_0) + A\Delta x + B\Delta y + .. + K(\Delta x)^3 + ..$
Equivalently,
$T_N(X, Y) = \ldots + K(\Delta X)^3/\Lambda^{3-a}$
For finite ΔX integer, T is also integer, and no proliferation of defect lines $\Rightarrow K = 0$.
Same is true for all higher powers.
For a non-trivial dependence on x, y, if quadratic term is not zero, $a = 2$. Else, $a = 1$. Independent of dimension.

The function $\phi(x, y)$ is a piece-wise linear (quadratic) function of (x, y), but with an infinite number of pieces.

Dependence of the diameter Λ with N

This is much less constrained.

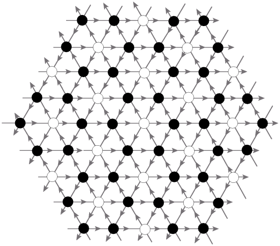
- If the initial background density is low enough everywhere,
  $\Lambda \sim N^{1/d}$
- If many sites have large heights
  $\Lambda = \infty \quad \text{for finite N}$
- For an in-between set of periodic backgrounds
  $\Lambda \sim N^{\alpha} \quad \text{for } 1/\mathrm{d} < \alpha \leq 1$

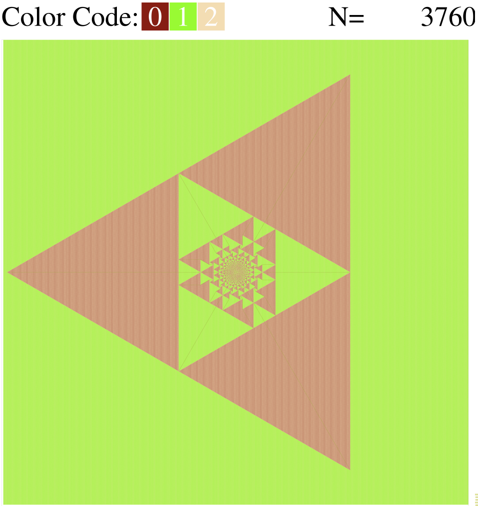
If $\Lambda \sim N^{\alpha}$, with $\alpha > 1/2$
We construct an infinite family of periodic backgrounds on the F-lattice that seem to have a different α for each member.

Patterns with fast-growing sandpiles

Directed traingular lattice with honeycomb background pattern

Diameter $\sim N$





Examples of patterns with fast-growing sandpiles

The ‘Bat-pattern’ on F-lattice
Here $\Lambda \sim N^{\alpha}$, $\alpha \approx 0.55$

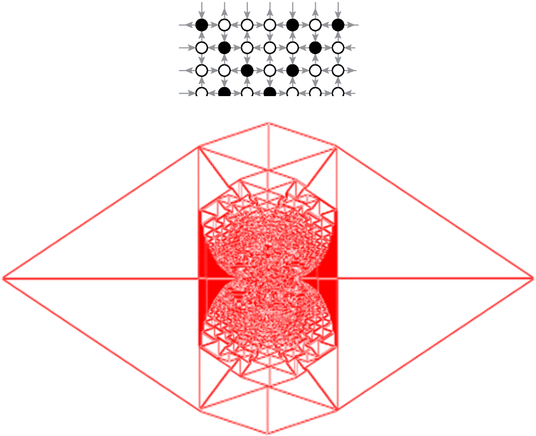


Figure: Only the boundaries of patches are shown.

Quantitative characterization

Consider the case with $a = 1$, say the triangular lattice pattern. The exact characterization of the patterns involves four steps:

- $\phi(\xi, \eta)$ is a piece-wise linear function, with rational slopes. Parameterize as $\phi_P(\xi, \eta) = a_P\, \xi + b_P\, \eta + c_P$
- The allowed values of (a_P, b_P) for different patches form a periodic hexagonal lattice.
- The condition that three patches meet at a point implies that c_P satisfies a Laplace equation on the adjacency graph of patches.
- Exact solution of these equations gives the exact boundaries of patches

For $a = 2$, the procedure is similar, but quadratic functions need six parameters per patch.

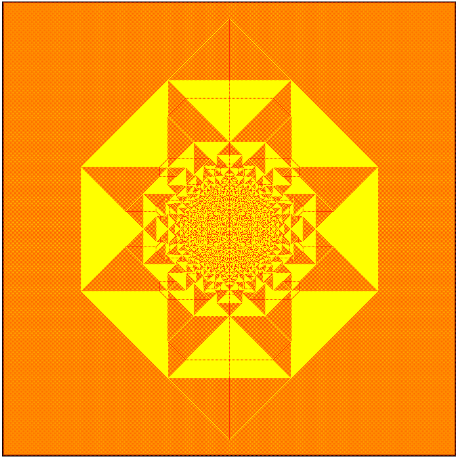


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard.

This pattern can be characterized exactly.

Pattern formation in a noisy background

In presence of noise, the function ϕ is no longer polynomial, but the proprtionate growth still holds.

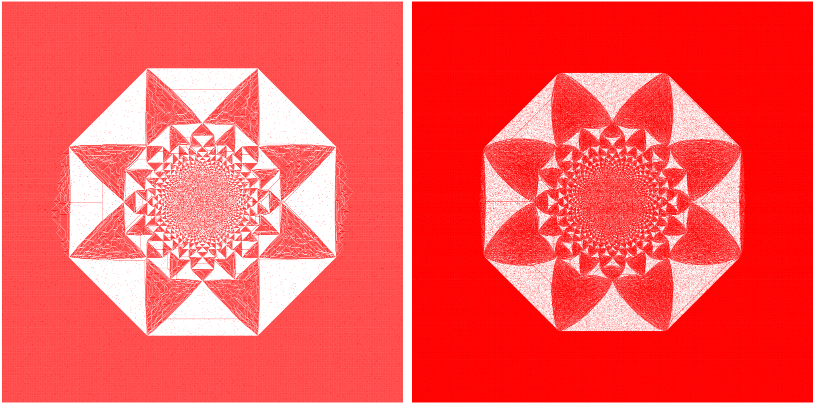


Figure: Pattern grown on the F-lattice with some heights 1 replaced by 0's. (a) 1% sites changed, N=228,000 , (b) 10% changed, N= 896,000.

If some 0's are also repalced by 1's, the effect is more dramatic.

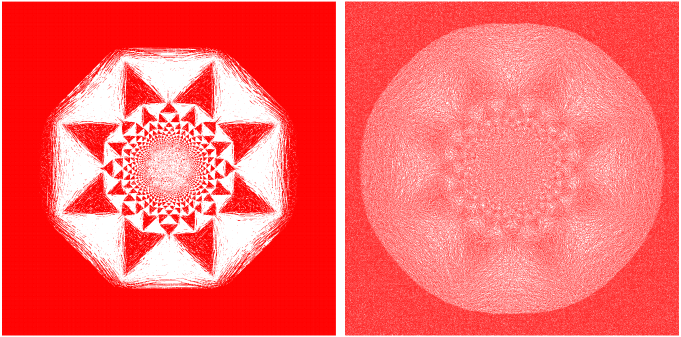


Figure: Pattern grown on the F-lattice with some heights flipped. (a) 1% sites changed (b) 10% sites changed

At higher noise level, the details of the pattern are not easy to see, but averaging over different realizations of noise brings out the pattern clearly.

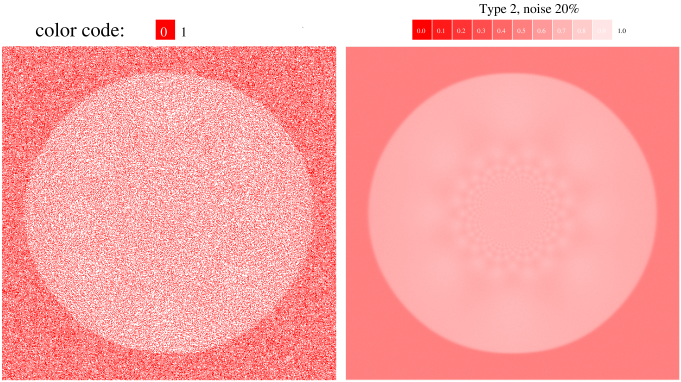


Figure: F-lattice, checkerboard with 20% sites flipped. $N = 57000$. (a) single realization (b) averaged height over 10^5 realizations.

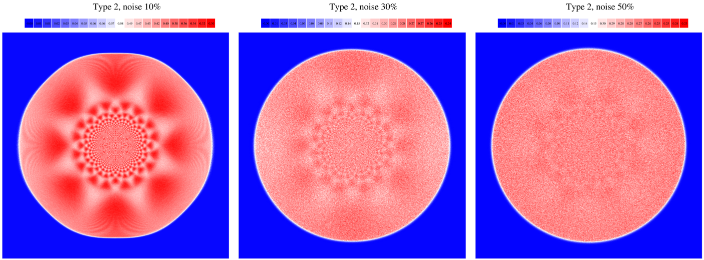


Figure: Averaged change in height with increasing noise strength 10%, 30%, 50% The color code for each pattern representing the height values are shown in the colorbar.

If we apply a $z \to 1/z^2$ transformation to these figures, we get

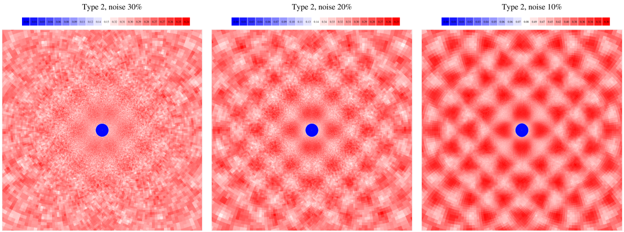


Figure: Result of applying $1/z^2$ transformation. Note the nearly gridlike pattern

This suggests that we can write the change in density as

$$\Delta\rho(x, y) \approx A(\rho)g(x, y), \tag{1}$$

Where ρ is the background density

This suggests that the simplest perturbation to the density field in the high noise limit is a periodic perturbation in the z'-coordinates.

$$g(x, y) = -\cos\frac{\pi x'}{2}\cos\frac{\pi y'}{2}, \qquad (2)$$

where $x' = \frac{2xy}{(x^2+y^2)^2}$, and $y' = \frac{x^2-y^2}{(x^2+y^2)^2}$.
A pictorial representation of this function is given below.

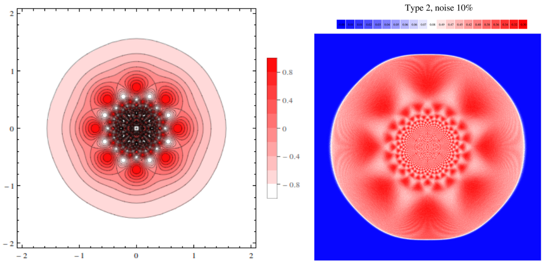


Figure: Density pattern using the function $g(x, y)$, compared to actual pattern. The black lines are contours of constant density.

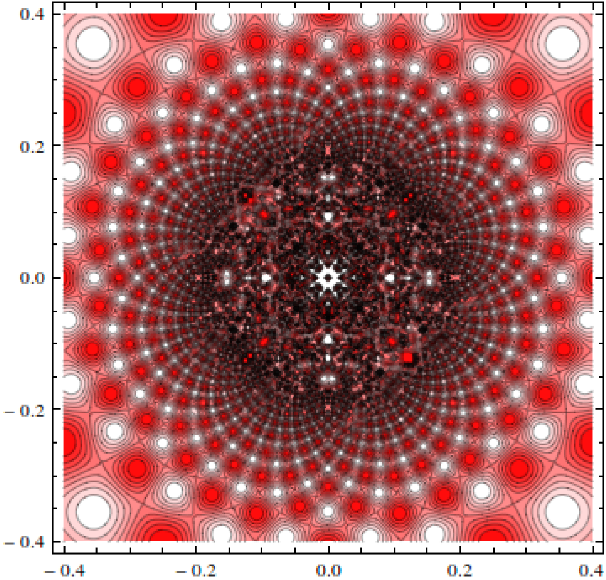
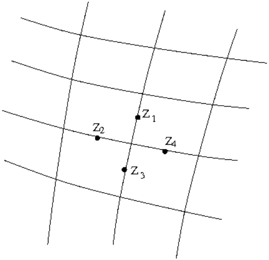


Figure: A zoom-in on the theoretical lowest -mode density perturbation $g(x, y)$.

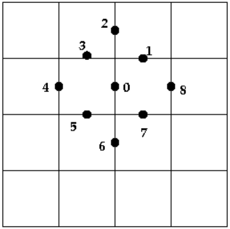
Discrete Analytic Functions

Functions defined only on discrete points in the complex $z-$ plane.



Discrete Cauchy-Riemann conditions:

$$\frac{F(z_1) - F(z_3)}{z_1 - z_3} = \frac{F(z_2) - F(z_4)}{z_2 - z_4}$$



On a square grid :

$$\Delta F_{13} + \Delta F_{35} + \Delta F_{57} + \delta F_{71} = 0$$

is equivalent to

$$\Delta F_{02} + \Delta F_{04} + \Delta F_{06} + \Delta F_{08} = 0$$

Discrete Laplace Equation.

Sum, but not product, of discrete analytic functions is also DA

We find that the coefficients of the linear terms in the toppling function define a discrete analytic function $d + ie$ of the complex variable $m + in$, where (m, n) is the patch label.
These conditions, and the asymptotic behavior for large $|m + in|$ determine the toppling function, and hence the pattern, completely.
The pattern on the F-lattice involves the discrete analytic function $D_{1/2}(z)$, which is analog of the anlytic function $z^{1/2}$, defined by $D_{1/2}(0) = 0$, and $D(z) \to z^{1/2}$, for $|z| \to \infty$.
Growing sandpiles near an edge /wedge involve other fractional powers of z^n.

DA's on multiple Riemann sheets.

These simple rules can give rise to patterns that are unexpectedly like true life.

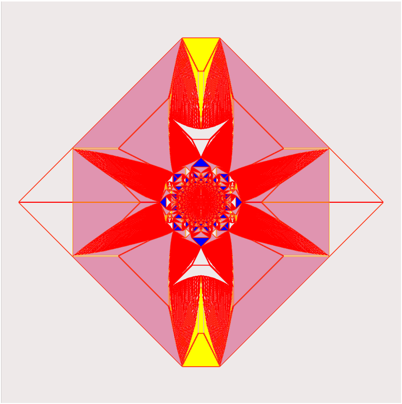


Figure: (a) A flower. (b) pattern produced by adding 256k particles on the F-lattice, with tilted squares backgound with spacing 4. Different colours denote different densities of particles, averaged over the unit cell of the background pattern

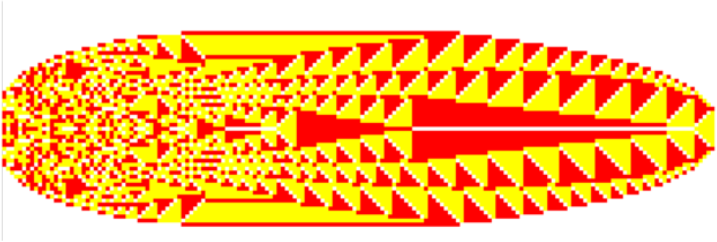


Figure: (a) A larva pattern. Produced on square lattice, with particle transfer on toppling only to up, down, right neighbors. Here $N = 10^4$. Particles are added at the left column center. Color code: 0=white, 1= red, 2=yellow.

Summary

- Growing sandpiles give a simple model showing complex patterns, and proportionate growth.
- Exact characterization of asymptotic pattern in some cases.
- Robustness to small noise in initial background.
- Characterization of patterns on noisy backgrounds ?

Thank You.

References

Modelling proprtionate growth, T. Sadhu and DD, Current Science, **103** (2012) 512.

A sandpile model for proportionate growth, DD and T. Sadhu (2013), to appear in JSTAT, [arXiv:1310.1359]

Connection to Tropical Mathematics

Define

$$a \oplus b = Max[a, b] \tag{3}$$

$$a \otimes b = a + b \tag{4}$$

Then standard properties of usual addition and multiplication (commutative, identity, distributive ..) contiue to hold.

Example: $3 \oplus 5 \oplus 2 = 5$

$3 \otimes 4 = 7$

Tropical polynomials: $a \otimes x \otimes x \oplus b \otimes x \oplus c$

Example: $x \otimes x \oplus 2 \otimes x \oplus 5 = Max[2x, x + 2, 5]$.

Fundamental theorem of tropical algebra.

A piecewise -linear convex function can be represented as a tropical polynomial.

Hence may be useful for describing the toppling function function in growing sandpiles where toppling function is piece-wise linear.

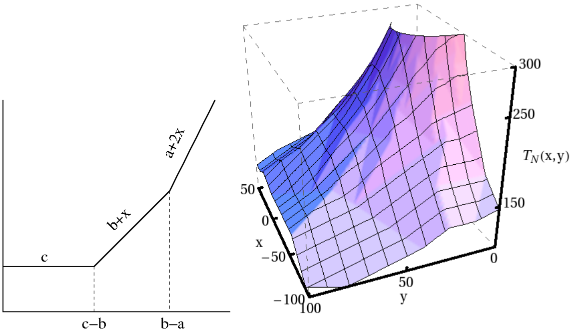


Figure: (a) A piece-wise linear function $c \oplus b \otimes x \oplus a \otimes x \otimes x$ (b) Graph of the toppling function for the fast-growing sandpile on triangular lattice

Or crumpled paper.

Figure: Crumpled paper. Picture taken from the internet