Quantum Breakdown of Thermodynamics: Many-Body Localization

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Based on work with
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- Hamiltonian $H = H^\dagger$ on finite-dim Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes_L}$.
- H looks like Markov generator, but we need all its eigenvectors, not just the lowest one.
- 'States' are vectors in $\Phi \in \mathcal{H} \sim$ phase-space points.
- Instead of 'functions on phasespace' we have 'operators O on $\mathcal{H}$': Natural pairing:

$$(\Phi, O\Phi) = \langle \Phi | O | \Phi \rangle = \text{'value of } O \text{ in } \Phi\text{'}$$

- Time-evolution: $\Phi(t) = \mathrm{e}^{-\mathrm{i}tH}\Phi(0)$.
- $\Rightarrow$ useful to know the eigenvectors and eigenvalues of H.
- Eigenvectors, eigenvalues denoted by Ψ, $E(\psi)$ ('Energies').
- Conserved quantities' O:

$$[O, H] = 0 \quad \Rightarrow \quad \langle \Phi(t) | O | \Phi(t) \rangle = \langle \Phi(0) | O | \Phi(0) \rangle$$

Transport through a quantum chain

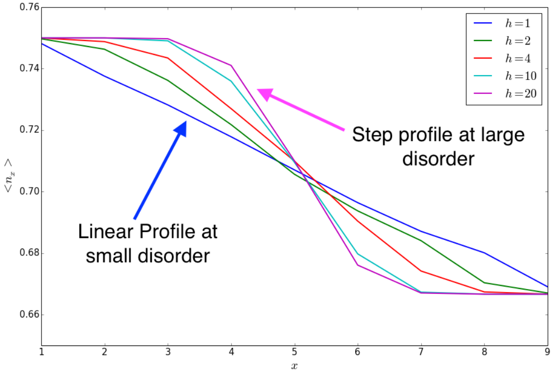
1. Hamiltonian dynamics of spin chain + Markov baths.
2. Magnetization S^z locally conserved $\Rightarrow$ study S^z profile.

Model of chain: disordered XXZ-chain

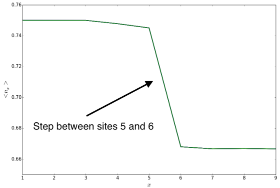
$$H = \frac{J}{2}\sum_{i=1}^{L}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + V S_i^z S_{i+1}^z + \sum_i h(i) S_i^z$$

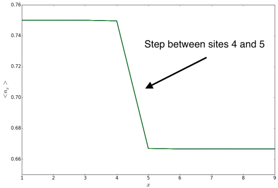
1. Pauli spin matrices S_i^x, S_i^y, S_i^z at site i.
2. Random fields $h(i) \in [-h, h]$ (uniform and i.i.d.)
3. Relevant disorder parameter: h/J.
4. Total magnetization: $\sum_i S_i^z$ is conserved: $[\sum_i S_i^z, H] = 0$.

Question: NESS-profile of $\langle S_i^z \rangle_{NESS}$ when baths set different boundary values.



$\langle S_i^z \rangle_{NESS}$-profile for $L = 9$ spins. **Average over disorder**.





Profile for two realizations at large disorder ($h = 10$)

Intuition: Two-site model

Drop $VS_i^z S_{i+1}^z$ and set $2H_I \equiv S_i^x S_{i+1}^x + S_i^y S_{i+1}^y$

$$H(J) = h_1 S_1^z + h_2 S_2^z + J H_I,$$

$J = 0$: Eigenstates: simple tensors $\eta = |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, \ldots.$

$$|\uparrow\uparrow\rangle \equiv |\uparrow\rangle_1 \otimes |\downarrow\rangle_2, \qquad S_i^z|\uparrow_i\rangle = |\uparrow_i\rangle, \quad S_i^z|\downarrow_i\rangle = -|\downarrow_i\rangle$$

Eigenvalues are

η	$\vert\uparrow\uparrow\rangle$	$\vert\uparrow\downarrow\rangle$	$\vert\downarrow\downarrow\rangle$	$\vert\downarrow\uparrow\rangle$
$E(\eta)$	$h_1 + h_2$	$h_1 - h_2$	$-h_1 - h_2$	$-h_1 + h_2$

$J > 0$: (Analytic) spectral perturbation theory applies if

$$\|JH_I\| \leq \min_{E,E'} |E - E'| \quad \Leftarrow \quad J \leq 2|h_1 - h_2| \quad (J \leq 2|h_1|, 2|h_2|)$$

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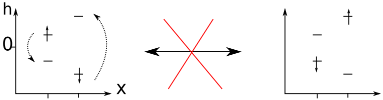
Conclude: $J \ll |h_1 - h_2| \Rightarrow J > 0$ eigenstates close to η.

Localization if $J \ll |h_1 - h_2|$. New eigenstates close to old η.



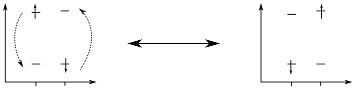
$|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ do not mix

Diagonalize $\widetilde{H} = UHU^\dagger$ in S_i^z-basis: $[\widetilde{H}, S_i^z] = 0$

$$U = 1 + \mathcal{O}(\tfrac{J}{|h_1 - h_2|}), \qquad \text{(PT)}.$$

Then construct quasilocal pseudo-spins

$$\widetilde{S}_i^z \equiv U^\dagger S_i^z U \quad \text{satisfying} \quad [H, \widetilde{S}_i^z] = 0, \qquad \widetilde{S}_i^z = S_i^z + \mathcal{O}(\tfrac{J}{|h_1 - h_2|})$$



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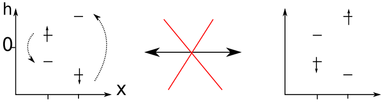

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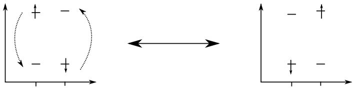
Delocalization if $J > |h_1 - h_2|$. No PT. η are resonant $\Rightarrow$ mix. $\widetilde{S}_i^z$ are non-local (fully spread over 2 sites)

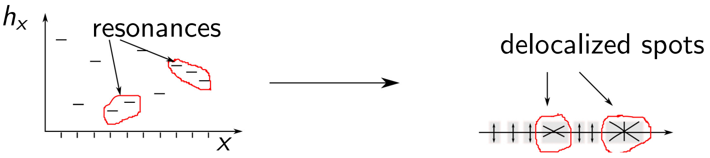


$|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ mix

Intuition for full chain

$$H = \sum_i h_i S_i^z + J H_I,$$

Bond $(i, i+1)$ resonant when $|h_i - h_{i+1}| \leq J$



- Strong disorder $J \ll h \Rightarrow$ Resonant bonds are sparse and isolated. PT works well away from resonances.
- At resonances: No PT, uncontrolled diagonalization, but at least they are isolated.
- Using uncontrolled but extremely plausible assumption, Imbrie (2014) proves that diagonalization can thus be done.

Many Body Localization: Precise Statement

Diagonalization: $\exists$ unitary U: $UHU^\dagger = \widetilde{H}$ such that

- $\widetilde{H}$ is diagonal in η-basis (S^z-basis).

$$\widetilde{H} = \sum g_i S_i^z + g_{i,i+1} S_i^z S_{i+1}^z + g_{i,i+1,i+2} S_i^z S_{i+1}^z S_{i+2}^z \dots$$

  Couplings $|g_{i,\dots,i+n}| \le \mathrm{e}^{-Cn}$ except at rare resonant spots.

- U quasilocal: $\widetilde{O} = UO_x U^*$ is quasilocal around x for local O_x.

Consequence: Conserved pseudo-spins $\widetilde{S}_i \equiv U^\dagger S_i^z U$:

$$[S_i^z, \widetilde{H}] = 0 \quad \Rightarrow \quad [\widetilde{S}_i^z, H] = 0.$$

$$\widetilde{H} = \sum g_i S_i^z + \dots \quad \Rightarrow \quad H = \sum g_i \widetilde{S}_i^z + \dots$$

We have a full set of (quasi)-local conserved quantities $\widetilde{S}_i^z$.
$H = H(\widetilde{S}^z)$ can be viewed as a chain of non-interacting pseudo-spins! This is a robust form of integrability.

Localization as a dynamical phase transitioin

- 2^L eigenvectors Ψ of H.
- Write $\sum_{\Psi\to e}\ldots$ for sum over all Ψ with energy density $E(\Psi)/L$ around e. There are $d(e)$ of them.
- The equilibrium-ensemble value is $\langle O\rangle_{\mathrm{e}} \equiv \frac{1}{d_e}\sum_{\Psi\to e}\langle\Psi|O|\Psi\rangle$
- Order parameter

$$\chi=\chi(\{h_i\})\equiv\frac{1}{d_e}\sum_{\Psi\to e}\left[\langle\Psi|S_i^z|\Psi\rangle-\langle S_i^z\rangle_e\right]^2$$

- 'Eigenstate Thermalization Hypothesis' (ETH) $\sim$ ergodic hypothesis. Eigenvectors behave as eigenvectors of random matrices (GOE) $\Rightarrow$ good concentration inqualities:

$$\langle\Psi|O|\Psi\rangle\simeq\langle O\rangle_e,\qquad \forall\Psi\to e,\ \forall\ \text{local } O$$

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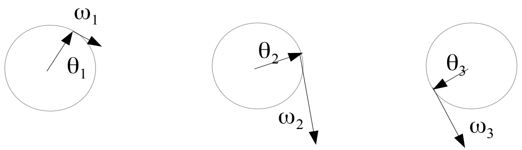
Ergodicity-Localization Phase Transition (in 1D!)

Small disorder: $\lim_L \mathbb{E}_h(\chi) = 0$ (ETH)
Large disorder: $\lim_L \mathbb{E}_h(\chi) > 0$ (Trivial if we replace $S_i^z \to \widetilde{S}_i^z$)

Chain of (classical) coupled rotors

$$H(\theta,\omega) = \sum_i H_i = \sum_i \frac{\omega_i^2}{2} + J\cos(\theta_i - \theta_{i+1}), \quad \theta_i \in S_1, \omega_i \in \mathbb{R}$$



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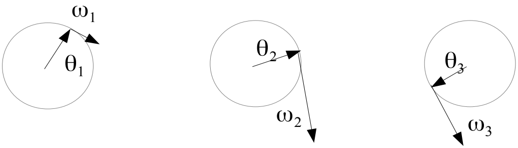

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- Rotors $i, i+1$ in resonance when $|\omega_i - \omega_{i+1}| \leq J$.
- Gibbs measure $\propto \mathrm{e}^{-\beta H}$: $\mathrm{Prob}_\beta(|\omega_i - \omega_{i+1}| \leq J) \sim J\sqrt{\beta}$
- Res. sparse if $J\sqrt{\beta} \ll 1 \Rightarrow$ intuition from quantum applies.
- Difference I: Even for 3 rotors, $\exists$ chaotic regions: KAM not in entire phase-space (Quantum: Can always diagonalize finite system)
- Difference II: Location of Res. can change (no quenched disorder) $\Rightarrow$ even sparse chaotic spots can transport.

'Asymptotic Localization' ($\sim$ Many-Body Nekoroshev)

We do not expect genuine MBL, but still some slow dynamics.

Theorem (DR-Huveneers, 2013)

Let $H_{[a,b]} \equiv \sum_{a \leq i < b} H_i$ and $\Delta H_{a,b}(t) \equiv H_{a,b}(t) - H_{a,b}(0)$.

$$\langle (\Delta H_{a,b}(t))^2 \rangle_\beta \leq C(m,\beta) J^{1/4}, \qquad t \leq J^{-m} C(m,\beta), \text{ any } m > 0$$

.

- For a 'normal' system, expect $\langle \cdot \rangle \propto Jt$ until saturation at $\langle \cdot \rangle \sim |b-a|$.
- Here, growth much slower: Initial fluctuations are frozen in for long time.
- Real Nekoroshev: H_x stable in all of phase space. Here: with large proba. (Indeed: small energies $\Rightarrow$ fast ballistic transport)
- We expect conductivity $\kappa \sim J^m$. Indirect Proof: add small noise $\mathcal{O}(J^m)$, then $\kappa = \mathcal{O}(J^m)$ indeed.