Combinatorics on words

CIRM, March 14-18, 2016

Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
09h30-11h **Introduction** Schedule discussion Presentation of topics	09h-11h **Mixed Problems** Frid Shur	09h-11h **k-abelian equivalence** Karhumäki Rao Puzynina Saarela Whiteland	09h-11h **Groups and languages** Diekert Ciobanu	10h-11h **Burrows-Wheeler transform** Kufleitner
11h30-12h30 **Sturmian words** Peltomäki	11h30-12h30 **Words and equations** Juhasz	11h30-12h30 Free discussion	11h30-12h30 **Ergodic theory and substitutions** Fickenscher	11h30-12h30 **Oldenburger word** Rodriguez Caballero
12h30-13h30 Lunch	12h30-13h30 Lunch	12h30-13h30 Lunch	12h30-13h30 Lunch	12h30-13h30 Lunch
15h-17h **Complexity, ergodicity** Monteil Emme	14h-18h **Calanques** (free time)	14h-14h45 **Equidistribution** Frid 15h-16h **Hippocampe** posters (in CIRM annex)	14h-15h **Weighted automata** Merlet	14h-17h Free discussion
17h-19h Free discussion	18h-19h **Invariant measures for train tracks** Hilion	16h-19h Free discussion	17h-19h **Discrete geometry** Brlek Labbé Jamet	17h-19h Free discussion
19h30-20h30 Dinner	19h30-20h30 Dinner	19h30-20h30 Dinner	19h30-20h30 **Bouillabaisse**	19h30-20h30 Dinner

Abstracts

- **Laura Ciobanu** : Conjugacy languages and growth series in groups.
  To any group G generated by a finite set X one can associate several languages of words over X, such as a language of normal forms, or of geodesics, and then determine whether such a language is regular, context-free etc., or analyze the generating function of the language and determine whether it is rational, algebraic or transcendental. There are several beautiful connections between the algebraic or geometric properties of the group and the languages and generating functions mentioned above.
  In this talk I will introduce a language which picks one word out of each conjugacy class, and show how for free groups these words are very similar to Lyndon words. Then I will explain how tools from various areas can help determine the transcendence of the conjugacy growth series in free groups.

- **Volker Diekert** : Sur la continuation d'un programme de Schützenberger
  Schützenberger was interested to characterize those regular languages where the groups in the syntactic monoids belong to a given variety of finite groups. He allowed operations on the language side which are union, intersection, concatenation and modified Kleene-star involving a mapping of a prefix code of bounded synchronization delay to a group G, but no (!) complementation.
  He succeeded to do so first for aperiodic monoids which is the case of the trivial group $\{1\}$. Thereby he showed a result which is, in some sense, more fundamental than his famous result StarFree = Aperiodic, but far less known. In a next step he generalized his result from the trivial group to abelian groups. Actually he also proved for every group G that the modified Kleene-star over prefix code of bounded synchronization delay does not introduce new groups. But he did not prove the converse. So, the programme remained unfinished.
  I will speak about recent developments that "concluded" that task. The talk is based on a joint paper with Tobias Walter.

- **Jordan Emme** : k-bonacci substitutions and thermodynamic formalism.
  We study k-bonacci substitutions. For each we define a renormalization operator associated to it and examine its iterates over potentials in a certain class. We also study the pressure function associated to potentials in this class and prove the existence of a freezing phase transition which is realized by the only ergodic measure on the subshift associated to the substitution.

- **Jon Fickenscher** : Ergodic measures for shifts with eventually constant complexity growth.
  We will consider (sub)shifts with complexity such that the difference from n to $n+1$ is constant for all large n. The shifts that arise naturally from interval exchange transformations belong to this class. An interval exchange transformation on d intervals has at most $d/2$ ergodic probability measures. We look to establish the correct bound for shifts with constant complexity growth. To this end, we give our current bound and discuss further improvements when more assumptions are allowed. This is ongoing work with Michael Damron.

- **Anna Frid** : Equidistributed sequences generated by binary morphisms.
  We associate with a infinite word w from a uniquely ergodic shift a number $\nu(w)$ from $[0, 1]$ equal to the measure of the set all words smaller than w. This mapping ν has been recently studied by Narbel and Lopez, and simultaneously by Avgustinovich, Puzynina and the author.

In the previous paper, we considered a family of fixed points of morphisms for which ν can be constructed directly. Here I show that with some additional techniques, given a primitive binary morphism with an aperiodic fixed point, we can explicitly construct $\nu(w)$ for all infinite words w from the shift of the fixed point of the morphism. To do it, we construct a whole equidistributed sequence on $[0, 1]$ corresponding to all shifts of w, with the help of a morphism on intervals.

- **Arnaud Hilion** : Invariant measures for stationary train-track towers.
  We will introduce train-track towers, and explain how they can be used to describe the invariant measures of their "legal subshifts" (or "laminations"). Applications to invariant measures of (non primitive) substitutions or invariant currents of some free group hyperbolic automorphisms will be given. This is a joint work with Nicolas Bédaride and Martin Lustig.

- **Arye Juhasz** : Equations and words.
  Let F be a free group freely generated by a finite set X and let R be a cyclically reduced non-empty word in F. Find the possible Dehn functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ of the group presentation $P = \langle X|R\rangle$. (In other words, suppose W is a word in F which represents 1 in the group presented by P. Then, of course, W is the product of k conjugates of R, for a minimally chosen k. In these terms the question is : what are the possible functions f as above , such that $k < f(|W|)$, for every such W ?)
  Conjecturally these are $f(t) \sim t$, $f(t) \sim t^2$, $f(t) \sim 2^t$ and the third Ackermann function. This problem is very difficult and in fact there is no known method in combinatorial and geometrical group theory by which one could attack the problem. There for it is reasonable to subdivide it into two problems :
  - Problem 1 : What are the possible non-polynomial Dehn functions ?
  - Problem 2 : What are the possible polynomial Dehn functions ?

  In the work that I would like to present I solve Problem 1 for positive words R (relators). I show that if R is positive then the only non-polynomial Dehn function is exponential. The method also gives partial solution to Problem 2, namely, that the polynomial has degree at most 4.

- **Glenn Merlet** : Separating words with tropical automata.

- **Jarkko Peltomäki** : Square root map on Sturmian words.
  In a recent paper, the author considered with M. Whiteland a square root map on Sturmian words. A Sturmian word s of slope $[0; a+1, b+1, \ldots]$ can be expressed as an infinite product of six distinct minimal squares : $s = X_1^2 X_2^2 \cdots$, where the X_i are one of the words $S_1 = 0$, $S_2 = 010^{a-1}$, $S_3 = 010^a$, $S_4 = 10^a$, $S_5 = 10^{a+1}(10^a)^b$, or $S_6 = 10^{a+1}(10^a)^{b+1}$. The square root of s is defined to be the word $\sqrt{s} = X_1 X_2 \cdots$. We proved that $\sqrt{s}$ has the same set of factors as s.
  The square root map can be generalized for so-called optimal squareful words. An optimal squareful word is an infinite aperiodic word whose all positions start one of the minimal squares S_i^2 above. The square root of such a word is well-defined. We constructed an infinite family of non-Sturmian optimal squareful words whose language is preserved by the square root map. However, the Sturmian subshift Ω_α of slope α satisfies $\sqrt{\Omega_\alpha} \subseteq \Omega_\alpha$, while the subshift generated by a constructed word fails to satisfy this property. This leads to the following open problem : if an aperiodic subshift Ω of optimal squareful words satisfies $\sqrt{\Omega} \subseteq \Omega$, does it follow that Ω contains only Sturmian words ? In other words, can this property be used to obtain a

characterization of Sturmian words in terms of the square root map ?
This question is linked to specific solutions to the word equation $X_1^2 X_2^2 \cdots X_n^2 = (X_1 X_2 \cdots X_n)^2$. A solution to this equation is a primitive word w such that w^2 can be written as a product of the minimal squares S_i^2, $w^2 = X_1^2 X_2^2 \cdots X_n^2$, satisfying the word equation. We proved that the solutions to the word equation in the language of a Sturmian word of slope α are exactly the reversed standard words of slope α. We also showed that an optimal squareful word may contain a solution to the word equation that cannot occur in a Sturmian word : for instance, if S is a long enough reversed standard word, then the word SLL, where L is S with the two first letters exchanged, is such a solution. Similarly, substituting into $S(L)^{2n}$ gives such solutions. Computer experiments suggest that all solutions to the word equation either occur in a Sturmian word or are obtainable using the construction given above (or with its variations). Thus we ask : are there other solutions to the word equations than the ones described ? If not, how to characterize all solutions ?
We believe that the proposed open problems are connected. Solving the latter can possibly lead to a solution of the former problem.

– **Svetlana Puzynina** : k-abelian palindromicity.

– **Michaël Rao** : k-abelian avoidability.

– **Aleksi Saarela** : k-abelian complexity and fluctuation.

– **Markus Whiteland** : k-abelian singletons in connection with Gray codes for Necklaces.
This work is based on [1]. We are interested in the equivalence classes induced by k-abelian equivalence, especially in the number of the classes containing only one element, k-abelian singletons. By characterizing k-abelian equivalence with k-switchings, a sort of rewriting operation, we are able to obtain a structural representation of k-abelian singletons. Analyzing this structural result leads, through rather technical considerations, to questions of certain properties of sets of vertex-disjoint cycles in the de Bruijn graph $dB_\Sigma(k-1)$ of order $k-1$. Some problems turn out to be equivalent to old open problems such as Gray codes for necklaces (or conjugacy classes). We shall formulate the problem in the following.
Let $\mathcal{C} = \{V_1, \ldots, V_n\}$ be a cycle decomposition of $dB_\Sigma(n)$, that is, a partition of the vertex set Σ^n into sets, each inducing a cycle or a loop in $dB_\Sigma(n)$. Let us then define the quotient graph $dB_\Sigma/\mathcal{C}$ as follows. The set of points are the sets in $\mathcal{C}$. For distinct sets $X, Y \in \mathcal{C}$, we have and edge from X to Y if and only if there exists $x \in X$, $y \in Y$ such that $(x, y) \in dB_\Sigma(n)$.
An old result shows that the size of a cycle decomposition of $dB_\Sigma(n)$ is at most the number of necklaces of length n over Σ (see [2]). We call a cycle decomposition *maximal*, if its size is maximal. In particular, the cycle decomposition given by necklaces is maximal.
Conjecture 1. *For any Σ and $n \in \mathbb{N}$, there exist a maximal cycle decomposition $\mathcal{C}$ of $dB_\Sigma(n)$ such that $dB_\Sigma(n)/\mathcal{C}$ contains a hamiltonian path.*
A natural candidate to study here is the cycle decomposition given by necklaces. This has been studied in the literature in the connection of Gray codes for necklaces. Concerning this, there is an open problem since 1997 [3] : Let $\Sigma = \{0, 1\}$, n odd, and $\mathcal{C}$ be the cycle decomposition given by necklaces of length n over $\{0, 1\}$. Does $dB(n)/\mathcal{C}$ contain a hamiltonian path ?
The answer to the above has been verified to be "yes" for $n \leq 15$ ([1]). The case of $n \geq 4$ and n even, the graph is bipartite with one partition larger than the other. On the other hand, we can find other maximal cycle decompositions of $dB_\Sigma(4)$, $dB_\Sigma(6)$, and $dB_\Sigma(8)$ for the binary alphabet which all admit hamiltonian quotient graphs.

We concluded in [1] that Conjecture 1 is equivalent to the following Θ-estimation of the number of k-abelian singletons of length n.

Conjecture 2. *The number of k-abelian singletons of length n over alphabet Σ is of order $\Theta(n^{N_\Sigma(k-1)-1})$, where $N_\Sigma(l)$ is the number of necklaces of length l over Σ.*

Références

[1] J. Karhumäki, S. Puzynina, M. Rao, and M. A. Whiteland : *On the cardinalities of k-abelian equivalence classes*, (submitted, 2016)

[2] J. Mykkeltveit : *A Proof of Golomb's Conjecture for the de Bruijn Graph*, Journal of Combinatorial Theory (B) 13, pp. 40–45 (1972)

[3] C. Savage : *A Survey of Combinatorial Gray Codes*, SIAM Review 39.4, pp. 605–629 (1997)