Periodicity, Stratified Surgery, and Multiaxial Manifolds

Min Yan, Hong Kong University of Science and Technology
(joint with S. Cappell and S. Weinberger)

1. Periodicity

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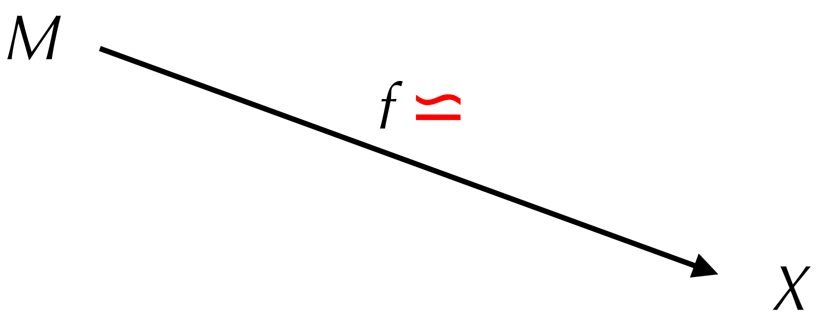

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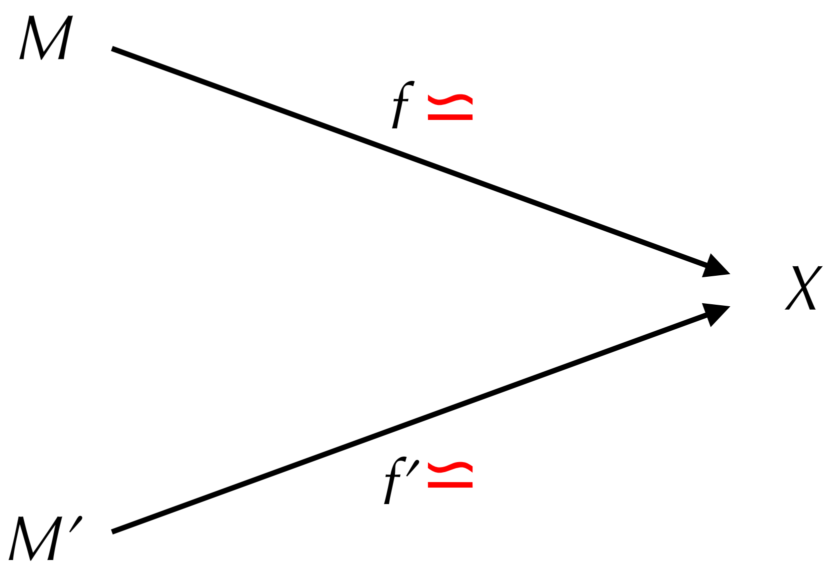

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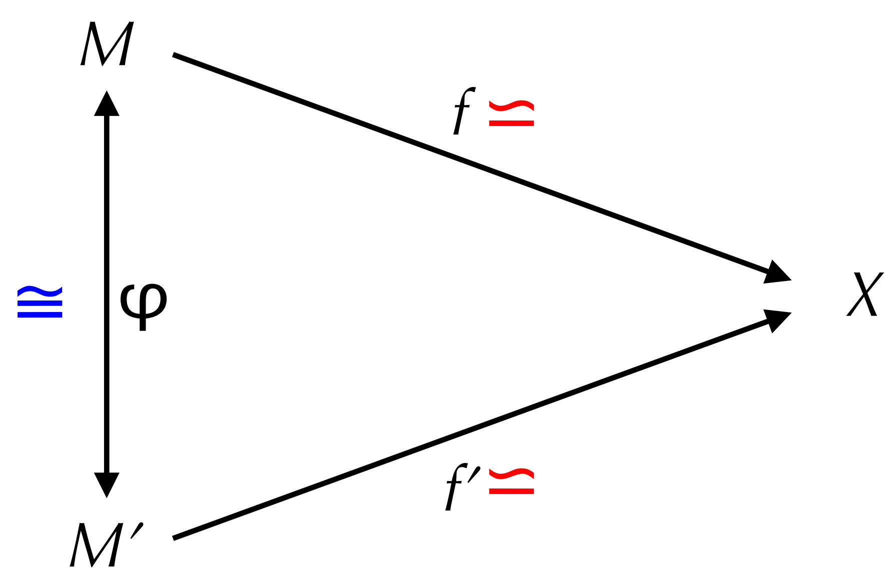

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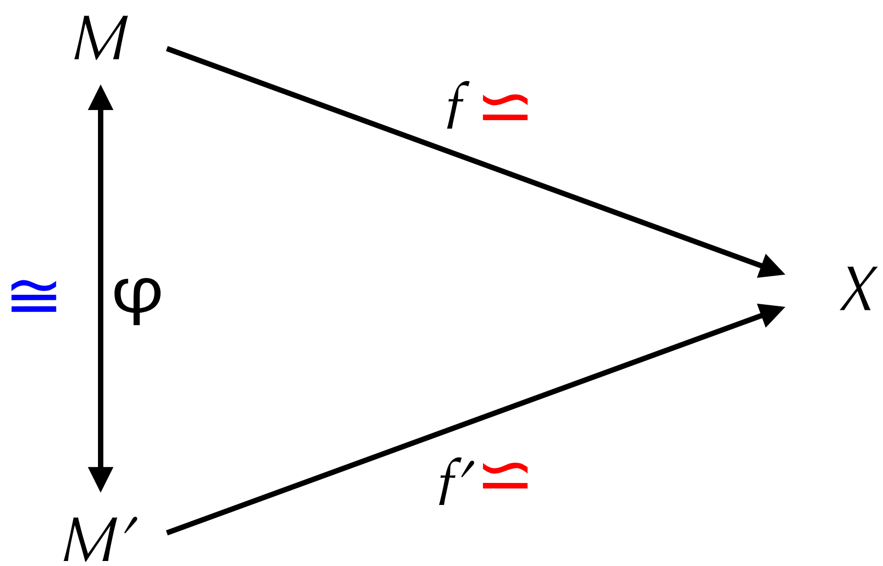

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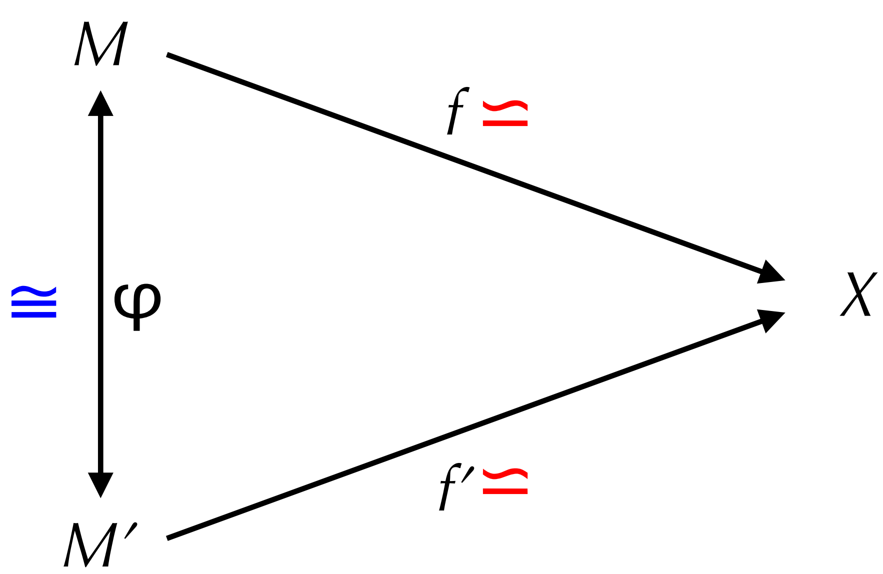

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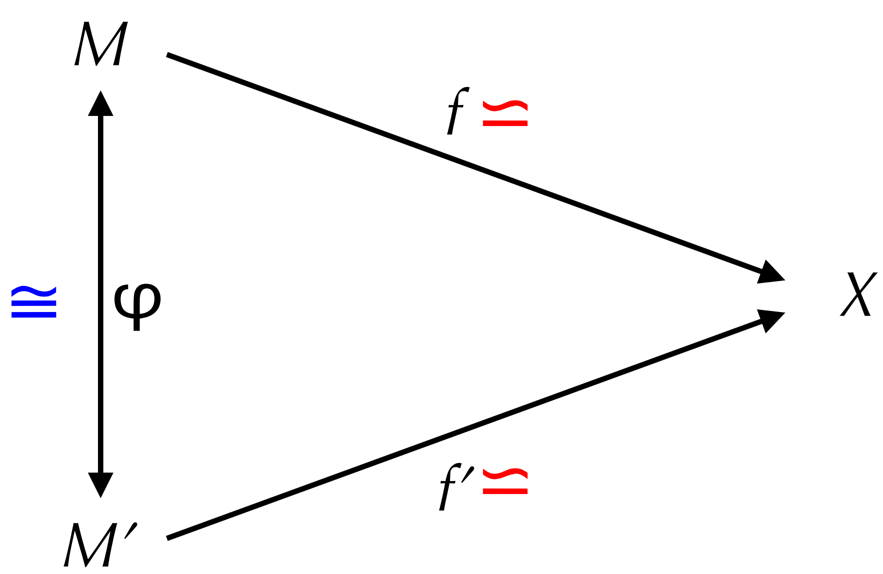

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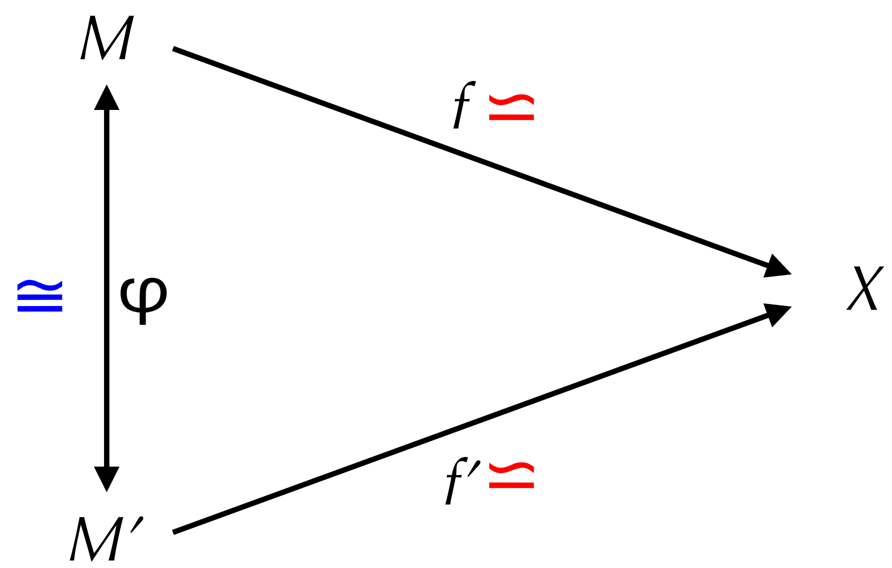

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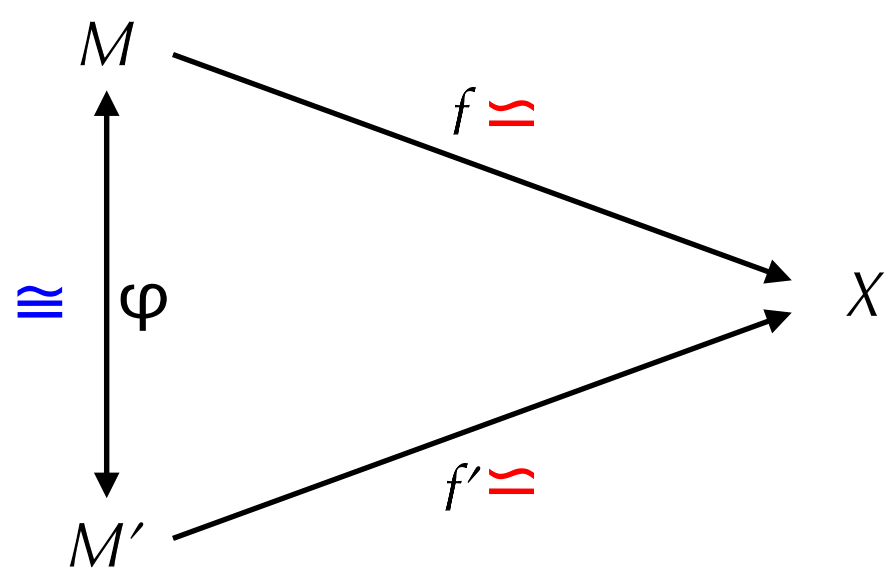

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3. Fake complex projective space [<1970]: $S^{\text{top}}(\mathbf{CP}^n) = \mathbf{Z}^{n/2} \oplus (\mathbf{Z}/2\mathbf{Z})^{n/2}$.

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[Wall 1971]: geometrical interpretation of $L_n(X)$, and 4-fold periodicity given by multiplying a manifold of signature 1 ($\mathbf{CP}^{\mathrm{even}}$, $\mathbf{HP}^n$, etc.)

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Surgery exact sequence is the long exact sequence of homotopy groups of a fibration

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$\Rightarrow$ periodicity

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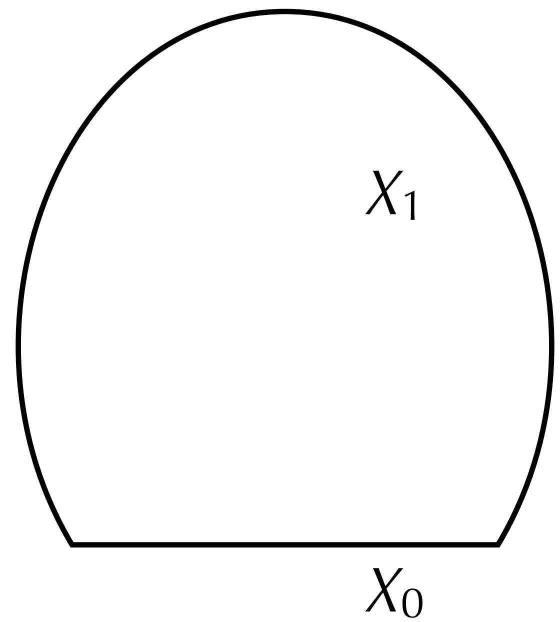
$\boldsymbol{L}(X)$ is a spectrum valued covariant functor.

$\boldsymbol{H}_*(X; \boldsymbol{L}(\bullet)) \to \boldsymbol{L}(X)$ is the assembly map for the covariant functor $\boldsymbol{L}$.

The structure $\boldsymbol{S}(X)$ measures the lack of additivity of the functor $\boldsymbol{L}$.

2. Stratified Space

2-Strata Space



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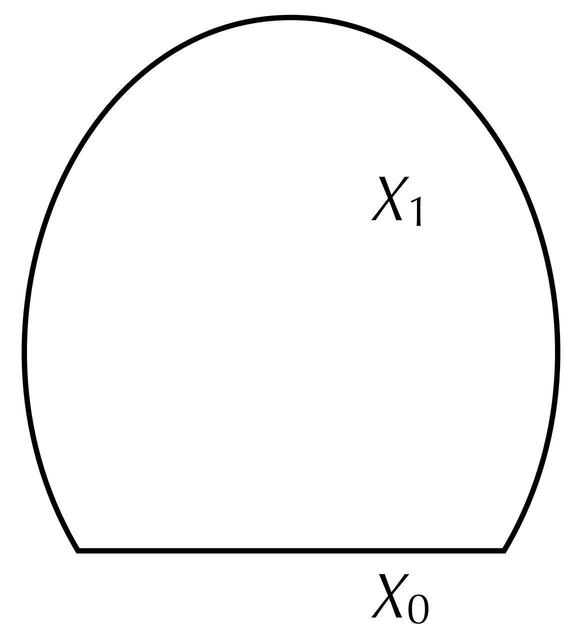

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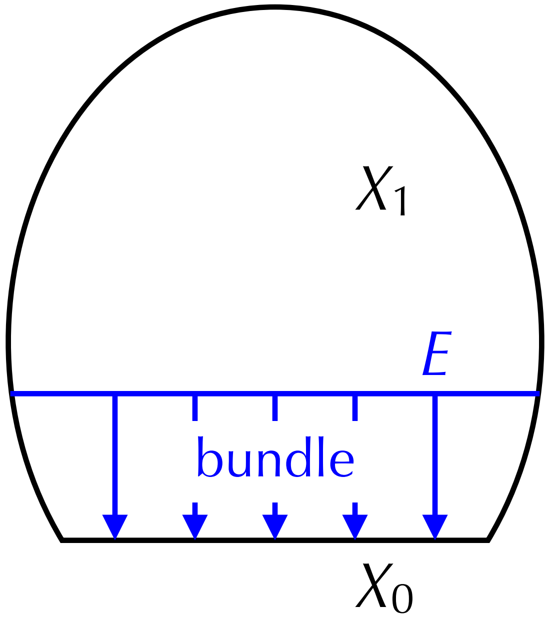

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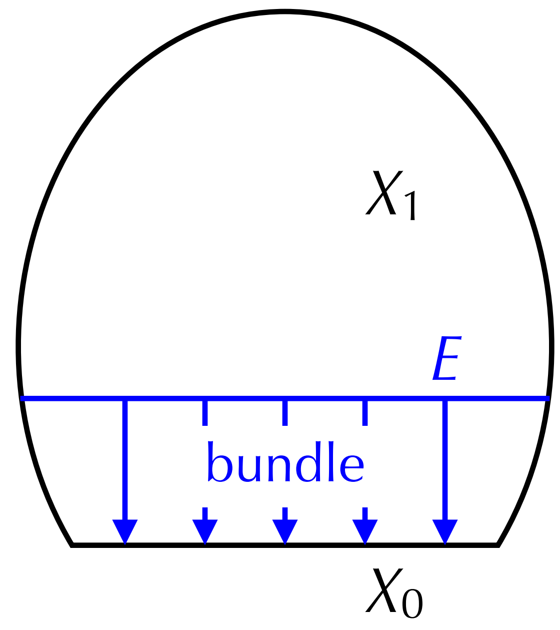

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[Browder-Quinn 1975]: stratified surgery

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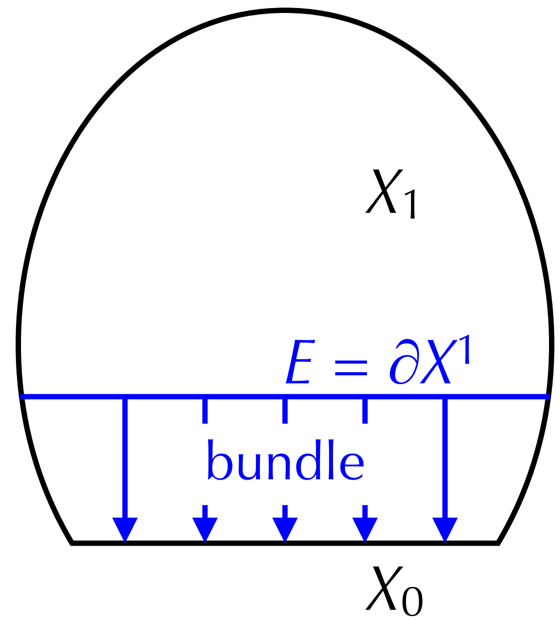

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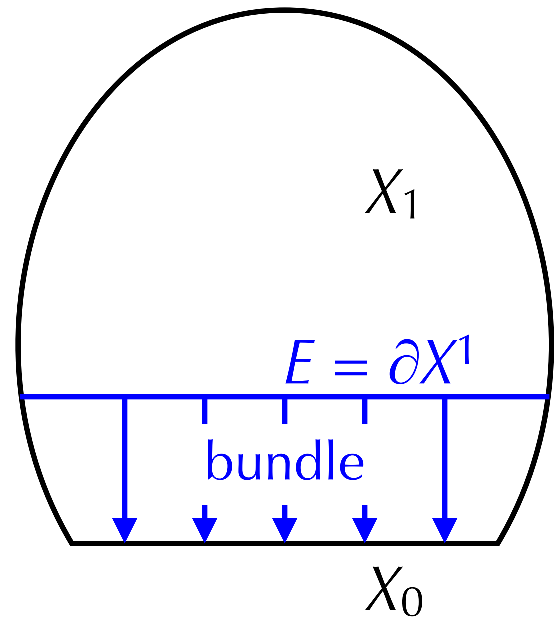

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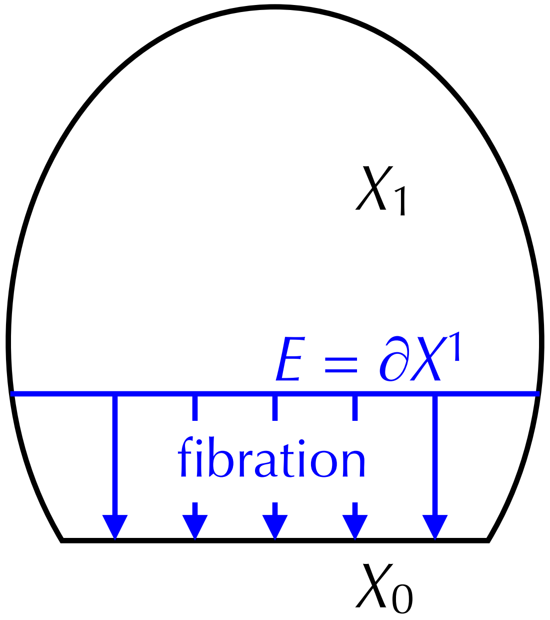

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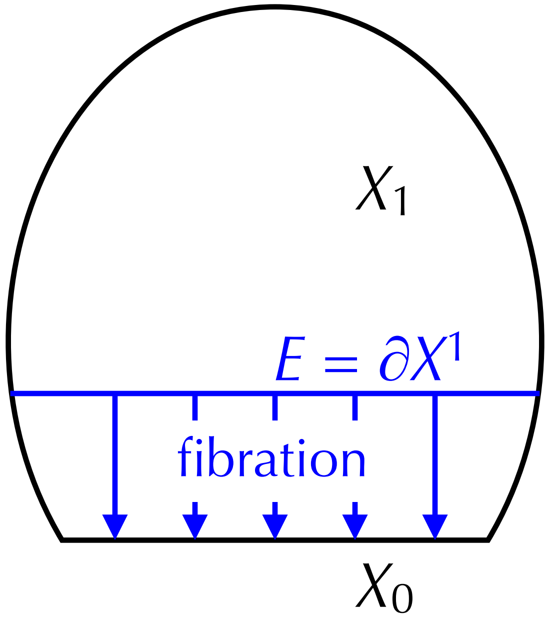

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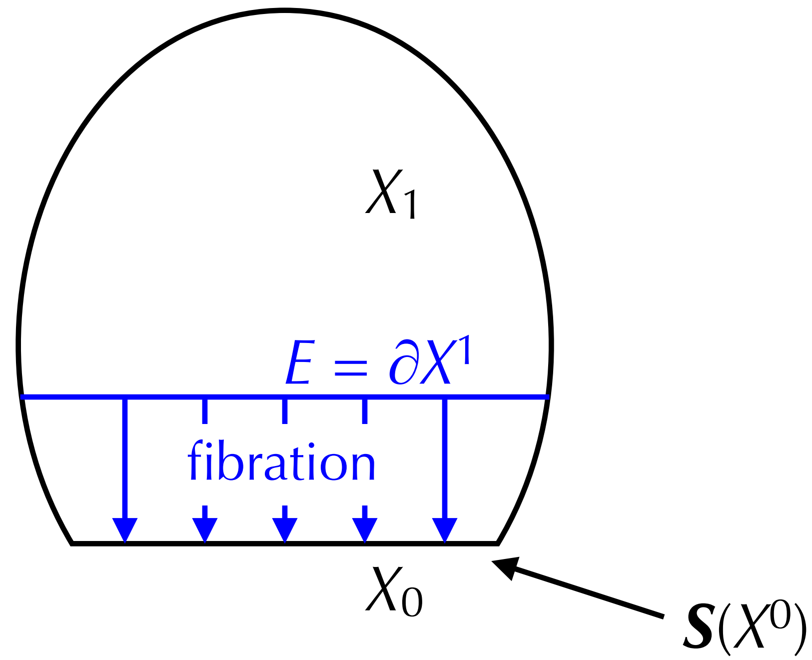

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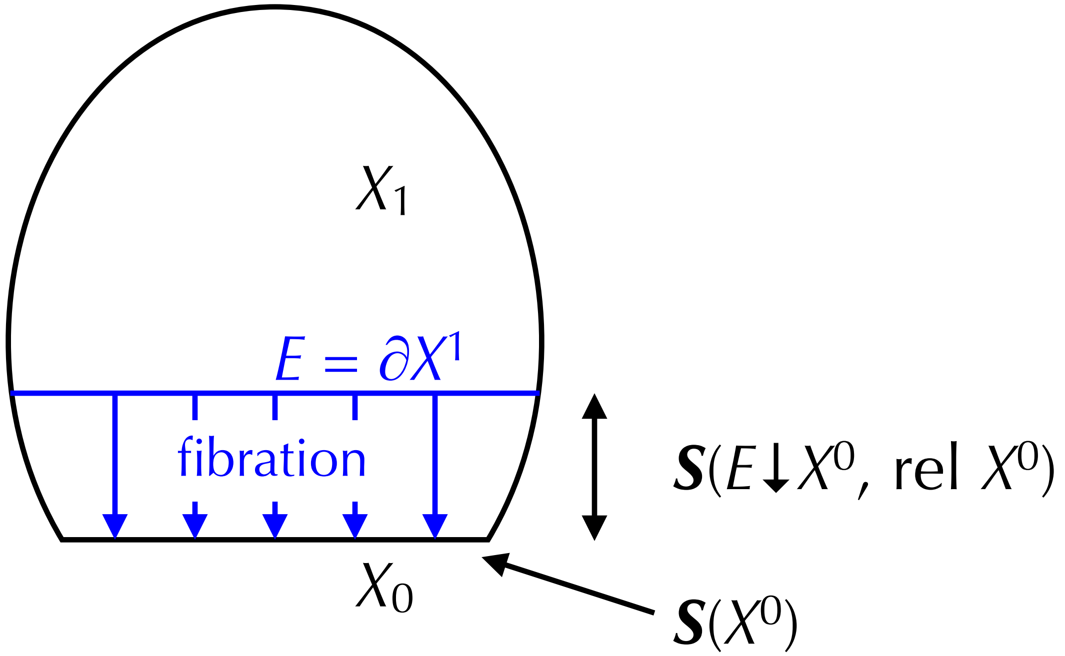

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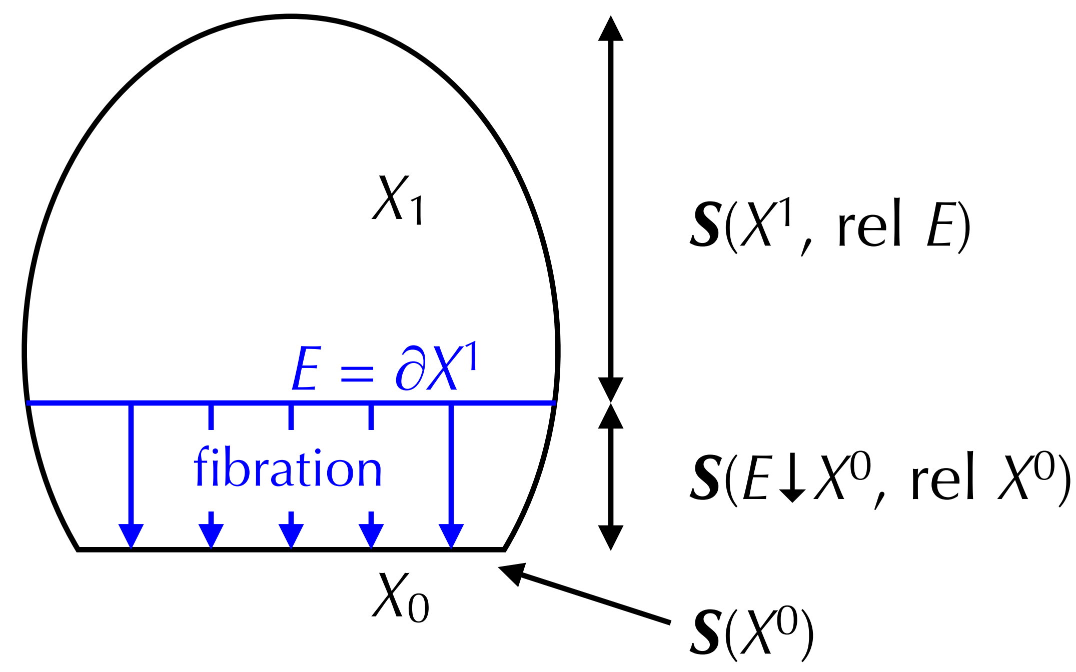

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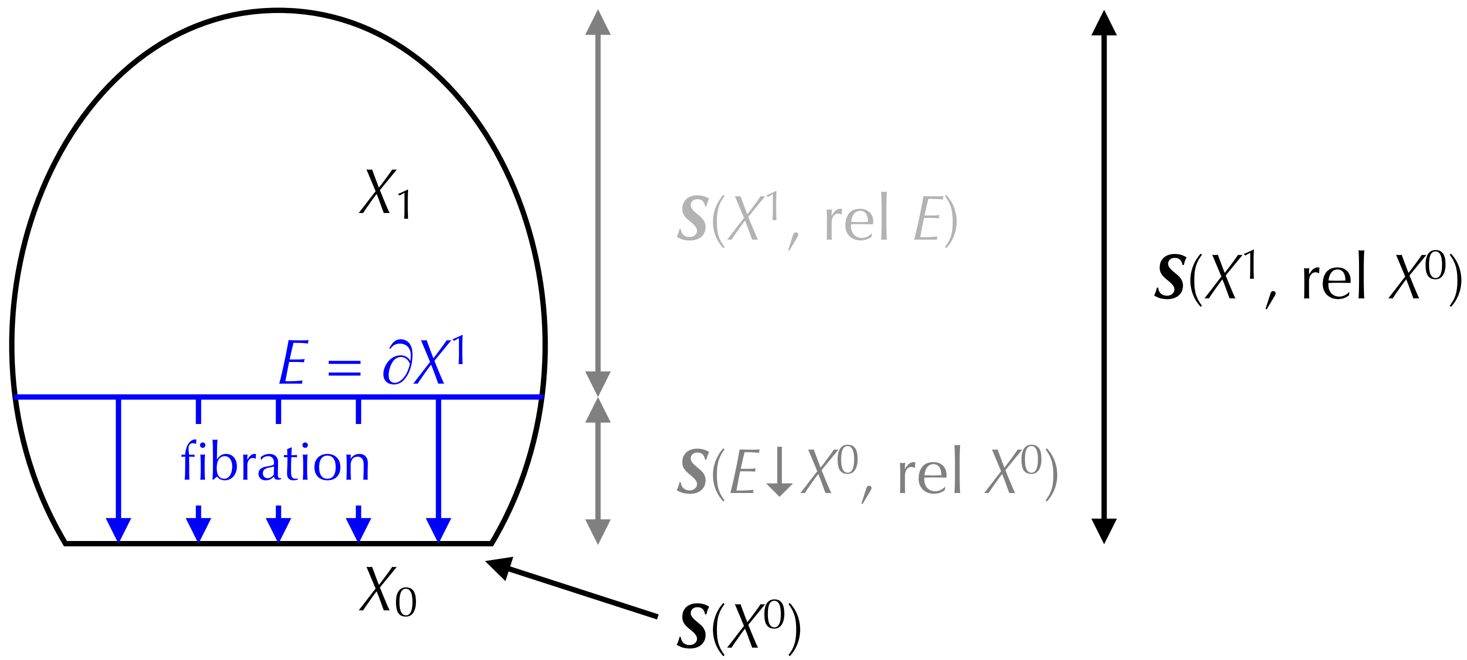

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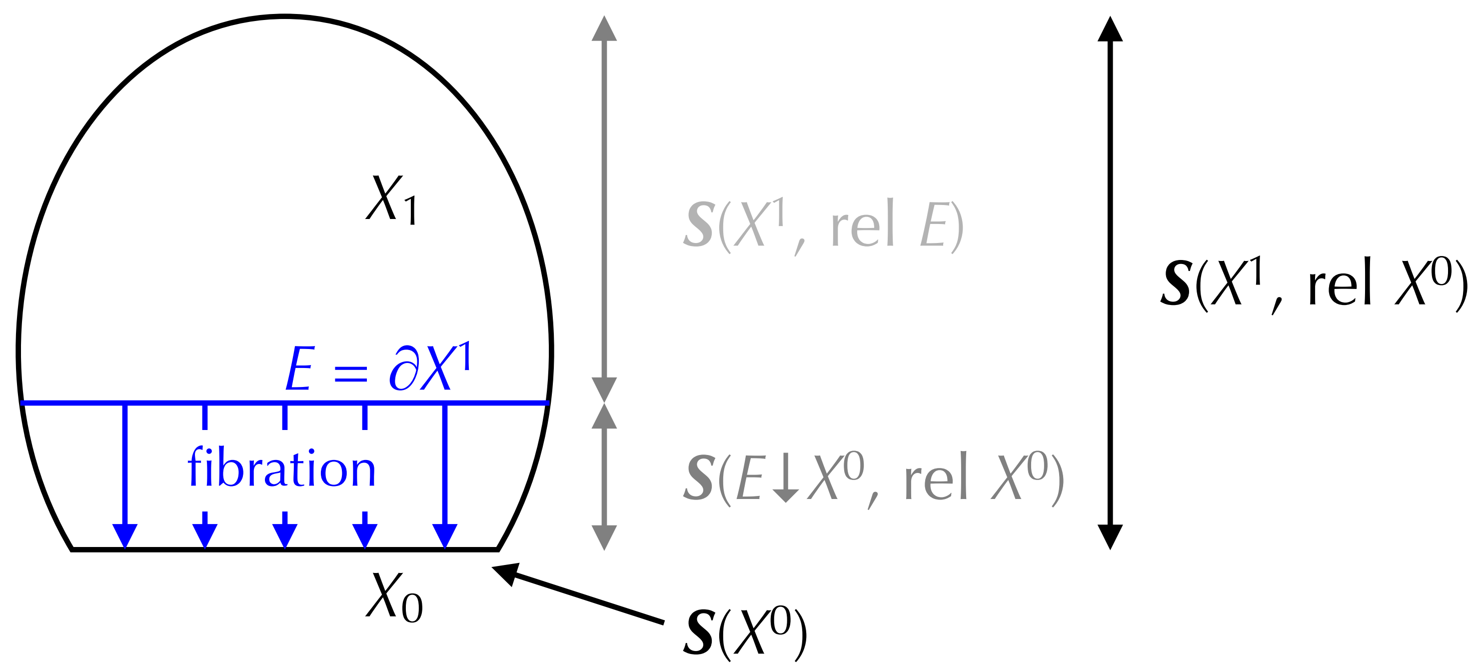

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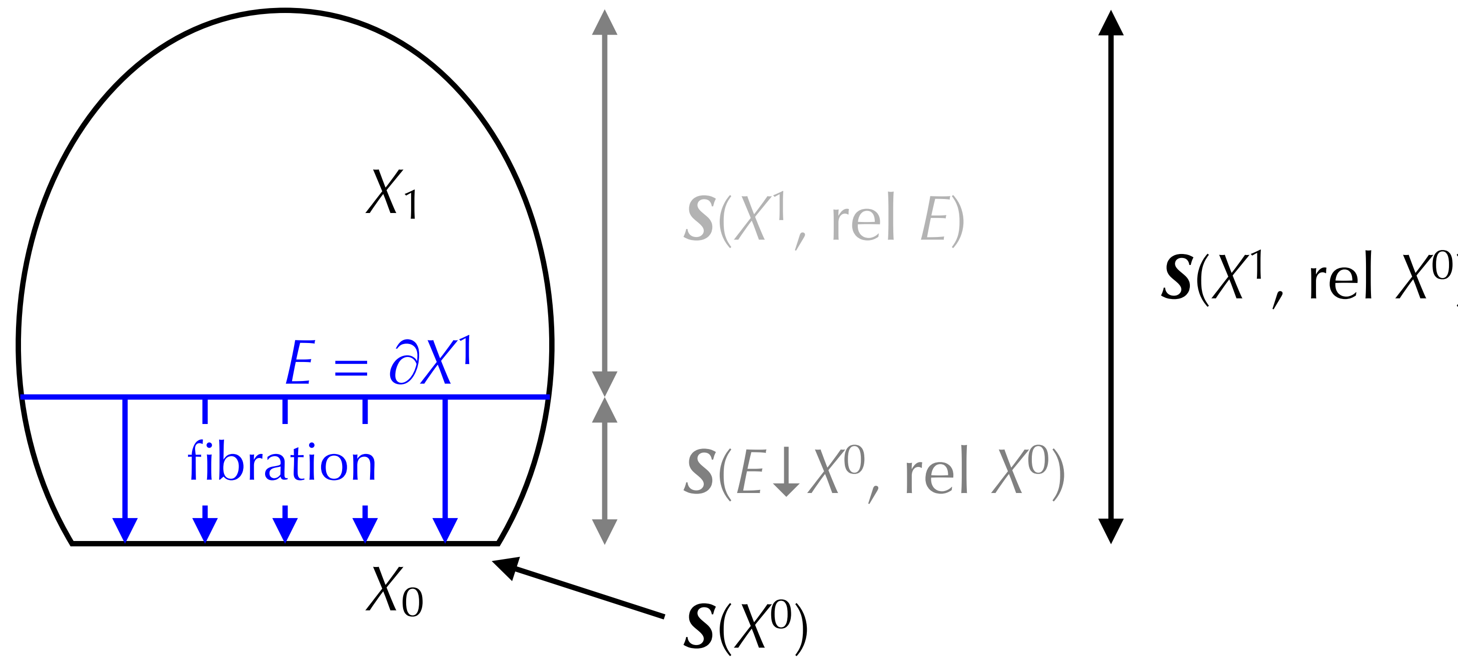

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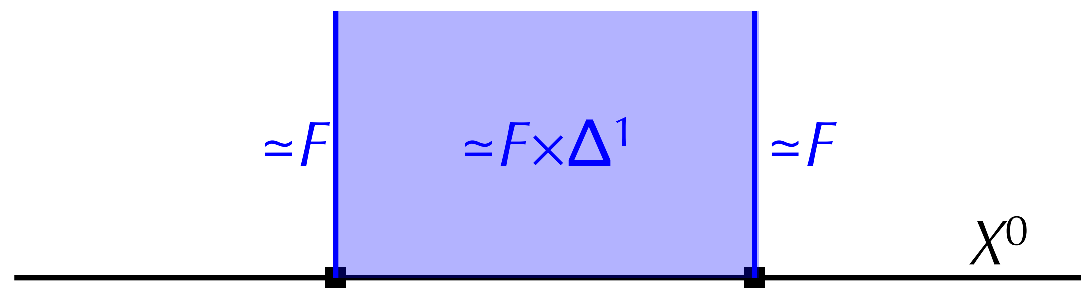

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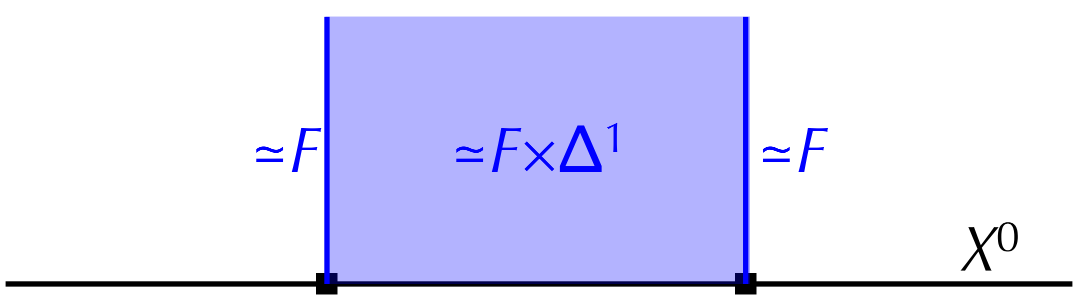

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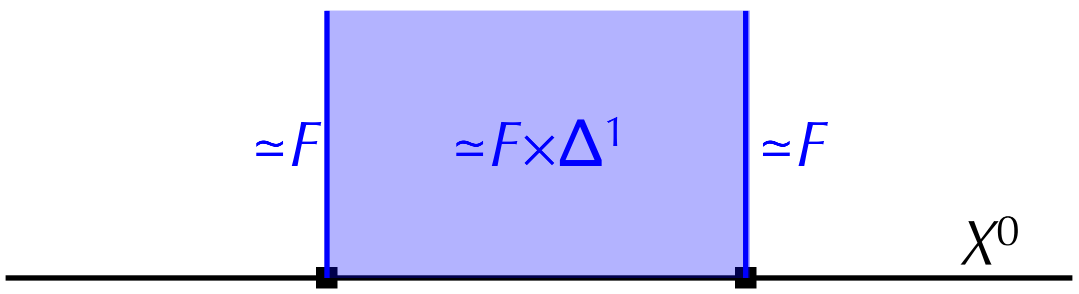

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$$\boldsymbol{S}(E{\downarrow}X^0, \text{rel } X^0) \to \boldsymbol{H}_*(E; \boldsymbol{L}(\bullet)) \to \boldsymbol{H}_*(X^0; \boldsymbol{L}(F))$$

Then combine with

$$\boldsymbol{S}(X^0) \to \boldsymbol{H}_*(X^0; \boldsymbol{L}(\bullet)) \to \boldsymbol{L}(X^0)$$

$$\boldsymbol{S}(X^1, \text{rel } E) \to \boldsymbol{H}_*(X^1; \boldsymbol{L}(\bullet)) \to \boldsymbol{L}(X^1)$$

Stratified Surgery Theory

Stratified Surgery Theory

[Weinberger]: stratified surgery fibration (modulo topological *K*-theory)

$$\boldsymbol{S}^{\mathrm{htp}}(X) \to \boldsymbol{H}_{*}(X;\, \boldsymbol{L}^{\mathrm{BQ}}(\mathrm{loc}X)) \to \boldsymbol{L}^{\mathrm{BQ}}(X)$$

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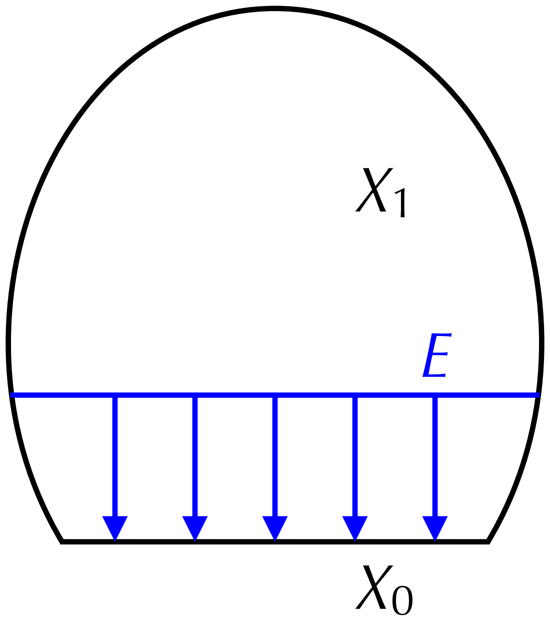

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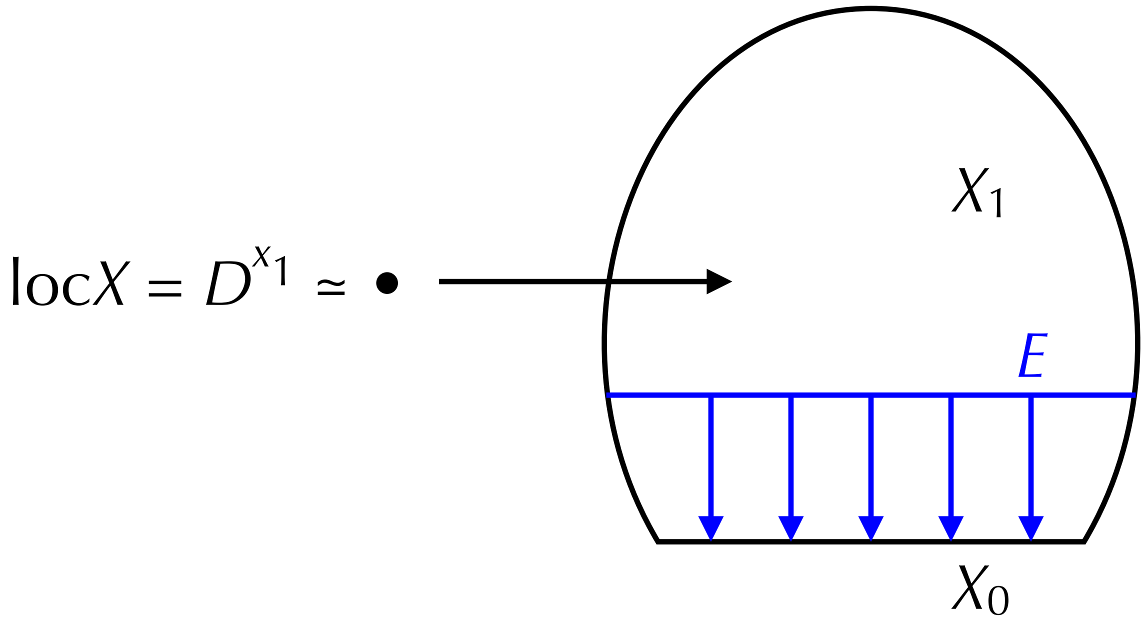

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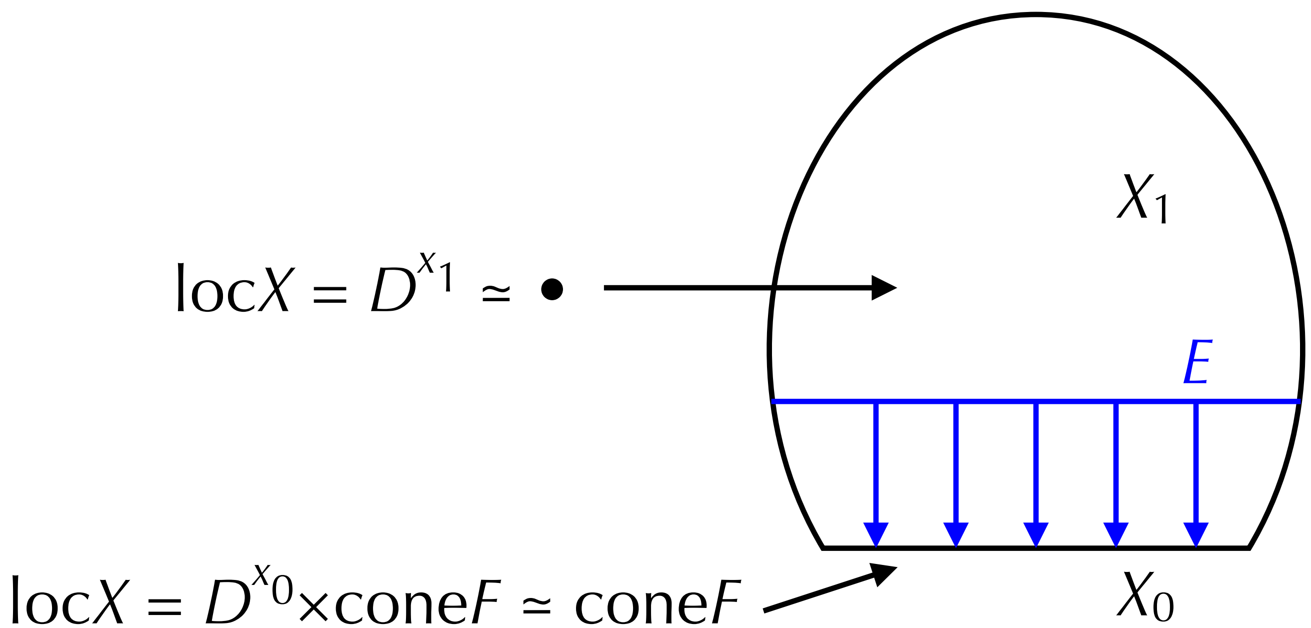

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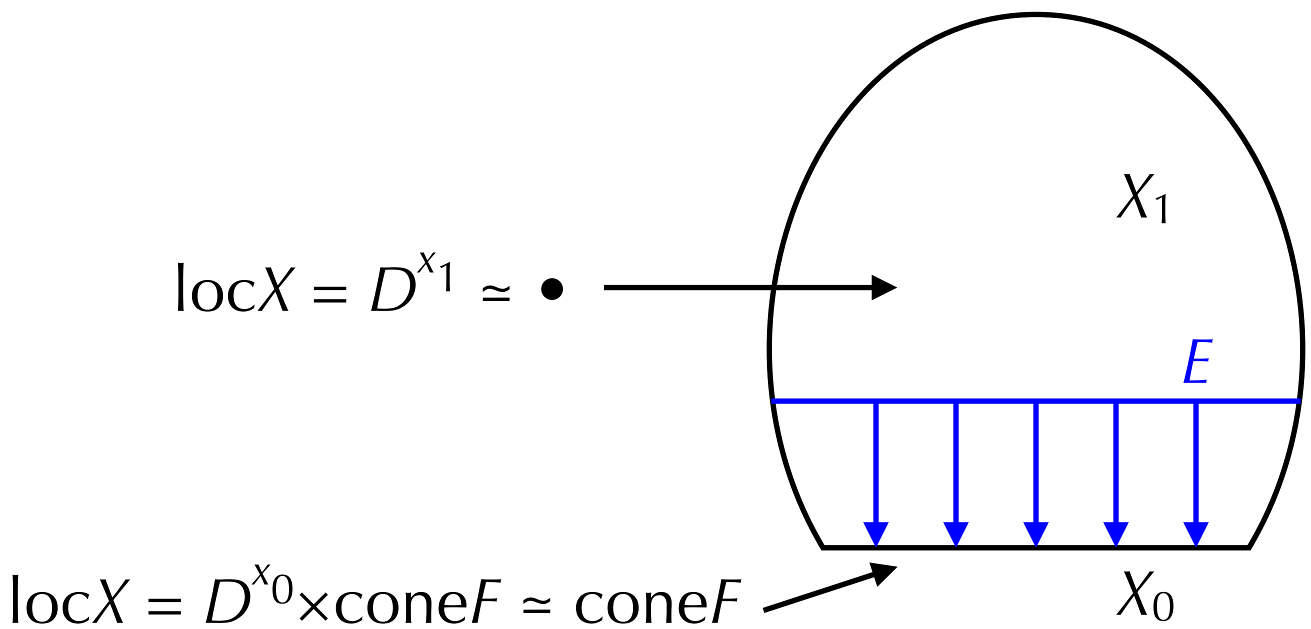

[Browder-Quinn]: stratified surgery fibration

$$\boldsymbol{S}^{\mathrm{geom}}(X) \to \boldsymbol{H}^{*}(X;\, \boldsymbol{L}(\bullet)) = \mathrm{Maps}(X,\, \boldsymbol{L}(\bullet)) \to \boldsymbol{L}^{\mathrm{BQ}}(X)$$

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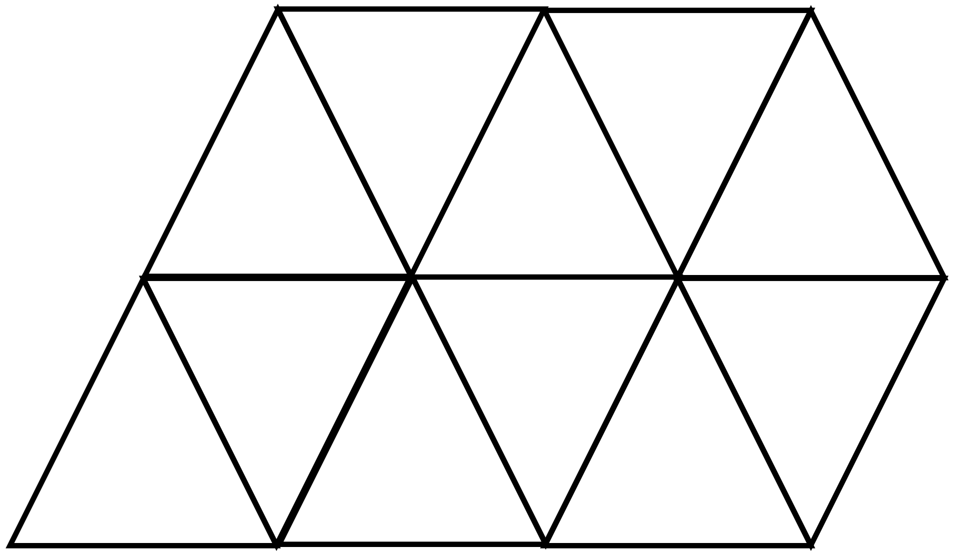

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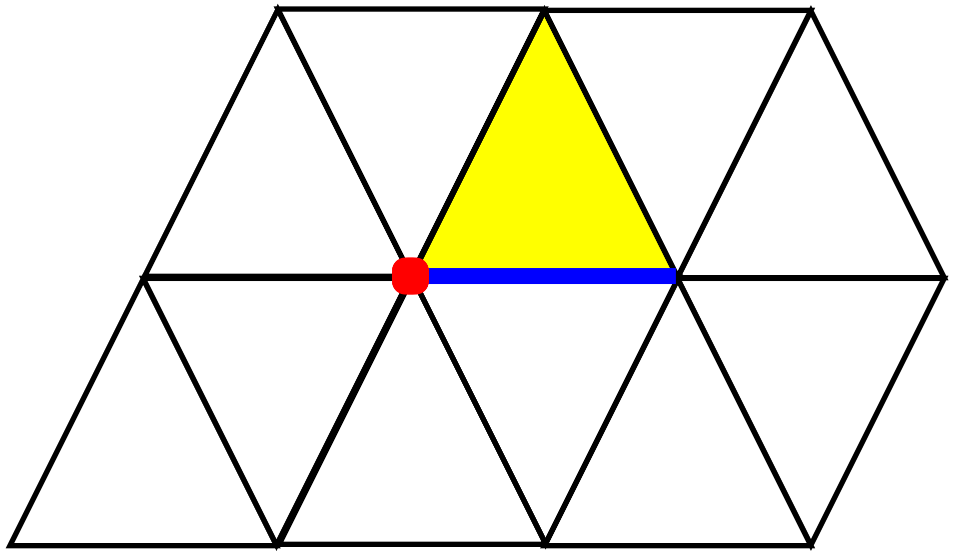

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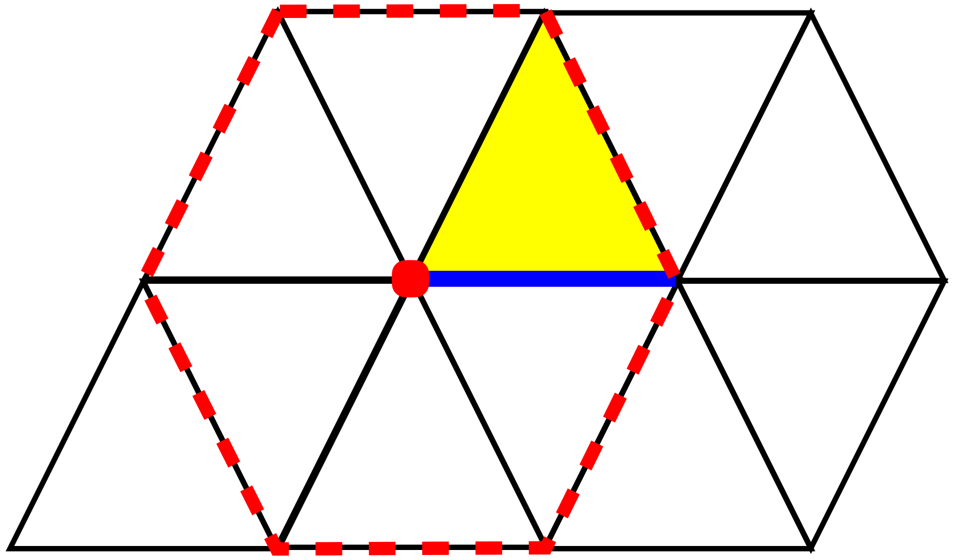

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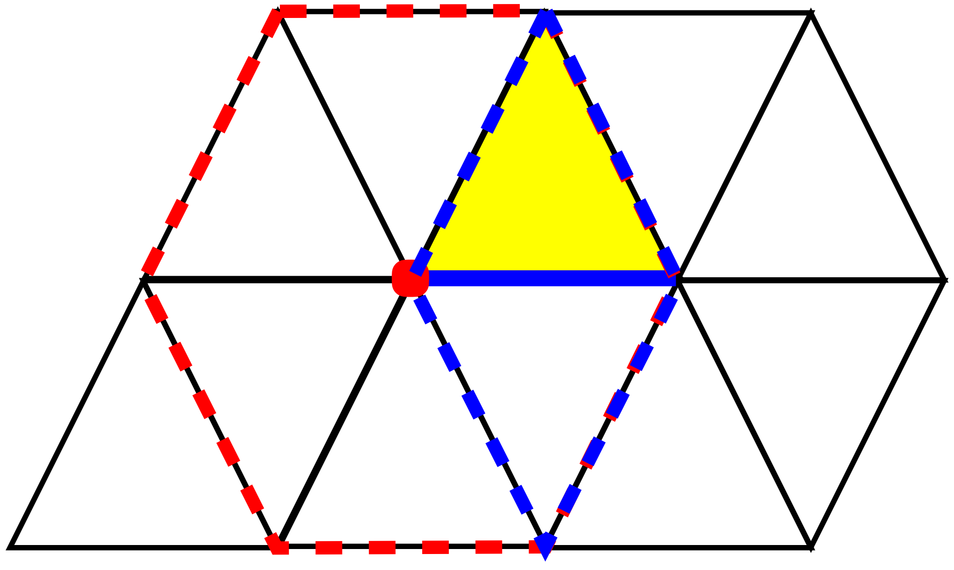

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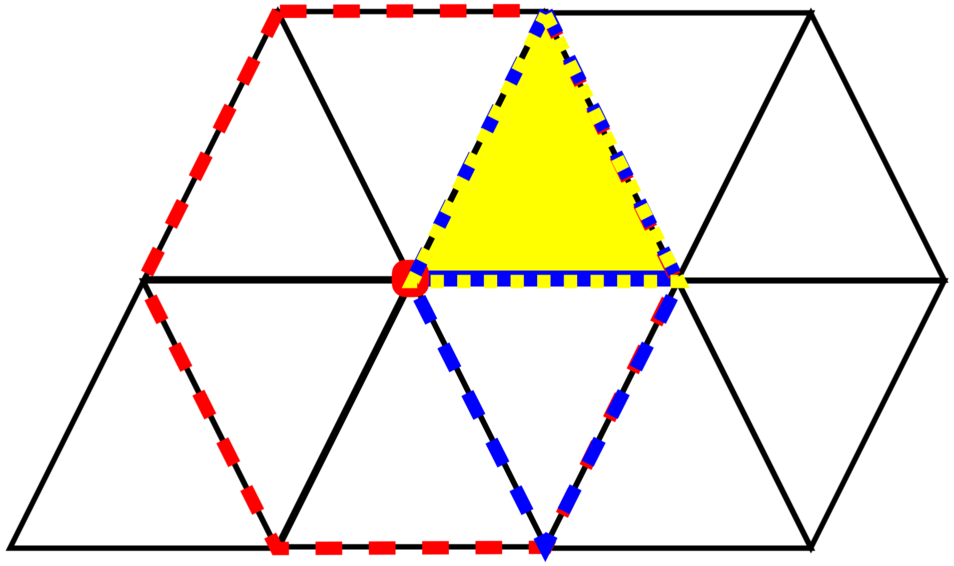

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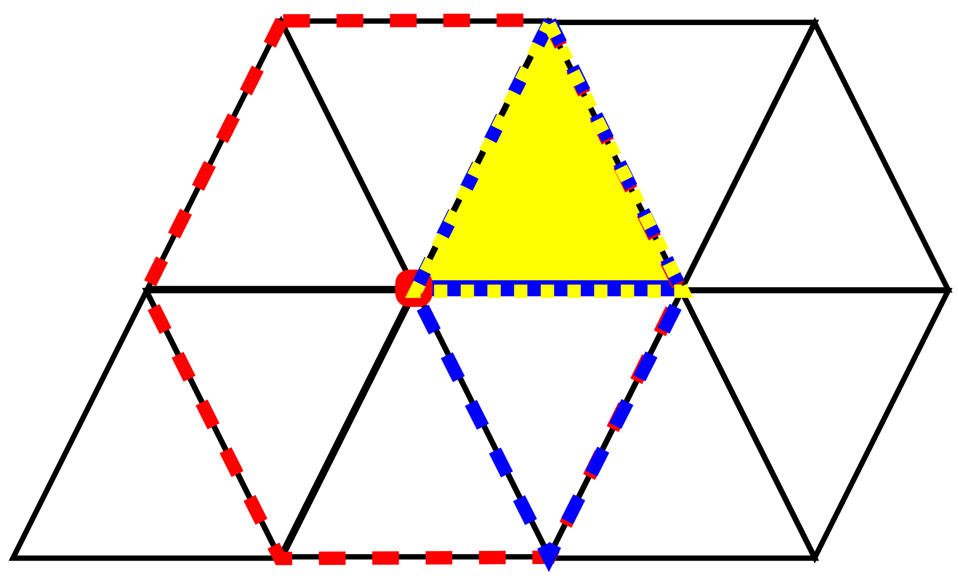
Homology is homotopy pushout: $\boldsymbol{H}_*(X;\, \boldsymbol{L}) = \cup\, \boldsymbol{L}(\text{loc}_\sigma X) \times \sigma \,/ \sim$.

The assembly map is canonical by compatible maps $\boldsymbol{L}(\text{loc}_\sigma X) \to \boldsymbol{L}(X)$.



Stratified Surgery Theory

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Homology (spectrum) $\boldsymbol{H}_*(X; \boldsymbol{L})$, with local coefficient system (or cosheaf) $\boldsymbol{L}$.

Satisfying usual homology axioms.

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Homology (spectrum) $\boldsymbol{H}_{*}(X; \boldsymbol{L})$, with local coefficient system (or cosheaf) $\boldsymbol{L}$.

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Homology $H_n(X; \boldsymbol{L}) = \pi_n \boldsymbol{H}_{*}(X; \boldsymbol{L})$

Stratified Surgery Theory

Homology (spectrum) $\boldsymbol{H}_{\ast}(X; \boldsymbol{L})$, with local coefficient system (or cosheaf) $\boldsymbol{L}$.

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Homology $H_n(X; \boldsymbol{L}) = \pi_n \boldsymbol{H}_{\ast}(X; \boldsymbol{L}) \Leftarrow$ spectral sequence $E^2{}_{p,q} = H_p(X, \pi_q \boldsymbol{L})$.

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What is $H_p(X; L)$? [$L = \pi_q \boldsymbol{L}$]

The local coefficient system L:

assigns an abelian group L_σ to each simplex σ of X,

assigns a homomorphism $L_\tau \to L_\sigma$ for $\sigma \subset \tau$, all homomorphisms compatible.

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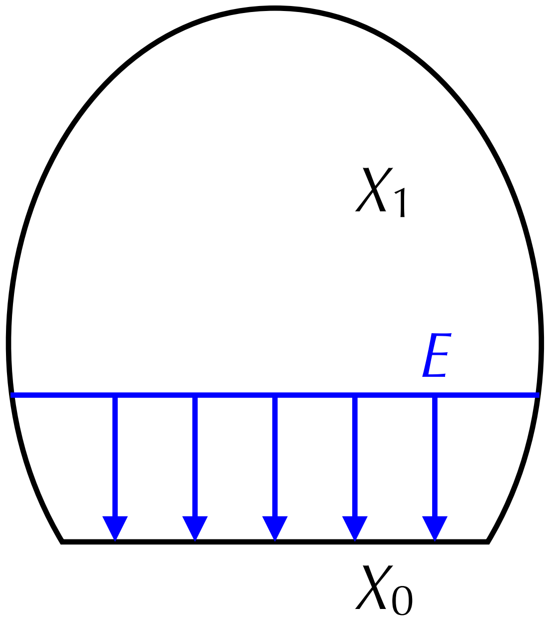
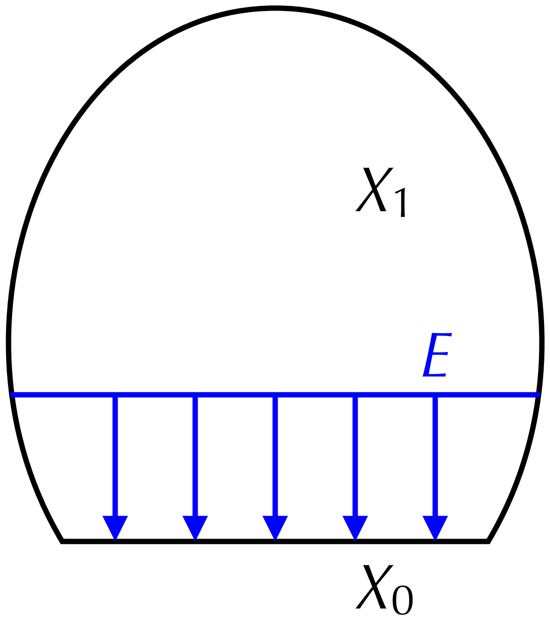
The local coefficient system in usual textbook assumes $L_\tau \to L_\sigma$ are all isomorphic.

Proof of Stratified Surgery



Proof of Stratified Surgery

Recall $\boldsymbol{S}(E{\downarrow}X^0, \text{rel } X^0) \to \boldsymbol{H}_*(E; \boldsymbol{L}(\bullet)) \to \boldsymbol{H}_*(X^0; \boldsymbol{L}(F))$



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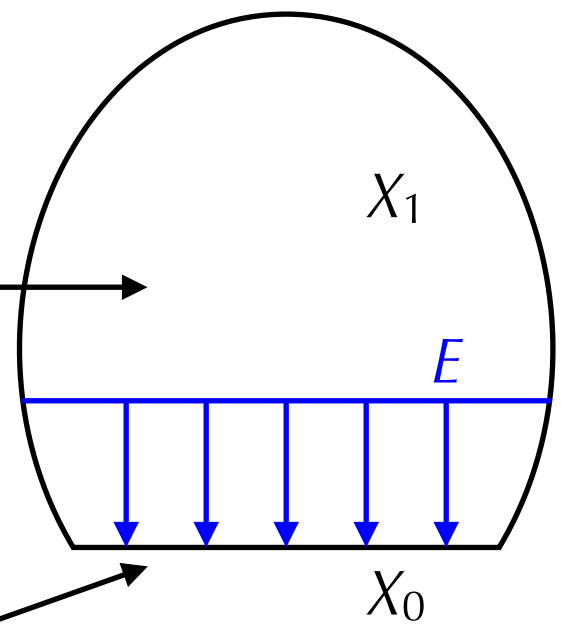
$\to \boldsymbol{H}_*(X^0; \boldsymbol{L}(F)) = \boldsymbol{H}_*(X^0; \boldsymbol{L}(\text{loc}X, \text{rel } X^0)) = \boldsymbol{H}_*(\text{nd}(X^0); \boldsymbol{L}(\text{loc}X, \text{rel } X^0))$

$\boldsymbol{L}(\text{loc}X, \text{rel } X^0) = \boldsymbol{L}(\bullet)$



$\boldsymbol{L}(\text{loc}X, \text{rel } X^0)$

$= \boldsymbol{L}(\text{cone}F, \text{rel cone point})$

$= \boldsymbol{L}(F)$

Proof of Stratified Surgery

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So $\boldsymbol{S}(E{\downarrow}X^0, \text{rel } X^0) = \boldsymbol{H}_*(\text{nd}(X^0), E; \boldsymbol{L}(\text{loc}X, \text{rel } X^0)) = \boldsymbol{H}_*(X, X^1; \boldsymbol{L}(\text{loc}X, \text{rel } X^0))$

[modulo shifting of dimension]

$\boldsymbol{L}(\text{loc}X, \text{rel } X^0) = \boldsymbol{L}(\bullet)$

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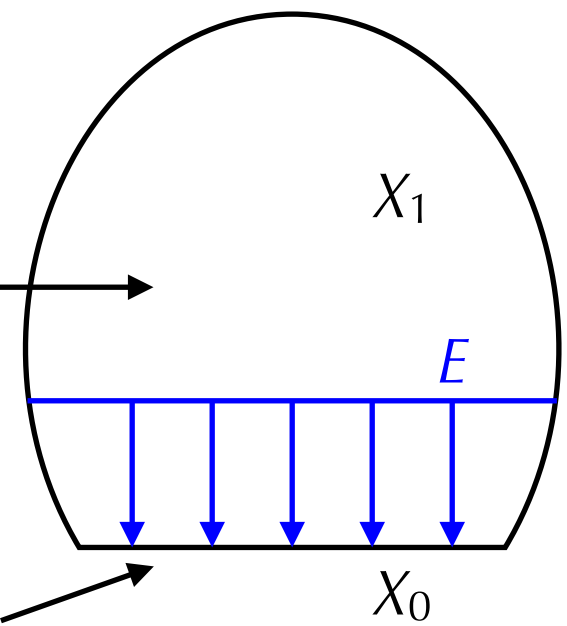

Proof of Stratified Surgery

$\boldsymbol{S}(X, \text{rel } X^0) \longrightarrow \boldsymbol{H}(X, \boldsymbol{L}(\bullet)) \longrightarrow \boldsymbol{L}(X, \text{rel } X^0)$

Proof of Stratified Surgery

$$\begin{array}{ccccc}
\boldsymbol{S}(X^1, \text{rel } E) & & & & \\
\downarrow & & & & \\
\boldsymbol{S}(X, \text{rel } X^0) & \longrightarrow & \boldsymbol{H}(X, \boldsymbol{L}(\bullet)) & \longrightarrow & \boldsymbol{L}(X, \text{rel } X^0) \\
\downarrow & & & & \\
\boldsymbol{S}(E\!\downarrow\! X^0, \text{rel } X^0) & & & &
\end{array}$$

Proof of Stratified Surgery

$$
\begin{array}{ccccc}
\boldsymbol{S}(X^1, \text{rel } E) & \longrightarrow & \boldsymbol{H}_*(X^1, \boldsymbol{L}(\bullet)) & \longrightarrow & \boldsymbol{L}(X^1) \\
\downarrow & & & & \\
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Apply Stratified Surgery to Group Action

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group G acting on manifold M

⇒ orbit space $X = M/G$ stratified by conjugate classes of isotropy groups

⇒ $S_G(M) = S(M/G)$.

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Applying stratified surgery to homotopically stratified group actions on topological manifolds, we get

- equivariant periodicity: $S_G(M\times\mathbf{D}V, \text{rel } M\times\mathbf{S}V) = S_G(M)$.
- homotopy replacement of fixed point: $N \simeq_G M \Rightarrow N^G \simeq M^G$.
- functoriality: $M \to_G N \Rightarrow S_G(M) \to S_G(N)$.

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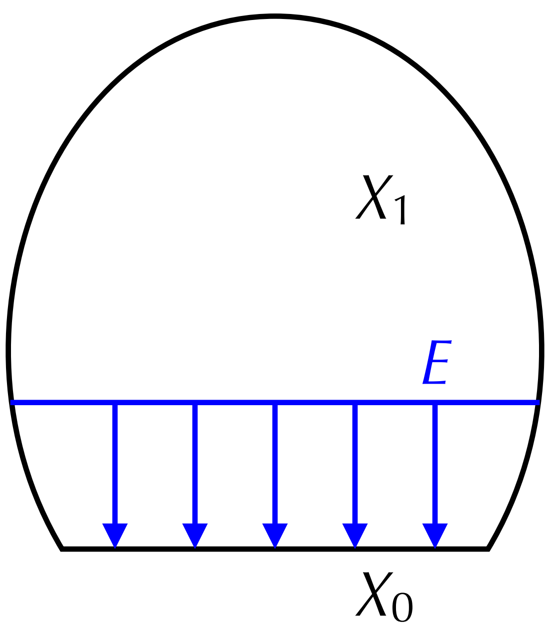
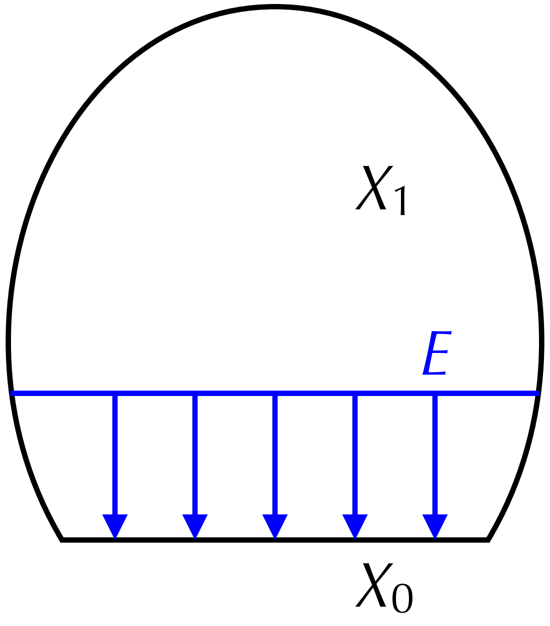
3. Multiaxial Manifold

Stratified Interpretation of Periodicity



Stratified Interpretation of Periodicity

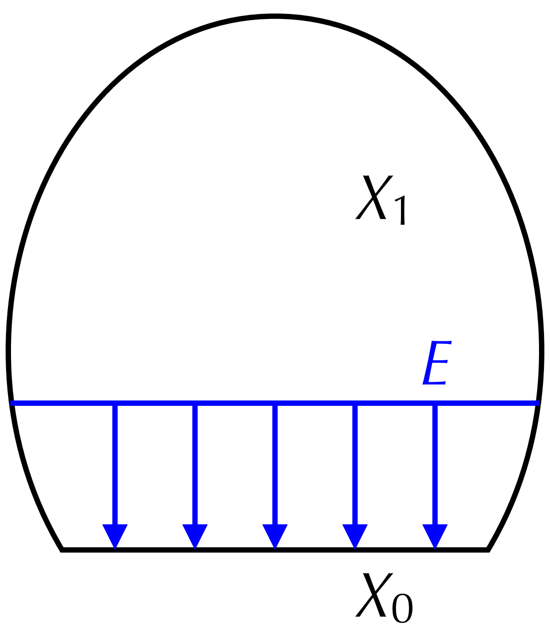
Consider 2-strata space.



Stratified Interpretation of Periodicity

Consider 2-strata space.

Relative surgery obstruction $\boldsymbol{L}(X^1, E)$ [not rel E] for manifold X^1 with $\partial X^1 = E$.



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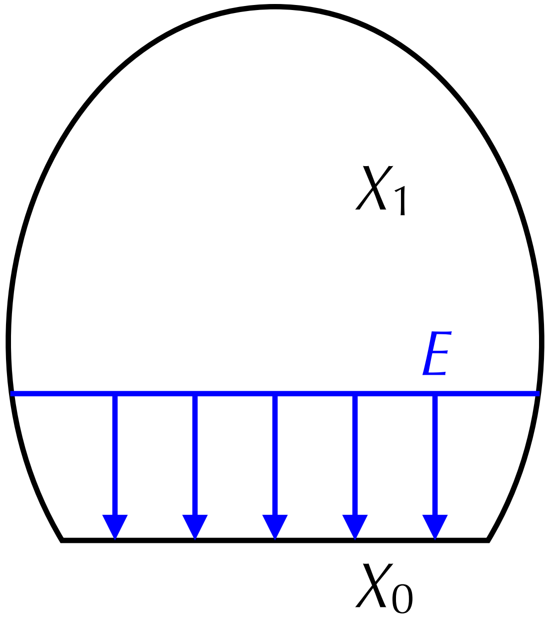
$E = X^0 \times (\mathbf{CP}^{\text{even}}, \mathbf{HP}^n, \ldots)$, or more generally bundle $E \to X^0$ with signature 1 fibre

$\Rightarrow$ by periodicity, $\times$(signature 1): $\boldsymbol{L}(X^0) \to \boldsymbol{L}(E)$ is isomorphism



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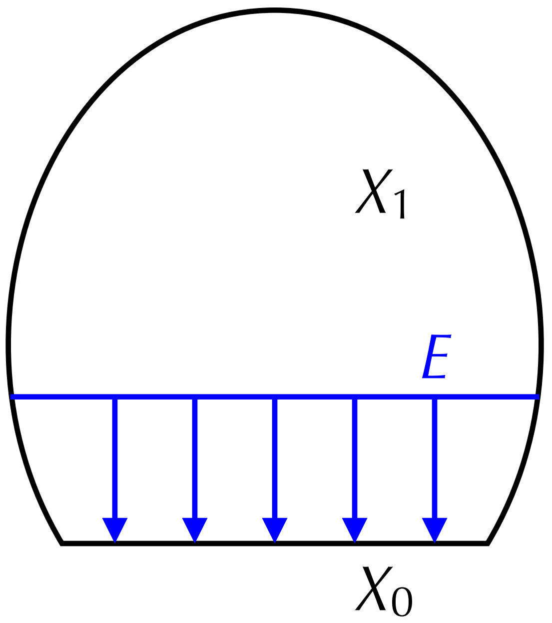

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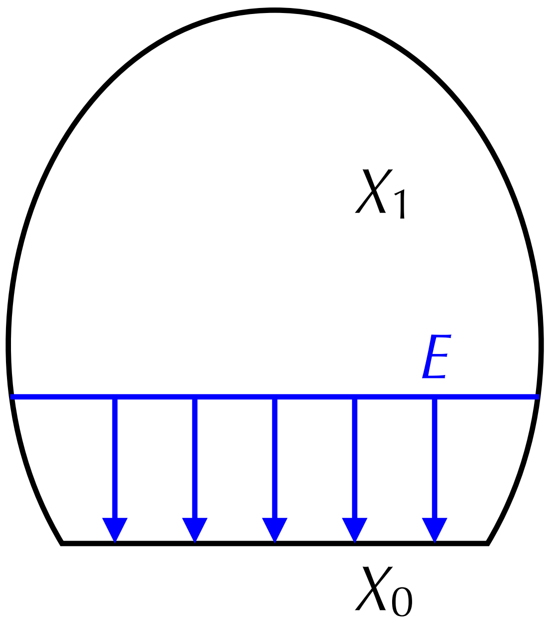
This happens when X is the orbit space of semi-free circle action on manifold M, such that $\dim M^{\mathbf{S}^1} = \dim M + 2$ (4).



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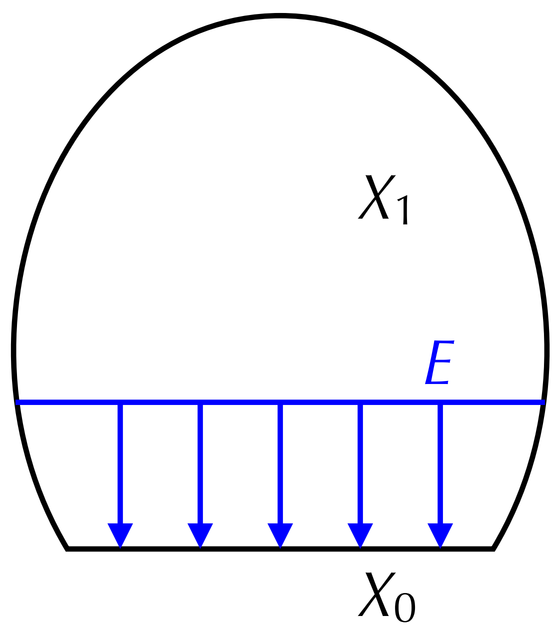

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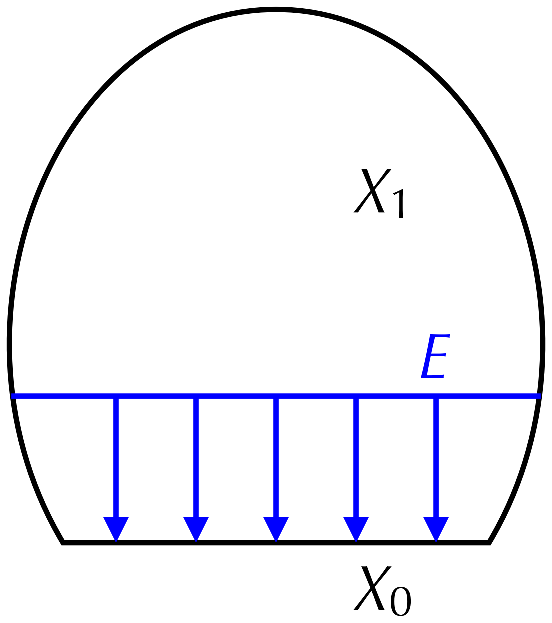

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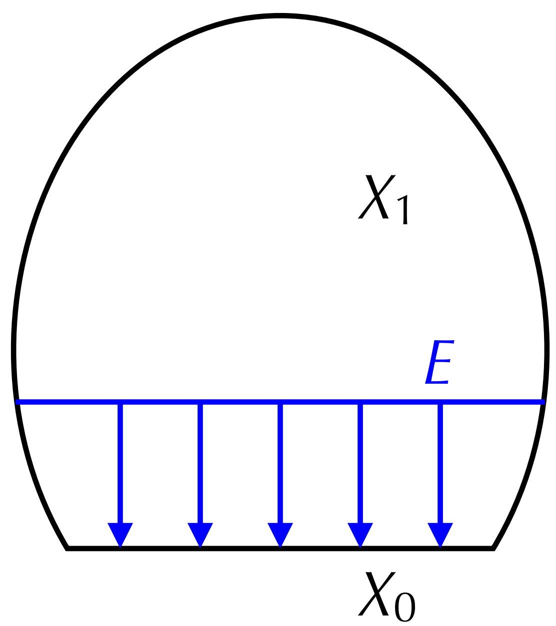

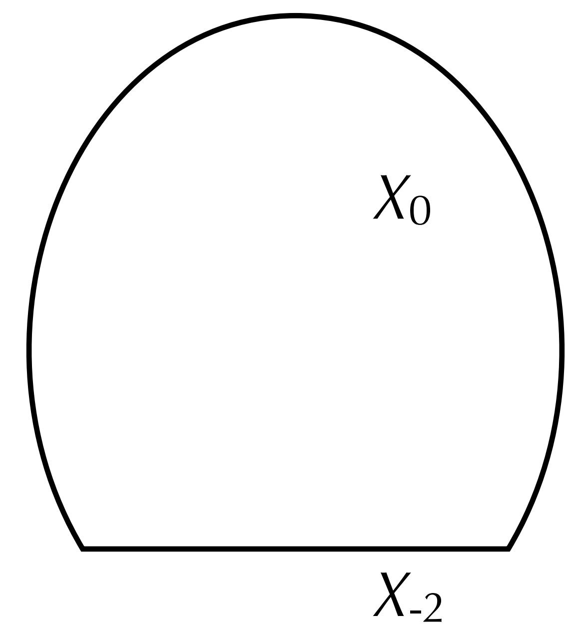

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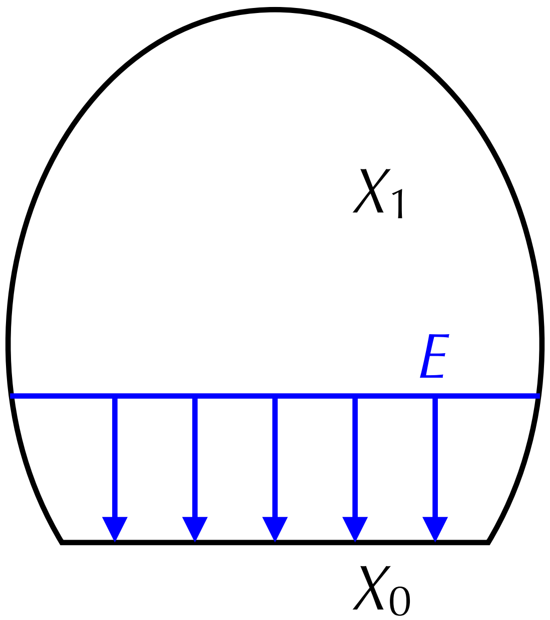

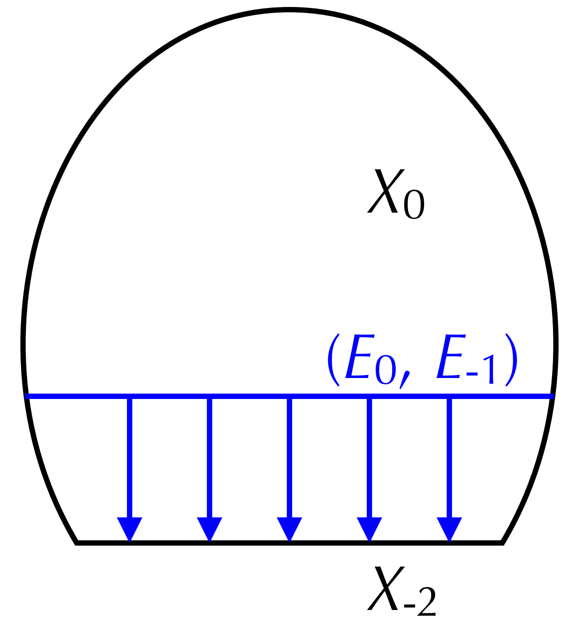

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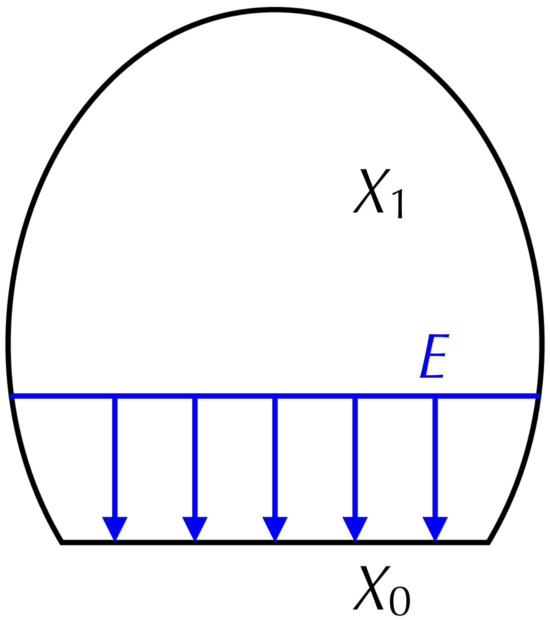

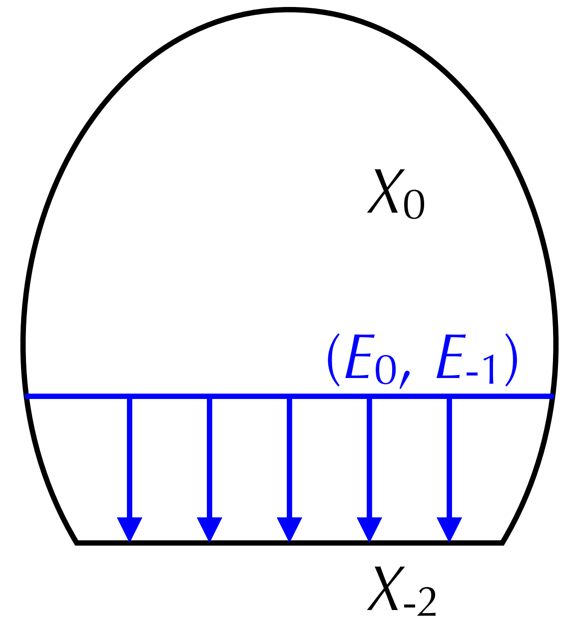

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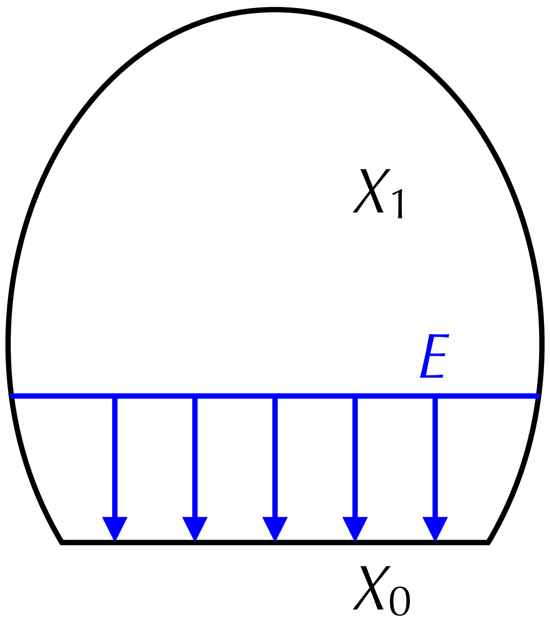

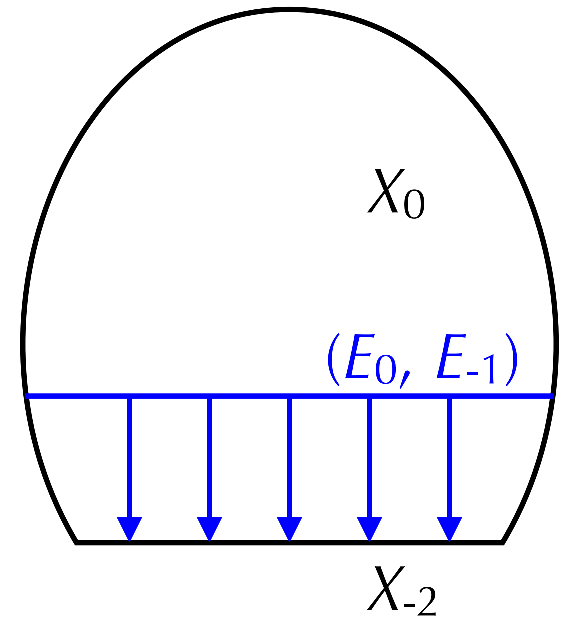

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$G_i = U(i)$, $N_{U(i+1)}U(i)/U(i)$ = circle, get adjacent links $\mathbf{CP}^r$, $\mathbf{CP}^{r+1}$, $\mathbf{CP}^{r+2}$, …

So splitting happens to half of adjacent links (half have signature 1).

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Consider $X = M/G$ and $X_{-i} = GM^{G_i}/G_i$, for increasing G_i. If the "difference" between G_i and G_{i+1} is always a circle, then the adjacent links are always $\mathbf{CP}^{\#}$.

Definition: A $U(n)$-manifold is multiaxial, if any isotropy group is conjugate to a unitary subgroup $U(i)$ [plus some "locally flat" condition].

$G_i = U(i)$, $N_{U(i+1)}U(i)/U(i)$ = circle, get adjacent links $\mathbf{CP}^r$, $\mathbf{CP}^{r+1}$, $\mathbf{CP}^{r+2}$, …

So splitting happens to half of adjacent links (half have signature 1).

Example: manifold modeled on $U(n)$-representation $k\rho_n \oplus j\boldsymbol{\varepsilon} = (\mathbf{C}^n)^k \oplus \mathbf{R}^j$.

Multiaxial $U(n)$-manifold

Multiaxial $U(n)$-manifold

Theorem [S. Cappell, S. Weinberger, M. Yan 2015]

The structure set of the unit sphere $S_{U(n)}(\ \mathbf{S}(k\rho_n \oplus j\varepsilon)\ ) = \mathbf{Z}^A \oplus \mathbf{Z}_2{}^{B.}$

$A = \Sigma_{0\leq 2i<n} A_{n-2i,k}$ ($k-n$ even), and $A = A_{n,k\text{-}1} + \Sigma_{0\leq 2i\text{-}1<n} A_{n\text{-}2i+1,k}$ ($k-n$ odd),

$A_{n,k}$ is the number of Schubert cells of dim 0(4) in Grassmannian G(n,k),

B is the similar number of dim 2(4).

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This generalizes the classical result $S(\mathbf{C}P^n) = \mathbf{Z}^{n/2} \oplus (\mathbf{Z}/2\mathbf{Z})^{n/2}$. More results:

- Decomposition of the structure set $S_{U(n)}(M)$.
- Homotopy replacement of fixed point $M^{U(i)}$.
- suspension $*\mathbf{S}(\rho_n)$: $S_{U(n)}(\mathbf{S}(k\rho_n \oplus j\varepsilon)) \to S_{U(n)}(\mathbf{S}((k+1)\rho_\mathrm{n} \oplus j\varepsilon))$ is injective.
- similar result for simplecic group $Sp(n)$.

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ongoing: multiaxial $O(n)$-manifold.

Thank You