MODELS FOR EVOLUTION AND SELECTION.

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with P. Ferrari, E. Presutti, N. Soprano-Loto (in preparation)

Model with biological interpretation in terms of evolution and selection.

N **individuals** (this number will not change in time).

Each individual is in a state represented by a real number, the larger its value the better the state of the individual.

Individuals keep changing their state randomly modeled as

independent Brownian motion.

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Each individual is in a state represented by a real number, the larger its value the better the state of the individual.

Individuals keep changing their state randomly modeled as

independent Brownian motion.

The population goes towards a better fitness because:

- each individual independently of the others duplicates creating a new individual in its same position.

  as N is constant we must then

- delete the weakest individual, namely the leftmost particle.

We study the limit as $N \to \infty$ of the empirical averages and prove convergence to a deterministic function u related to the solution of the free boundary problem:

$$u_t = \frac{1}{2} u_{xx} + u, \qquad \text{in } (L_t, +\infty)$$

$$u(x,0) = \rho_0(x) \qquad \int \rho_0 = 1$$

with boundary conditions at the left edge L_t (the free boundary)

$$u(L_t, t) = 0, \qquad \frac{1}{2} u_x(L_t, t) = \int_{L_t}^{\infty} u(x,t)dx$$

and therefore the total mass is conserved so that $\frac{1}{2} u_x(L_t, t) = 1$

$$\frac{d}{dt} \int_{L_t}^{\infty} u(x,t)dx = -\frac{1}{2} u_x(L_t, t) + \int_{L_t}^{\infty} u(x,t)dx$$

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This process is in the family of branching- selection models introduced by **Brunet-Derrida** to study the effect of microscopic noise in front propagation.

- E. Brunet, B. Derrida, *Shift in the velocity of a front due to a cutoff.* Phys. Rev. E, **56** 2597–2604 (1997)

- E. Brunet, B. Derrida, *Effect of macroscopic noise on front propagation.* J. Stat. Phys., **103** 269–282 (2001)

Maillard studied extensively this issue obtaining accurate results for the speed.

- P. Maillard *Branching Brownian motion with selection* (2012)

R. Durrett and D. Remenik considered a different process in the same class.

Each of the N particles gives rise to a child at rate one.

The position of the child of a particle at $x \in \mathbb{R}$ is $x + y$ with y chosen according to a symmetric probability $p(x, y)$.

At each birth event the left-most particle is removed.

They prove that the empirical averages is described in the limit $N \to \infty$ by the solution $u(\cdot, t)$, $t \geq 0$ of an integro-differential equation with a free boundary.

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We have developed a theory which includes our model but applies also to a large class of systems: models for heat conduction, queuing theory, propagation of fire, interface dynamics.....

Main feature in the microscopic-particle version is that the interaction has a topological nature and the particles evolution is a free boundary problem.

Gioia Carinci, Cristian Giardinà, Pablo Ferrari, Errico Presutti

Particles move in $d = 1$ and there are two edges or one edges. An edge is a rightmost or a leftmost particle.

The rule of dynamics are the usual ones: particles are either free (independent random walks or Brownian motions) or they have some local interaction (for instance simple exclusion) and on top of that particles may duplicate via a branching process.

The topological interaction refers to the fact that the edges are special as they may disappear at some given rate being then replaced by new edges, the rightmost and leftmost particles among those which have survived.

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For instance in the problem of heat conduction the macroscopic equation is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad \text{in } [L_t, 0]$$

with $u(x,0) = \rho_0(x) \geq 0$ and boundary conditions

$$u(L_t, t) = 0 \qquad \frac{1}{2}\frac{\partial u}{\partial x}\Big|_{r=L_t} = \frac{1}{2}\frac{\partial u}{\partial x}\Big|_{x=0} = 1$$

the condition in 0 means that there is a mass source at 0 of rate 1. Thus the total mass is preserved.

$$\int_{L_t}^{\infty} u(x,t)dx = \int \rho_0(x)dx$$

Microscopic evolution: Brownians (or random walks) with births (rate 1) at the origin and deaths of the leftmost particle.

In all these models:

- an important tool in the proofs is played by *mass transport inequalities*
- the proof of convergence in the macroscopic limit does not use theorems on the limit PDE .

Strategy for the proof: introduce a family of upper and lower barriers in an order defined by mass transport and prove that the barriers have a unique separating element.
The upper and lower barriers squeeze the solution we are looking for.

The proofs exploit extensively probabilistic ideas and techniques based on the well known relation between heat equation and Brownian motion and between the hitting distribution at the boundaries and the Dirichlet condition.

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Branching BM with selection

$\underline{x}(t) = (x_1(t), .., x_N(t))$ = particle configuration at time t.

Independently particles move as Brownians and at rate one create a new particle at their position. At each branching time remove leftmost particle.

Given the initial macroscopic profile $\rho_0(x) \geq 0, \int \rho_0(x)dx = 1$ distribute the initial positions $x_1(0), .., x_N(0)$ in $\mathbb{R}$ independently with same law $\rho_0(x)dx$.

Empirical mass density at time $t \geq 0$

$$\pi_t^{(N)}(dx) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}(x)dx$$

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Branching BM with selection

Theorem

There is a function ρ such that for any $t \geq 0$ and any $\varepsilon > 0$

$$\lim_{N\to\infty} P^{(N)}\left[\sup_{r\geq 0}\left|\int_r^\infty \pi_t^{(N)}(dr') - \int_r^\infty \rho(r',t)dr'\right| > \varepsilon\right] = 0$$

$\rho(r,t)$ coincides with the solution $u(r,t)$ of the free boundary problem till when the latter exists (see later).

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Mass transport inequalities.

Partial order among functions:

Definition

$$u \preccurlyeq v \quad \text{iff} \quad \int_r^\infty u(x)dx \le \int_r^\infty v(x)dx, \qquad \forall r$$

The order among configurations is analogous:

Definition

$$\underline{x} \preccurlyeq \underline{y} \quad \text{iff} \quad |\underline{x} \cap [r,\infty)| \le |\underline{y} \cap [r,\infty)| \qquad \forall r$$

having regarded $\underline{x} = (x_1, \ldots, x_N)$ and $\underline{y} = (y_1, \ldots, y_N)$ as subsets of $\mathbb{R}$.

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Stochastic barriers

It is more convenient to remove particles at the end!

Split $[0, t]$ in the union of the time intervals $[(k-1)\delta, k\delta)$, $\delta > 0$, $k = 1, .., \delta^{-1}t$.

We define two stochastic **barriers** $\underline{x}^{\pm}(t)$ iteratively:

supposing to have defined $\underline{x}^{\pm}(t)$ for $t \leq (k-1)\delta$ we define it till time $t \leq k\delta$.

Initially

$$\underline{x}^{-}(0) = \underline{x}(0) = \underline{x}^{+}(0)$$

Stochastic barriers

Assume in the time interval $[0, \delta]$ there are m branching events.

- $\underline{x}^+(t)$ is defined by letting particles branch and move independently for $t \in [0, \delta)$.
  At time δ delete the m leftmost particles.
- At time 0 delete the m leftmost particles from $\underline{x}(0)$. Then $\underline{x}^-(t)$ is defined by letting the $N - m$ remaining particles move and branch independently for $t \in [0, \delta)$.

Then iterate.

Theorem

$$\underline{x}^-(k\delta) \preccurlyeq \underline{x}(k\delta) \preccurlyeq \underline{x}^+(k\delta)$$

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$\underline{x}(\delta) \preccurlyeq \underline{x}^+(\delta)$

Assume only one branching event: $m = 1$ and at $t_1 \in (0, \delta)$ particle i of the true process at position x_i creates a new particle at x_i. At time t_1 color green the leftmost particle which is deleted for the true process.

- Particles with same label move the same.

- In the process $\underline{x}^+(t)$ there is also the green particle at all times $t \in (0, \delta)$.
- At time δ the leftmost particle is removed in the $\underline{x}^+$-process but it is present in the true process.
- Then $\underline{x}^+$ “has more mass on the right".

$\underline{x}^-(\delta) \preccurlyeq \underline{x}(\delta)$

Color red the leftmost particle at time 0.

- In the process $\underline{x}^-(t)$ the red particle is never present while it is present in the true process.
- Assume the particle i (the one that branches) is not the green one. Then at time t_1 the $\underline{x}^-$ process will have N particles included the green one which is the leftmost. Thus the true process “has more mass on the right".

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Similar inequalities are valid in:

- heat conduction model previously described
- in the Durrett and Remenik model
- in the SSEP where the leftmost particle and rightmost hole are killed at rate ε
- two species of random walks (colored) with change of color of the leftmost particle and rightmost particle of the two species
- interface dynamics on a sticky substrate.

Deterministic barriers

The free evolution T_t is defined as the solution of $u_t = \frac{1}{2}u_{xx} + u$

$$T_t u = e^t G_t \star u, \qquad G_t(r, r') = \frac{1}{\sqrt{2\pi t}} e^{-(r-r')^2/2t}$$

Observe that T_t amplifies by e^t the mass of u.

The cut operator C_m is defined as

$$C_m u(x) = u(x)\mathbf{1}_{x \geq V_{m,u}}, \qquad \int_{V_{m,u}}^{\infty} u(x)dx = m$$

Deterministic barriers

The upper barrier S^+ is defined by first applying the free evolution in a time interval of length δ and then cutting mass to the left in such a way to have total mass 1.

$$S^+_{k\delta}(u) = C_1 T_\delta ....C_1 T_\delta C_1 T_\delta u, \qquad k \text{ times}$$

The lower barrier S^- is defined by first cutting mass to the left in such a way to have total mass $e^{-\delta}$ and then applying the free evolution in a time interval of length δ.

$$S^-_{k\delta}(u) = T_\delta C_{e^{-\delta}} ......T_\delta\, C_{e^{-\delta}}\, T_\delta C_{e^{-\delta}} u, \qquad k \text{ times}$$

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Hydrodynamic limit for the approximating processes

The hydrodynamic limits of the processes $\underline{x}^{\pm}(t)$ are $S_t^{\pm}$.

Theorem

For all r

$$\lim_{N\to\infty} |\underline{x}^{\pm}(t) \cap [r, \infty)| = \int_r^{\infty} S_t^{\pm}(\rho_0), \qquad \textit{in probability}$$

$t = k\delta$

Limit of the deterministic barriers

- $S_t^{\delta,-}(u) \preccurlyeq S_t^{\delta,+}(u), \quad \|S_t^{\delta,-}(u) - S_t^{\delta,+}(u)\|_{L^1} \leq c\delta$
- $S_t^{\delta,-}(u)$ is non decreasing and $S_t^{\delta,+}(u)$ is non increasing in δ (in the sense of $\preccurlyeq$).
- There exists a function $S_t u(r)$ continuous in (r,t) for $t > 0$ such that for all $t > 0$

$$\lim_{n\to\infty} \|S_t^{2^{-n}t,+}u(r) - S_t u(r)\|_\infty = 0$$

- $S_t^{\delta,-}u \preccurlyeq S_t u \preccurlyeq S_t^{\delta,+}u$

$$\int_r^\infty S_t u = \inf_{\delta: t=k\delta, k\in\mathbb{N}} \int_r^\infty S_t^{\delta,+}u = \sup_{\delta: t=k\delta, k\in\mathbb{N}} \int_r^\infty S_t^{\delta,-}u$$

The free boundary problem

$$u_t = \tfrac{1}{2} u_{xx} + u \quad \text{in } (L_t, \infty), \quad u(r,0) = \rho_0(r), \quad \int \rho_0 = 1$$

$$u(L_t, t) = 0, \qquad \frac{1}{2} \frac{\partial u}{\partial r}\Big|_{r=L_t} = 1$$

Local (in time) existence in the literature (e.g Fasano).

Write u in terms of a standard Brownian motion $\{B_t\}$.
Define the stopping time

$$\tau^L = \inf\{t \geq s : B_t \geq L_t\}, \quad \text{and} = \infty \text{ if the set is empty}$$

Then

$$\int_r^\infty u(x,t)dx = e^t \int \rho_0(r') P_{r'}(B_t > r; \tau^L > t) dr'$$

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L_t, $t \geq 0$ is such that

$$e^t \int \rho_0(x)P_x\big(\tau_0^L > t\big)dx = 1$$

Theorem

$$S_t^{\delta,-}\rho_0 \preccurlyeq u \preccurlyeq S_t^{\delta,+}\rho_0$$

The analogous theorem has been proved in most of the cases I mentioned before.