k-Abelian Complexity and Fluctuation

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This talk is based on the following articles:

- Karhumäki, Saarela, Zamboni:
  On a generalization of Abelian equivalence and complexity of infinite words (2013)
- Cassaigne, Karhumäki, Saarela:
  On growth and fluctuation of k-abelian complexity (2015)

Outline

1. Background
2. Low complexity
3. High complexity
4. Fluctuating complexity

k-abelian equivalence

For $k \geq 1$, words u and v are *k-abelian equivalent* if $|u|_t = |v|_t$ for all words t such that $|t| \leq k$.

Example

0011001 and 0001011 are 2-abelian equivalent, but not 3-abelian equivalent.

k-abelian equivalences form a hierarchy of equivalence relations (and congruences) between abelian equivalence ($k = 1$) and equality ($k \to \infty$).

Factor complexity

Let $F_n(w)$ be the set of factors of w of length n.

The *factor complexity* of w is the function

$$\mathcal{P}_w : \mathbb{Z}_+ \to \mathbb{Z}_+,\ \mathcal{P}_w(n) = \#F_n(w).$$

The *k-abelian complexity* of w is the function

$$\mathcal{P}_w^k : \mathbb{Z}_+ \to \mathbb{Z}_+,\ \mathcal{P}_w^k(n) = \#\{[u]_k \mid u \in F_n(w)\},$$

where $[u]_k$ is the k-abelian equivalence class of u.

Outline

Equality

Theorem (Morse and Hedlund)

- $\mathcal{P}_w(n) < n+1$ *for some* $n \Leftrightarrow w$ *ult. per.* $\Leftrightarrow \mathcal{P}_w(n) = O(1)$
- $\mathcal{P}_w(n) = n+1$ *for all* $n \Leftrightarrow w$ *Sturmian*

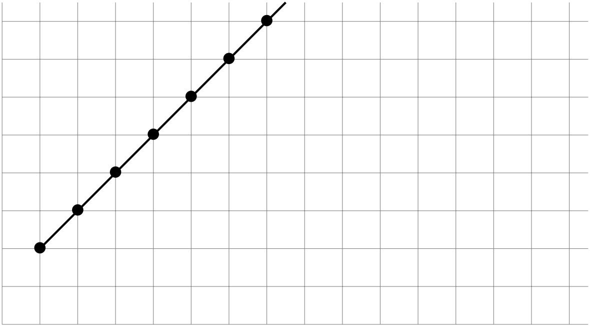


Figure : The factor complexity of Sturmian words.

Abelian equivalence

Theorem (Morse and Hedlund; Coven and Hedlund)

- $\mathcal{P}^1_w(n) < 2$ *for some* $n \Leftrightarrow w$ *per.* $\Rightarrow w$ *ult. per.* $\Rightarrow \mathcal{P}^1_w(n) = O(1)$
- $\mathcal{P}^1_w(n) = 2$ *for all* n *and* w *aper.* $\Leftrightarrow w$ *Sturmian*

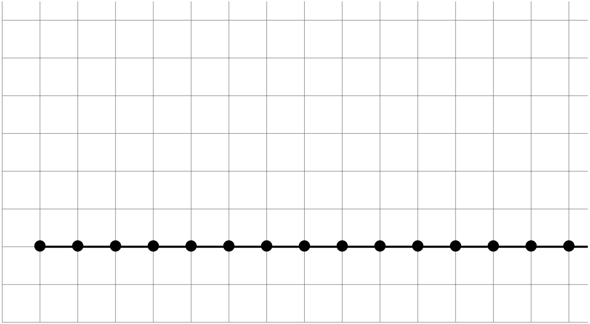


Figure : The abelian complexity of Sturmian words.

k-abelian equivalence

Theorem

- $\mathcal{P}_w^k(n) < \min(2k, n+1)$ *for some* $n \Rightarrow w$ *ult. per.* $\Rightarrow \mathcal{P}_w^k(n) = O(1)$
- $\mathcal{P}_w^k(n) = \min(2k, n+1)$ *for all* n *and* w *aper.* $\Leftrightarrow w$ *Sturmian*

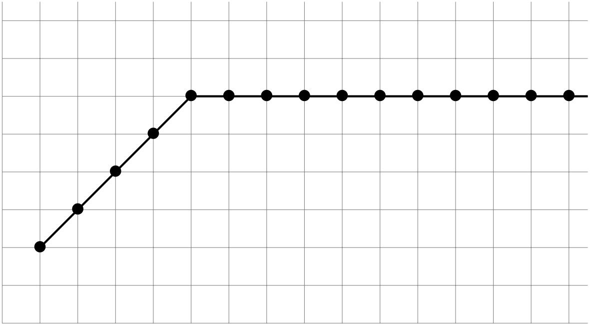


Figure : The 3-abelian complexity of Sturmian words.

Outline

Number of equivalence classes

Let $\mathrm{nec}_k(n)$ be the number of k-abelian equivalence classes of words of length n (alphabet size m is constant).

Theorem

$\mathrm{nec}_k(n) = \Theta(n^{(m-1)m^{k-1}})$.

Proof.

Can be proved by using Rauzy multigraphs and linear algebra, or by a combinatorial argument using a different “minimal” definition of k-abelian equivalence. □

Corollary

$\mathcal{P}_w^k(n) = O(\mathrm{nec}_k(n)) = O(n^{(m-1)m^{k-1}})$.

Outline

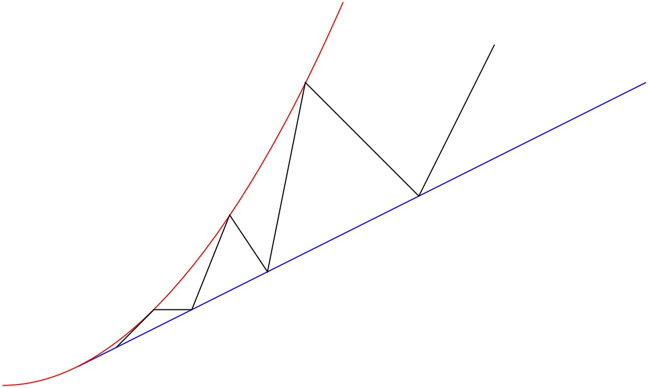
Notation

- $m \geq 2$, $\Sigma_m = \{0, \ldots, m-1\}$ m-letter alphabet.
- $f(n) = O'(g(n))$ if $\exists \alpha > 0$: $f(n) < \alpha g(n)$ for infinitely many n.
- $f(n) = \Omega'(g(n))$ if $\exists \alpha > 0$: $f(n) > \alpha g(n)$ for infinitely many n.

Fluctuation

For how slowly growing f and fast growing g can we find words w such that

$$\mathcal{P}_w^k(n) = O'(f(n)) \qquad \text{and} \qquad \mathcal{P}_w^k(n) = \Omega'(g(n))?$$



Maximal fluctuation

We can have

- $f(n) = O(1)$ and $g(n)$ *almost* $\Theta(\mathrm{nec}_k(n))$ or
- $f(n) = O(n)$ and $g(n) = \Theta(\mathrm{nec}_k(n))$.

We cannot have

- $f(n) = o(n)$ and $g(n) = \Theta(\mathrm{nec}_k(n))$.

(Recall that $\mathrm{nec}_k(n) = \Theta(n^{m^k - m^{k-1}})$ is the maximal k-abelian complexity.)

From $2k$ to logarithmic

If $\mathcal{P}_w^k(n) < 2k$ for some $n \geq 2k-1$, then w is ultimately periodic.

Theorem

Let $k \geq 1$. There exists $w \in \Sigma_2^\omega$ such that

$$\liminf \mathcal{P}_w^k = 2k \qquad \text{and} \qquad \mathcal{P}_w^k(n) = \Omega'(\log n).$$

Proof.

Construction: Image of the period-doubling word under the morphism $h : \Sigma_2^* \to \Sigma_2^*$ defined by $h(0) = 0^{k-1}1$ and $h(1) = 0^k1$. □

From bounded to almost maximal

Theorem

Let $k \geq 1$. Let f be a function such that

$$f(n) = o(\mathrm{nec}_k(n)) = o(n^{(m-1)m^{k-1}}).$$

There exists $w \in \Sigma_m^\omega$ such that

$$\mathcal{P}_w^k(n) = O'(1) \qquad \textit{and} \qquad \mathcal{P}_w^k(n) = \Omega'(f(n)).$$

Proof.

Construction for $k = 1$: Let $l_1, l_2, l_3, \ldots$ be a sequence of numbers that grows fast enough. Let u_i be a word that has a factor in every abelian equivalence class of words in $\Sigma_m^{l_i}$. Then w can be defined by a Toeplitz construction using the words u_i. □

From linear to maximal

Theorem

Let $k \geq 1$. There exists $w \in \Sigma_m^\omega$ such that

$$\mathcal{P}_w^k(n) = O'(n) \qquad \text{and} \qquad \mathcal{P}_w^k(n) = \Omega'(\mathrm{nec}_k(n)) = \Omega'(n^{(m-1)m^{k-1}}).$$

Proof.

Construction for $k = 1$: Let $u_0 = 0$ and, for $j \geq 0$,

$$u_{j+1} = \prod_{(n_0,\ldots,n_{m-1})} \prod_{i=0}^{m-1} i^{|u_j|+n_i},$$

where the outer product is taken over all sequences $(n_0, \ldots, n_{m-1})$ of non-negative integers such that $\sum_{i=0}^{m-1} n_i = m|u_j|$. Let $w = u_0 u_1 u_2 \cdots$. □

Not from sublinear to maximal

Theorem

Let $k \geq 1$. There does not exist $f(n) = o(n)$ and $w \in \Sigma_m^\omega$ such that

$$\mathcal{P}_w^k(n) = O'(f(n)) \qquad \text{and} \qquad \mathcal{P}_w^k(n) = \Omega'(\mathrm{nec}_k(n)) = \Omega'(n^{(m-1)m^{k-1}}).$$

Thank You!