An Obata-Lichnerowicz theorem for stratified spaces

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Analysis, Geometry and Topology of Stratified Spaces, CIRM

A classical result

Theorem (Obata-Lichnerowicz)

Let (M^n, g) be a compact Riemannian manifold of dimension n and assume $Ric_g \geq (n-1)g$. Then the first non-zero eigenvalue $\lambda_1(M)$ of the Laplacian Δ_g is larger or equal than n, with equality if and only if (M^n, g) is isometric to the round sphere $(\mathbb{S}^n, can)$.

Question : Is it possible to obtain such a lower bound on the spectrum of the Laplacian on a stratified space ?

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Stratified space

A compact stratified space X is a compact metric space which admits a finite decomposition in strata Σ^j, $j = 0, \dots n$ such that :

$$X = \Omega \sqcup \bigcup_j \Sigma^j$$

- Ω is an open smooth manifold, of dimension n, dense in X ;
- Σ^j is a smooth manifold of dimension j ; $\Sigma^{n-1} = \emptyset$;
- For any $x \in \Sigma^j$ there exists a neighbourhood $\mathcal{U}_x \subset X$ and a homeomorphism

$$\varphi_x : \mathbb{B}^j(\delta_x) \times C_{[0,\delta_x)}(Z_j) \to \mathcal{U}_x.$$

where Z_j is a stratified space, called link of the stratum Σ^j.

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Metrics and angles

A model metric g_0 on X such that :

- g_0 is a Riemannian metric on Ω ;
- for any $x \in \Sigma^j$, the metric on $\mathcal{U}_x$ has the form :

$$g_0 = h + dr^2 + r^2 k_j,$$

where h is a Riemannian metric on $\mathbb{R}^j$, $r \in (0,1)$ and k_j is a model metric on the link Z_j.

For $\mathbf{j = n - 2}$: the link Z_j is a circle $\mathbb{S}^1$ and for α fixed we consider metrics :

$$g_0 = h + dr^2 + (\alpha/2\pi)^2 r^2 d\theta^2,$$

We call α the **angle** of the stratum Σ^{n-2}.

An admissible metric g, satisfying : there exist $\Lambda > 0$, $\gamma > 1$ such that for any $x \in \Sigma^j$ and for any j, the metric g on $\mathcal{U}_x$ satisfies :

$$\|\varphi_x^* g - g_0\|_{g_0} \leq \Lambda r^\gamma, \qquad r \in (0, \delta_x].$$

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Analytic and geometric tools

One can define

- **Sobolev space $\mathbf{W^{1,2}(X)}$** : the closure of the Lipschitz functions on X with the usual norm.
- **Laplacian Δ_g** : the Friedrichs extension of the quadratic form on $C_0^\infty(\Omega)$ given by the Dirichlet energy.

For any point $x \in \Sigma^j$ define :

- the **tangent cone** at x : $C(S_x) = \mathbb{R}^j \times C(Z_j)$.
- the **tangent sphere** at x : $S_x = \left[0, \frac{\pi}{2}\right] \times \mathbb{S}^{j-1} \times Z_j$.

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A lower bound for the curvature

Definition (Ricci lower bound)

We say that (X^n, g) *stratified space has Ricci tensor bounded below by* $(n-1)$ *if :*

- $Ric_g \geq (n-1)g$ *on the regular set* Ω *;*
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Singular Lichnerowicz theorem

Theorem (M., 2014)

Let (X^n, g) be a stratified space with Ricci tensor bounded below by $(n-1)$. Then the first non-zero eigenvalue $\lambda_1(X)$ is larger than or equal to the dimension n.

Remarks :

- If $\alpha > 2\pi$, this theorem does not hold. Counterexample :

$$\mathbb{S}^n_\alpha = [0, \pi/2] \times \mathbb{S}^{n-2} \times \mathbb{S}^1,$$

$$g_\alpha = d\varphi^2 + \cos^2(\varphi) g_{\mathbb{S}^{n-2}} + (\alpha/2\pi)^2 \sin^2(\varphi) d\theta^2.$$

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Obata singular theorem

Theorem (Myers singular theorem, M. 2015)

Let (X^n, g) be a stratified space with Ricci tensor bounded below by $(n-1)$. Then its diameter is less than or equal to π. Moreover, $\lambda_1(X) = n$ if and only if diam$(X) = \pi$.

Theorem (M., 2015)

Let (X^n, g) be a stratified space with Ricci tensor bounded below by $(n-1)$. Then $\lambda_1(X) = n$ if and only if there exists a stratified space (Γ^{n-1}, h) such that (X^n, g) is isometric to the warped product $([-\pi/2, \pi/2] \times \Gamma, dt^2 + \cos^2(t)h)$.

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Outline of proof

$\Leftarrow$ is trivial : for $([-\pi/2, \pi/2] \times \Gamma, dt^2 + \cos^2(t)h)$, we have an explicit eigenfunction relative to n, $\varphi(t) = \sin(t)$.

As for $\Rightarrow$...

Step 0. Induction on the dimension : true for $n = 1$; assume that the result holds for any $k \leq (n-1)$.

Step 1. Locally g is a warped product.

Step 2. On the regular set g is a warped product.

Step 3. Globally g is a warped product.

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Indeed, if $\Delta_g \varphi = n\varphi$ and $\Gamma_0 = \varphi^{-1}(0) \cap \Omega$, for any $x \in \Gamma_0$ there exist neighbourhoods $\mathcal{W}_x \subset \Omega$, $\mathcal{V}_x \subset \Gamma_0$, an interval I_x and $E : I_x \times \mathcal{V}_x \to \mathcal{W}_x$ such that

$$E^* g = dt^2 + \cos^2(t) h, \quad \text{where } h = g|_{\Gamma_0}.$$

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For any $x \in \Gamma_0$ we have $I_x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (use the induction hypothesis on tangent spheres) $\Rightarrow$ Extend E from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \Gamma_0$ to Ω.

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Obata singular theorem

Some remarks :

- In dimension $n = 2$ result by R. Mazzeo, H. Weiss 2015.
- C. Ketterer, 2014 : Obata-Lichnerowicz theorem for metric measure spaces satisfying the curvature dimension condition $RCD^*(K, n)$ (uses N. Gigli's Splitting theorem).

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Some consequences on the Yamabe problem

The Yamabe problem : Given a stratified space (X^n, g) does there exist a metric $\tilde{g} = u^{4/n-2} g$, conformal to g, with constant scalar curvature $S_{\tilde{g}}$?

The Yamabe constant :

$$Y(X, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_X S_{\tilde{g}} dv_{\tilde{g}}}{\mathrm{Vol}_{\tilde{g}}(X)^{\frac{n-2}{n}}}.$$

The Yamabe constant of a ball :

$$Y(B(p,r)) = \inf \left\{ \int_X S_{\tilde{g}}, u \in W_0^{1,2}(B(p,r) \cap \Omega), \|u\|_{\frac{2n}{n-2}} = 1 \right\}.$$

The **local Yamabe constant** :

$$Y_\ell(X) = \inf_{p \in X} \lim_{r \to 0} Y(B(p,r)).$$

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Theorem (K. Akutagawa, G. Carron, R. Mazzeo, 2012)

Let (X^n, g) be a stratified space. If S_g belongs to L^q with $q > \frac{n}{2}$ and if the Yamabe constant $Y(X, [g])$ is ***strictly smaller*** *than the local Yamabe constant $Y_\ell(X)$ then there exists a conformal metric with constant scalar curvature.*

Problem : value of the local Yamabe constant ?

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Problem : value of the local Yamabe constant ?

The singular Lichnerowicz theorem allows one to

- give a lower bound for the Yamabe constant of stratified spaces with Ricci tensor bounded below.

$$Y(X,[g]) \geq \frac{n(n-2)}{4} \mathrm{Vol}_g(X)^{\frac{2}{n}};$$

- compute the **local Yamabe constant** when the links carry an Einstein metric.

For example, for (X^n, g) stratified space with one singular stratum Σ^{n-2} of angle $\alpha < 2\pi$, the local Yamabe constant is :

$$Y_\ell(X) = \left(\frac{\alpha}{2\pi}\right)^{\frac{2}{n}} Y(\mathbb{S}^n, [\mathrm{can}]).$$

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Einstein metrics and the Yamabe problem

Thanks to the Obata-Lichnerowicz singular theorem :

Theorem (M., '15)

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- Lower bound on $Y(X, [g])$ attained if $Ric_g = (n-1)g$.
- $\exists \tilde{g} = u^{4/n-2} g$ non-trivial, $S_{\tilde{g}} = const \Rightarrow \exists \varphi$ s.t. $\Delta_g \varphi = n\varphi$.

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