Analytic torsion and dynamical zeta function on closed locally symmetric spaces

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Notation

- X compact connected oriented manifold without boundary.
- Take $\rho : \pi_1(X) \to U(m)$. Let $F = \widetilde{X} \times_\rho \mathbf{C}^m$ be the associated flat vector bundle.
- $(\Omega^\bullet(X,F), d)$ the de Rham complex with values in F.
- $H^\bullet(X,F)$ the corresponding de Rham cohomology.
- Assumption: $H^\bullet(X,F) = 0$.

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Hodge Laplacian

- g^F: Hermitian metric on $F = \widetilde{X} \times_\rho \mathbf{C}^m$ induced by the canonical Hermitian metric on $\mathbf{C}^m$.
- g^{TX}: Riemannian metric on X.
- d^* formal adjoint of d.
- Hodge Laplacian $\square^X = dd^* + d^*d : \Omega^\bullet(X,F) \circlearrowleft$.
  - Hodge: $H^\bullet(X,F) = \ker \square^X$.
  - $H^\bullet(X,F) = 0 \Longleftrightarrow \square^X$ is invertible.

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Analytic torsion

- Ray-Singer (1971): the analytic torsion is given by

$$T_X(\rho) = \prod_{i=1}^{\dim X} \Big(\underbrace{\det \left(\square^X|_{\Omega^i} \right)}_{\text{regularized det.}} \Big)^{(-1)^i i} \in \mathbf{R}_+^*.$$

- $T_X(\rho)$ is a topological invariant. (ind. of g^{TX})
- Cheeger (1978), Müller (1978):

$$T_X(\rho) = \text{Reidemeister torsion.}$$

- Müller (1992): the case $\rho : \pi_1(X) \to \mathrm{SL}_m(\mathbf{C})$.
- Bismut-Zhang (1992): the case for ρ arbitrary and g^F arbitrary.

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Fried's conjecture

- When $X = \mathbb{S}^1$ and $\rho : n \in \mathbf{Z} \to e^{in\theta} \in U(1)$, then
  - $H^\bullet(X, F) = 0 \iff e^{i\theta} \neq 1$.
  - $T_X(\rho) = \left(1 - e^{i\theta}\right)^{-1} \left(1 - e^{-i\theta}\right)^{-1}$.
  - Milnor's observation: $\log T_X(\rho) = \sum_{n \in \mathbf{Z} \setminus \{0\}} \frac{e^{in\theta}}{|n|}$.
- Fried (1986): hyperbolic manifold.
- Fried's conjecture (1987): for locally homogenous space,

$$\underbrace{\log T_X(\rho)}_{\text{topological invariant}} = \underbrace{\sum_{\text{non trivial closed geodesics on } X}}_{\text{dynamical invariant}} \cdots .$$

- Moscovici-Stanton (1991), S. (2016): X is a closed locally symmetric reductive manifold (non positive curvature).

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The V-invariant of Bismut-Goette

- V-invariant is defined for compact manifolds S equipped with $\mathbb{S}^1$-action.
- V-invariant has a Poincaré-Hopf type formula.

Proposition (Bismut-Goette, 2004)

Let $f : S \to \mathbf{R}$ be an $\mathbb{S}^1$-invariant Morse-Bott function with critical submanifold $B_f \subset S$. Then

$$V(S) = (-1)^{\mathrm{ind}(f)} V(B_f) + \cdots ,$$

where $\mathrm{ind}(f) : B_f \to \mathbf{Z}$ *is the Morse index (locally constant).*

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Analytic torsion as V-invariant

- $LX = C^\infty(\mathbb{S}^1, X)$: free loop space of X. $\mathbb{S}^1 \curvearrowright LX$
- $\Gamma = \pi_1(X)$ and $[\Gamma]$ = freely homotopy space of X. Then

$$LX = \coprod_{[\gamma]\in[\Gamma]} (LX)_{[\gamma]}.$$

- By an argument of path integral (Witten, Atiyah, Bismut ...),

$$\log T_X(\rho) = \sum_{[\gamma]\in[\Gamma]} \mathrm{Tr}\,[\rho(\gamma)]\, V\big((LX)_{[\gamma]}\big).$$

- Assume X is of non positive curvature. $E(x_\cdot) = \frac{1}{2}\int_0^1 |\dot{x}_s|^2 ds$ is Morse-Bott on LX, s.t., all the critical points are local minima.
- B_E = {closed geodesics on X} $= \coprod_{[\gamma]\in[\Gamma]} B_{[\gamma]}$.
- Reformulation of the formal Fried conjecture:

$$\log T_X(\rho) = \sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \mathrm{Tr}\,[\rho(\gamma)]\, V\big(B_{[\gamma]}\big).$$

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Reductive group and symmetric space

- G: connected real reductive Lie group. That is $G \subset \mathrm{GL}_N(\mathbf{R})$ s.t., $g \in G \Longrightarrow g^t \in G$.
  - $K = G \cap \mathrm{O}(N)$ maximal compact.
  - Cartan decomposition: $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, and

$$(Y, k) \in \mathfrak{p} \times K \simeq e^Y k \in G.$$

  - For $Y_1, Y_2 \in \mathfrak{g}$, set $B(Y_1, Y_2) = \mathrm{Tr}[Y_1 Y_2]$. Then

$$B|_{\mathfrak{p}} > 0, \qquad B|_{\mathfrak{k}} < 0, \qquad \mathfrak{p} \perp_B \mathfrak{k}.$$

- e.g. $G = \mathrm{SL}_n(\mathbf{R}), \mathrm{SO}^0(n, 1) \ldots$
- $\widetilde{X} = G/K$ symmetric space.
  - $\widetilde{X}$ contractible .
  - $G \to \widetilde{X}$ is a K-principal bundle. $T\widetilde{X} = G \times_K \mathfrak{p}$.
  - $\exists\, g^{T\widetilde{X}}$ of non positive curvature (induced by B).

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The main result of S. 2016

$X = \Gamma\backslash\widetilde{X}$ loc. symmetric space, where $\Gamma \subset G$ is discrete cocompact and torsion free.

- $\pi_1(X) = \Gamma$.
- $B_{[\gamma]}$ is a compact manifold. (loc. symmetric)
- the elements in $B_{[\gamma]}$ have the same length $l_{[\gamma]}$.

Theorem (S. 2016)

For $\mathrm{Re}(\sigma) \gg 1$, *we define a Ruelle-type dynamical zeta function by*

$$R(\sigma) = \exp\Big(\sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \mathrm{Tr}[\rho(\gamma)] V(B_{[\gamma]}) e^{-\sigma l_{[\gamma]}} \Big).$$

$R(\sigma)$ *has a mero. extension on* $\mathbf{C}$, *which is holomorphic at* 0, *s.t.,*

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The trace formula: Selberg, Bismut...

- Recall $\Gamma\backslash G \to X = \Gamma\backslash G/K$ is a K-principal bundle. Let $\tau : K \to \mathrm{GL}(E)$ be a rep. of K and let $\mathscr{E} = \Gamma\backslash G \times_K E$.
- $C^{\mathfrak{g}} \in U(\mathfrak{g})$ the Casimir operator. It acts on $C^\infty(X, \mathscr{E} \otimes F)$, which is denoted by $C^{\mathfrak{g},\tau}$.
- Selberg: $\exp(-tC^{\mathfrak{g},\tau})$ is of trace class, such that

$$\mathrm{Tr}\left[\exp(-tC^{\mathfrak{g},\tau})\right] = \sum_{[\gamma]\in[\Gamma]} \mathrm{Tr}[\rho(\gamma)]\mathrm{vol}(B_{[\gamma]})O_{[\gamma]}.$$

- Evaluation of orbital integral $O_{[\gamma]}$: Harish-Chandra, Bismut's explicit formula (2011).
- If $E = \Lambda^{\cdot}(\mathfrak{p}^*)$ and τ is induced by adjoint action, then

$$\mathscr{E} = \Lambda^{\cdot}(T^*X), \qquad C^{\mathfrak{g},\tau} = \square^X.$$

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Proof: the case $\delta(G) \neq 1$

- Set $\delta(G) = \mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)$.
  - $\mathfrak{t} \subset \mathfrak{k}$ Cartan subalgebra. Set

$$\mathfrak{b} = \{Y \in \mathfrak{p} : [Y, \mathfrak{t}] = 0\}.$$

  - $\dim \mathfrak{b} = \delta(G)$.
- If $\delta(G) = 0$, $\nexists F$ with $H^{\bullet}(X, F) = 0$, since

$$\chi(Z, F) = m\chi(Z) = (-1)^{\frac{\dim X}{2}} m \frac{|W_G|}{|W_K|} \frac{\mathrm{vol}(X)}{\mathrm{vol}(\widetilde{X}^d)} \neq 0.$$

- If $\delta(G) \geqslant 2$, then $T_X(\rho) = 1$ and $V(B_{[\gamma]}) = 0$.
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The case: $\delta(G) = 1$

- We have the orthogonal decomposition $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{c} \oplus \mathfrak{d}$ such that

$$\mathfrak{b} \oplus \mathfrak{c} = \{Y \in \mathfrak{p} : [Y, \mathfrak{b}] = 0\}.$$

- Let $K_M \subset K$ be the connected component of the identity of the centralizer of $\mathfrak{b}$ in K.

Proposition (S. 2016)

The actions of K_M on $\mathfrak{c}$ and $\mathfrak{d}$ lift uniquely to elements in the real representation ring $R(K)$ of K.

- We have an identity in $R(K)$,

$$\sum_{i=1}^{\dim \mathfrak{p}} (-1)^i i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{\dim \mathfrak{d}} \underbrace{\sum_{i=0}^{\dim \mathfrak{c}} (-1)^{i+j-1} \Lambda^i(\mathfrak{c}^*) \otimes \Lambda^j(\mathfrak{d}^*)}_{\text{denoted by } \tau_j = \tau_j^+ - \tau_j^- \in R(K)}.$$

- Put $T_j(\sigma) = \det(\sigma + C^{\mathfrak{g}, \tau_j^+}) / \det(\sigma + C^{\mathfrak{g}, \tau_j^-})$.

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Meromorphic extension of $R(\sigma)$

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Proposition (S. 2016)

There exits an odd polynomial $P(\sigma)$ and $\lambda_j \in \mathbf{R}$ such that

$$R(\sigma) = \exp(P(\sigma)) \prod_{j=0}^{\dim \mathfrak{d}} T_j\left((\sigma + \lambda_j)^2 - \lambda_j^2\right)$$

- Since $\sum_i (-1)^i i \Lambda^i(\mathfrak{p}^*) = \sum_j \tau_j$, we have

$$\prod_{i=1}^{\dim X} \det\left(\sigma + \square^X|_{\Omega^i}\right)^{(-1)^i i} = \prod_{j=0}^{\dim \mathfrak{d}} T_j(\sigma).$$

- If every T_j is holomorphic at $\sigma = 0$, then

$$R(0) = \prod_{j=0}^{\dim \mathfrak{d}} T_j(0) = T_X(\rho).$$

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The end of the proof: regularity of T_j at 0

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$T_j(\sigma)$ is holomorphic at $\sigma = 0$.

Proof.

It is enough to show $r_j = \dim\ker(C^{\mathfrak{g},\tau_j^+}) - \dim\ker(C^{\mathfrak{g},\tau_j^-}) = 0$. If

$$L^2(\Gamma\backslash G, F) = \bigoplus_\pi n(\pi)\pi,$$

then $r_j = \sum_{\pi, C^{\mathfrak{g},\pi}=0} n(\pi)\left(\dim(\pi\otimes_{\mathbf{R}}\tau_j^+)^K - \dim(\pi\otimes_{\mathbf{R}}\tau_j^-)^K\right)$.
Using Hecht-Schmid Character formula,

$$r_j = \sum_{\pi,\chi_\pi=0} n(\pi)\left(\dim(\pi\otimes_{\mathbf{R}}\tau_j^+)^K - \dim(\pi\otimes_{\mathbf{R}}\tau_j^-)^K\right).$$

By the classification theory of Vogan-Zuckerman and Salamanca-Riba, if $H^\cdot(X,F)=0$, then $\chi_\pi = 0 \Longrightarrow n(\pi) = 0$. □

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By the classification theory of Vogan-Zuckerman and Salamanca-Riba, if $H^{\cdot}(X,F)=0$, then $\chi_\pi = 0 \Longrightarrow n(\pi)=0$. □

The end of the proof: regularity of T_j at 0

Proposition (S. 2016)

$T_j(\sigma)$ is holomorphic at $\sigma = 0$.

Proof.

It is enough to show $r_j = \dim\ker(C^{\mathfrak{g},\tau_j^+}) - \dim\ker(C^{\mathfrak{g},\tau_j^-}) = 0$. If

$$L^2(\Gamma\backslash G, F) = \bigoplus_{\pi} n(\pi)\pi,$$

then $r_j = \sum_{\pi, C^{\mathfrak{g},\pi}=0} n(\pi)\left(\dim(\pi\otimes_{\mathbf{R}}\tau_j^+)^K - \dim(\pi\otimes_{\mathbf{R}}\tau_j^-)^K\right)$.
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Bibliography

S. Shen, *Analytic torsion, dynamical zeta functions and orbital integrals*, arXiv:1602.00664.
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