Open problems on square root map on Sturmian words

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- Background on the square root map
- Open problems

Minimal squares

- Let s be a Sturmian word of slope $[0; a+1, b+1, \ldots]$.
  - Between two blocks 1, there is 0^a or 0^{a+1}.
  - Between two blocks 10^{a+1}, there is $(10^a)^b$ or $(10^a)^{b+1}$.
- Any position in s begins with one of the six squares:

$$\begin{aligned} S_1^2 &= 0^2, & S_4^2 &= (10^a)^2, \\ S_2^2 &= (010^{a-1})^2, & S_5^2 &= (10^{a+1}(10^a)^b)^2, \\ S_3^2 &= (010^a)^2, & S_6^2 &= (10^{a+1}(10^a)^{b+1})^2, \end{aligned}$$

- The squares are *minimal*: they do not have proper square prefixes.
- The *square roots* in the case $a = 1$, $b = 0$ appear in the footer of every slide.

The square root map

- Every Sturmian word s is a product of these six minimal squares.
- $s = X_1^2 X_2^2 X_3^2 \cdots$
- $\sqrt{s} = X_1 X_2 X_3 \cdots$
- The square root map deletes half of each square.

Example: Fibonacci

$$f = (010)^2 \cdot (100)^2 \cdot (10)^2 \cdot (01)^2 \cdot 0^2 \cdot (10010)^2 \cdots ,$$
$$\sqrt{f} = 010 \cdot 100 \cdot 10 \cdot 01 \cdot 0 \cdot 10010 \cdots$$

Theorem (P.-W.)

If s is a Sturmian word, then $\mathcal{L}(s) = \mathcal{L}(\sqrt{s})$. That is, the square root map preserves the language of a Sturmian word.

Generalization

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Question

Can we characterize Sturmian words among optimal squareful words with $\sqrt{}$?

Idea for a characterization

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Theorem (P.-W.)

There exists a non-Sturmian optimal squareful word Γ such that $\sqrt{\Gamma} = \Gamma$.

Idea for the construction

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- $w = 01010010$ is a solution.

Idea for the construction

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Idea for the construction

- Fixed points of $\sqrt{\ }$: $01c_\alpha$ and $10c_\alpha$.
- $01c_\alpha$ and $10c_\alpha$ are fixed points because they have arbitrarily long solutions to the word equation

$$X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$$

as square prefixes.

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- Let $\Gamma = \lim_{n\to\infty} w_n$.
- Then $\sqrt{\Gamma} = \Gamma$.

How it works

- Let S be a (long enough) seed solution of the word equation.
- Now $\sqrt{SS} = S$, $\sqrt{SL} = S$, $\sqrt{LS} = L$, $\sqrt{LL} = L$.
    - L is S with two first letters exchanged.

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- Iterate to get longer solutions.
- $S \to LSS \to \underline{SS}SLSSLSS \to \ldots$

- Sturmian subshift Ω_α satisfies the *strong* property $\sqrt{\Omega_\alpha} \subseteq \Omega_\alpha$.

Towards the open problems

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- Sturmian subshift Ω_α satisfies the *strong* property $\sqrt{\Omega_\alpha} \subseteq \Omega_\alpha$.
- What about the subshift Ω_Γ generated by the fixed point Γ?

Theorem (P.-W.)

There exists $w \in \Omega_\Gamma$ such that $\sqrt{w}$ is periodic.

- Ω_Γ does not satisfy the strong property.

Open problems

Conjecture

Let Ω be a minimal subshift containing optimal squareful words. If $\sqrt{\Omega} \subseteq \Omega$, then Ω is a Sturmian subshift.

Connection to solutions of the word equation

- We used a specific construction to obtain the fixed point Γ.

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- Other types of solutions might produce a counter example to the conjecture.
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Open problem

Characterize the solutions to the word equation $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$.

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- Suppose $\sqrt{\Omega} \subseteq \Omega$, with Ω minimal.
- Show that points of Ω contain arbitrarily long solutions to the word equation.
- If all solutions are Sturmian, then the conclusion is clear.
- If not, then by altering our proof, we can show that there is $w \in \Omega$ s.t. $\sqrt{w}$ is periodic.
- Thus $\sqrt{\Omega} \subseteq \Omega$ is not satisfied.

Thank You

Thank you for your attention!

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A square root map on Sturmian words
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