Avoiding k-abelian powers in words

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k-abelian equivalence

Two words u and v are k-**abelian equivalent** if:
for every $w \in A^*$ with $|w| \leq k$, $|u|_w = |v|_w$

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uv is a k-**abelian-square** if u and v are k-abelian equivalent uvw

is a k-**abelian-cube** if u, v, w are k-abelian equivalent

$u_1 u_2 \ldots u_n$ is a k-**abelian-**n**th-power** if $u_1, \ldots u_n$ are k-abelian equivalent

k-abelian equivalence

"∞-abelian-equivalence" $=$ word equality

$\Downarrow$

1-abelian-equivalence $=$ abelian-equivalence

k-abelian equivalence

"∞-abelian-equivalence" $=$ word equality

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

3-abelian-equivalence

$$\Downarrow$$

2-abelian-equivalence

$$\Downarrow$$

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What about avoidability of powers ?

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Avoidability of k-abelian-powers

Question (Huova, Karhumäki, Saarela, Saari 2011)

- *Is there a k such that k-abelian-squares avoidable on a ternary alphabet?*
- *Is there a k such that k-abelian-cubes avoidable on a binary alphabet?*

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Theorem (Huova, Karhumäki, Saarela, Saari 2011)

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Theorem (R. 2013)

2-*abelian-cubes are avoidable over 2 letters.*
3-*abelian-squares are avoidable over 3 letters.*

2-abelian-cube-free binary word

Let $h : A_3^* \to A_2^*$ be the following 47-uniform morphism.

$$h : \begin{cases} 0 \to 00100101001011001001010010011001001100101101011 \\ 1 \to 00100110010011001101100110110010011001101101011 \\ 2 \to 00110110101101001011010110100101001001101101011 \end{cases}$$

Theorem (R. 2013)

For every abelian-cube-free word $w \in A_3^*$, $h(w)$ is 2-abelian-cube-free.

$\Rightarrow$ 2-abelian-cubes are avoidable on a binary alphabet

3-abelian-square-free ternary word

Let $h : A_4^* \to A_3^*$ be the following 25-uniform morphism.

$$h : \begin{cases} 0 \to 0102012021012010201210212 \\ 1 \to 0102101201021201210120212 \\ 2 \to 0102101210212021020120212 \\ 3 \to 0121020120210201210120212 \end{cases}$$

Theorem (R. 2013)

For every abelian-square-free word $w \in A_4^*$, $h(w)$ is 3-abelian-square-free.

$\Rightarrow$ 3-abelian-squares are avoidable on a ternary alphabet

Idea of the proof (1/2)

- Sufficient conditions for a morphism h to be k-abelian-nth-power-free
- i.e. for every abelian-nth-power-free word w, $h(w)$ is k-abelian-nth-power-free

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- Similar to [Carpi 1993] (sufficient conditions for a morphism to be abelian-nth-power-free)
- 1st condition: the prefixes (resp. suffixes) of length $k-1$ in the images of h are the same.
- 2nd condition: 'generalized' Parikh matrix M of h has full rank:
  $\forall x \in \Sigma, w \in \Sigma^k$
  $$M[x, w] = |h(a)p|_w$$
  where p is the prefix of $h(x)$ of length $k-1$

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  - either the pre-image in w has an abelian-nth-power
- If yes : h is k-abelian-nth-power-free

Avoidability of k-abelian-nth-powers over ℓ letters (sum.)

"∞-abelian-equivalence" $=$ word equality

$\ell \backslash n$	2	3
2	No	Yes (Thue 1906)
3	Yes (Thue 1906)	Yes

k-abelian-equivalence, $3 \leq k$

$\ell \backslash n$	2	3
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2-abelian-equivalence

$\ell \backslash n$	2	3
2	No	Yes (R. 2013)
3	No (Huova, Karhumäki, Saarela, Saari 2011)	Yes
4	Yes	Yes

1-abelian-equivalence $=$ abelian-equivalence

$\ell \backslash n$	2	3	4
2	No	No	Yes (Dekking 79)
3	No	Yes (Dekking 79)	Yes
4	Yes (Keränen 92)	Yes	Yes

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Negative answer to the first Mäkelä's question

Question (Mäkelä 2003)

Can you avoid abelian-cubes of the form uvw where $|u| \geq 2$, over two letters? - You can do this at least for words of length 250.

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Proof:

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Proof:

- Exhaustive search.
- One can restrict w.l.o.g. on Lyndon words.
- Largest Lyndon word: 290.

Avoidability of long abelian repetitions

Weak version of Mäkelä's questions:

Question 1

Can we avoid long abelian-cubes over two letters?

Question 2

Can we avoid long abelian-squares over three letters?

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$\Rightarrow$ Yes ! One can avoid abelian squares uv with $|u| \geq 6$

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Weak version of Mäkelä's questions:

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$\Rightarrow$ Still open... but period at least 3

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Can we avoid long abelian-squares over three letters?

$$\text{Let } h_6 : \begin{cases} a \to & ace \\ b \to & adf \\ c \to & bdf \\ d \to & bdc \\ e \to & afe \\ f \to & bce \end{cases} \text{ and } \varphi : \begin{cases} a \to & \texttt{bbbaabaaac} \\ b \to & \texttt{bccacccbcc} \\ c \to & \texttt{ccccbbbcbc} \\ d \to & \texttt{cccccccaa} \\ e \to & \texttt{bbbbbcabaa} \\ f \to & \texttt{aaaaaabaa}. \end{cases}$$

Theorem (R., Rosenfeld 2015)

The sequence obtained by applying φ to the fixed-point of h_6, $\varphi(h_6^\infty(a))$, does not contain any abelian-square of period more than 5.

What about long k-abelian-repetitions?

Theorem (R., Rosenfeld 2014 & 2015)

For every $k \geq 2$, one can avoid long k-abelian-squares on binary words.

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Let $g(k)$ be the minimal number of k-abelian-squares in an infinite binary word

Note: $g(1) = \infty$ [Entringer, Jackson & Schatz 1974]
$g(\infty) = 3$ [Fraenkel & Simpson 1995]

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Theorem (R., Rosenfeld 2014 & 2015)

- $5 \leq g(2) \leq 707$ — *period* ≤ 60
- $g(3) = g(4) = 4$ — 0^2, 1^2, $(01)^2$, $(10)^2$
- $g(k) = 3$ *for* $k \geq 5$ — 0^2, 1^2, $(01)^2$

$k = 2$

$$\varphi : \begin{cases} a \to & bbbaabaaac \\ b \to & bccacccbcc \\ c \to & ccccbbbcbc \\ d \to & ccccccccaa \\ e \to & bbbbbcabaa \\ f \to & aaaaaaabaa. \end{cases} \qquad h_2 : \begin{cases} a \to & 11100000000 \\ b \to & 11010001010 \\ c \to & 11111101010. \end{cases}$$

$h_2(\varphi(h_6^\infty(a)))$ has no 2-abelian squares of period more than 60

$k = 3$ and $k = 4$

$$h_3 : \begin{cases} 0 \to & 0001100101001101011000101010001011101011000101 \\ 1 \to & 0001100101001101011001110101011100011101011000101 \\ 2 \to & 0001100101001110001010001100101100011101011000101 \\ 3 \to & 0001100101001110010101001110001011001010110001 01. \end{cases}$$

For every abelian-square-free word w, $h_3(w)$ contains only 4 distinct 3-abelian-squares: 0^2, 1^2, $(01)^2$ and $(10)^2$

$k \geq 5$

h_5 :

$0 \to$
110001100101100010111001011001110001011000111001011000101110010110011100010111001011000101110001100101100011100101100111000110010110001011100101100111000101110010110001011100011001011000101110010110001110010111000110010110001011100101100111000101100011100101110001100101100010111001011000111001011100011001011000111001011001110001100101100011100101110001100101100010111001011000111001011100011001011000111001011001110001011000111001011000101110010110011100010111001011000101110001100101100011100101100111000101110010110001110010111000110010110001110010110011100011001011100010110011100010111001011000101110001100101110001011001110001100101100011100101110001011000111001011001110001011000111001011

$1 \to$
110001100101100010111001011001110001011000111001011000101110010110011100010111001011000101110001100101100011100101100111000110010110001011100101100111000101100011100101100111000110010110001110010110011100010111001011000111001011100011001011000111001011001110001100101100011100101110001011000111001011001110001100101100011100101100111000101100011100101100010111000110010110001110010110011100011001011000111001011100011001011000101110010110001110010111000110010110001110010110011100011001011000111001011100011001011000101110010110011100010110001110010110011100011001011100010110011100010111001011000101110001100101110001011001110001100101100011100101110001011000111001011001110001011000111001011

$2 \to$
110001100101100010111001011001110001011000111001011000101110010110011100010111001011000101110001100101100011100101100111000110010110001011100101100111000101100011100101100111000110010110001110010110011100010111001011000111001011100011001011000101110010110011100010110001110010110001011100101100111000101110010110001110010111000110010110001011100101100111000101110010110001011100011001011000101110010110011100010110001110010110011100011001011000111001011100011001011000101110010110011100010111001011000101110001100101100010111001011001110001100101100011100101100111000110010111000101100111000101110010110001011100011001011100010110011100011001011000111001011100010110001110010110011100010110001110010111

$3 \to$
110001100101100010111001011001110001011000111001011000101110010110011100010111001011000101110001100101100011100101100111000110010110001011100101100111000101100011100101100010111000110010110001011100101100011100101110001100101100011100101100111000110010110001110010111000101100011100101100111000110010110001110010110011100010110001110010110001011100011001011000101110010110011100010111000101110010110001110010111000110010110001011100101100111000101100011100101100111000110010110001110010110011100010110001110010110001011100101100111000101100011100101100111000110010111100010110011100010111001011000101110001100101110001011001110001100101100011100101110001011000111001011001110001011000111001011

For every abelian-square-free word w, $h_5(w)$ contains only 3 distinct 5-abelian-squares: 0^2, 1^2, $(01)^2$

Open questions : abelian equivalence

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Question

What is the minimal p such that one can avoid abelian-squares of period at least p over three letters?

$2 \leq p \leq 6$

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What is the minimal s such that one can construct infinite ternary word with only s abelian-squares ?

$3 \leq s \leq 22$

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Question

Can we avoid long abelian-cubes over two letters?

If yes:

- What is the minimum period ?
- What is the minimum number of cubes ?

Open questions : 2-abelian equivalence

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Question

What is the minimal p such that one can avoid 2-abelian-squares of period at least p over two letters?

$2 \leq k \leq 60$

Open questions : 2-abelian equivalence

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What is the minimal p such that one can avoid 2-abelian-squares of period at least p over two letters?

$2 \leq k \leq 60$

Question

What is $g(2)$? (i.e. smallest s such that one can construct an infinite binary word with only s 2-abelian-squares)

$5 \leq g(2) \leq 707$

Others questions

- Fractional k-abelian-repetitions ?
- k-abelian-repetitions patterns ?
- . . .