Alexander-type invariants of hypersurface complements

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- By contrast, Taubes (1992) showed that *every* finitely presented group is π_1 of a compact complex manifold (of $\dim_{\mathbb{C}} = 3$).
- Morgan (1978), Kapovich-Milson (1997), etc. found *infinitely many* non-isomorphic examples of *non-quasiprojective groups*.

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- E.g., many *knot groups* cannot be realized as $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ for a curve $\mathcal{C}$ (to be justified later).
- Slogan: Lots of obstructions on $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ can be derived by using knot theory invariants, e.g., Alexander-type invariants, or L^2/Novikov-type invariants.

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So it is natural to look for invariants of π_1 which are easier to handle and still capture a lot of the topology of the curve.

I. *Alexander-type invariants of plane curve complements*

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- there is a *central extension*:

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus \mathcal{C}) \to \pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) \to 0,$$

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- M is h.e. to a finite CW complex of real dimension 2.
- π is generated by meridian loops about the irreducible components of $\mathcal{C}$.
- $H_1(M) = H_1(\pi) = \mathbb{Z}^r$, for $r = \#$ of irred. components of $\mathcal{C}$.

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- **Slogan**: *Rigidity* properties of $\Delta_{\mathcal{C}}(t)$ impose obstructions on $\pi = \pi_1(M)$.

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- Monodromy theorem: the zeros of $\Delta_x(t)$ are roots of 1.

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Corollary

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ *be an irreducible degree* d *curve with only nodes and cusps as its singularities. If* $d \not\equiv 0 \pmod 6$*, then* $\Delta_{\mathcal{C}}(t) = 1$.

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- However, the class of possible π_1 of plane curve complements includes *braid groups*, or groups of *torus knots* of type (p, q).

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- If *the 6 cusps are on a conic*, then $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ is isomorphic to π_1 of the trefoil knot, and has Alexander polynomial $\Delta_{\mathcal{C}}(t) = t^2 - t + 1$.

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- If *the six cusps are not on a conic*, then $\pi_1(\mathbb{C}^2 - \mathcal{C})$ is abelian, so $\Delta_{\mathcal{C}}(t) = 1$.

(b) Twisted Alexander invariants

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- ported to the study of plane curve complements by Cogolludo-Florens, who found *new examples of Zariski pairs* which can be detected by the *twisted Alexander polynomial*, but which have the same classical Alexander polynomial.

Setup

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Definition (Twisted Alexander modules)

The *i-th twisted Alexander module of (M, ε, ρ)* is:

$$H_i^{\varepsilon,\rho}(M; \mathbb{F}[t^{\pm 1}]) = H_i(M_\varepsilon; \mathbb{V}_\rho) := H_i\left(C_*(M_\varepsilon, \mathbb{V}_\rho)\right),$$

where $C_*(M_\varepsilon, \mathbb{V}_\rho) := \mathbb{V} \otimes_{\mathbb{F}[\bar{\pi}]} C_*(M_\varepsilon)$ is the twisted chain complex of M_ε.

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For any pair (ε, ρ), the twisted Alexander modules $H_i^{\varepsilon,\rho}(M; \mathbb{F}[t^{\pm 1}])$ of $M = \mathbb{C}^2 \setminus \mathcal{C}$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules, for $i = 0, 1$.

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if $\varepsilon = lk$, $\mathbb{V} = \mathbb{C}$ and $\rho =$ trivial, get back the classical Alexander modules $H_i(M^c; \mathbb{C})$ of M. So the above result generalizes the Zariski-Libgober theorem.

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Definition

$\Delta_{\mathcal{C}}^{\varepsilon,\rho}(t) = \text{order} H_1^{\varepsilon,\rho}(M; \mathbb{F}[t^{\pm 1}])$ is *the twisted Alexander polynomial of $(\mathcal{C}, \varepsilon, \rho)$*.

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Theorem (M.-Wong)

Assume $\mathbb{F} = \mathbb{C}$, $\varepsilon = lk$, and $\rho : \pi \to \mathbb{V}$ an arbitrary representation. Let $\lambda_1, \cdots, \lambda_\ell$ be the eigenvalues of $\rho(x_0)^{-1}$. Then the roots of $\Delta_{\mathcal{C}}^{\varepsilon,\rho}(t)$ are contained in the splitting field of $\prod_{i=1}^{\ell}(t^d - \lambda_i)$ over $\mathbb{Q}$, which is cyclotomic over $\mathbb{Q}(\lambda_1, \cdots, \lambda_\ell)$.

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Theorem (Cogolludo-Florens, M.-Wong)

divisibility for twisted Alexander polynomials, relating the local and global ones.

II. *Novikov homology of plane curve complements*

Novikov-Betti and Novikov-torsion numbers

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- the *i-th Novikov-Betti number* $b_i(M,\xi)$ of (M,ξ) is the $\mathbb{Z}[\Gamma_\xi]$-rank of $H_i(M_\xi;\mathbb{Z})$, i.e.,

$$b_i(M,\xi) := \dim_{\mathbb{Q}_\xi} \mathbb{Q}_\xi \otimes_{\mathbb{Z}[\Gamma_\xi]} H_i(M_\xi;\mathbb{Z}) = rk_{R\Gamma_\xi} H_i(M;R\Gamma_\xi),$$

where $\mathbb{Q}_\xi := Frac(\mathbb{Z}[\Gamma_\xi])$, and $R\Gamma_\xi$ is the *rational Novikov ring* of Γ_ξ (a certain PID localization of $\mathbb{Z}[\Gamma_\xi]$).

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Remark

The above result holds more generally, for twisted Novikov-type invariants.

III. *L^2-Betti numbers of plane curve complements*

L^2-Betti numbers

Novikov-Betti numbers are special cases of L^2-Betti numbers, though the torsion-Novikov numbers do not have such an interpretation.

To any CW complex M and group homomorphism $\alpha : \pi_1(X) \to \Gamma$, we associate *$L^2$-Betti numbers*

$$b_i^{(2)}(M, \alpha) := \dim_{\mathcal{N}(\Gamma)} H_i \left(C_*(M_\alpha) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}(\Gamma) \right) \in [0, \infty],$$

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Remark (Friedl-M.)

if $\xi \in H^1(M; \mathbb{R})$, then

$$b_i(M, \xi) = b_i^{(2)}(M, \pi_1(M) \overset{\xi}{\twoheadrightarrow} Im(\xi))$$

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- Consider $b_p^{(2)}(M, \alpha)$ and $b_p^{(2)}(M^c, \bar{\alpha})$.
- A priori, there is no reason to expect $b_1^{(2)}(M^c, \bar{\alpha})$ to be finite (as M^c is an *infinite* CW complex).

Theorem (Friedl-Leidy-M.)

If $\alpha : \pi_1(M) \to \Gamma$ *is admissible, then*

$$b_i^{(2)}(M, \alpha) = \begin{cases} 0, & i \neq 2, \\ \chi(M), & i = 2. \end{cases}$$

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Corollary

$b_i^{(2)}(M, \alpha)$ *($i \geq 0$) depends only on the degree of* $\mathcal{C}$ *and on the local type of singularities, and is independent on* α *and on the position of singularities of* $\mathcal{C}$*. In fact,*

$$b_2^{(2)}(M, \alpha) = (d-1)^2 - \sum_{x \in \mathrm{Sing}(\mathcal{C})} \mu(\mathcal{C}, x).$$

Obstructions on the L^2-Betti numbers of curves

Theorem (Friedl-Leidy-M.)

If $\alpha : \pi_1(M) \to \Gamma$ is admissible, then $b_1^{(2)}(M^c, \bar{\alpha})$ is finite, and an upper bound is determined by the local type of singularities of $\mathcal{C}$:

$$b_1^{(2)}(M^c, \bar{\alpha}) \leq \sum_{x \in \mathrm{Sing}(\mathcal{C})} (\mu(\mathcal{C}, x) + n_x - 1) + 2g + d,$$

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Remark

$b_1^{(2)}(M^c, \bar{\alpha})$ depends in general on the position of singularities of $\mathcal{C}$ (this can be checked on Zariski's example of sextics with 6 cusps).

Consequences of finiteness property

Free groups $\mathbb{F}_m$ with $m \geq 2$ cannot be of the form $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$, for $\mathcal{C}$ a curve in general position at infinity, and similarly for groups of boundary links (i.e., those links whose components admit mutually disjoint Seifert surfaces).

Concluding remarks

All invariants of plane curve complements discussed today are dominated by the corresponding invariants of the link of $\mathcal{C}$ at infinity (i.e., Hopf link on d components) and, resp., by those of the boundary manifold of $\mathcal{C}$.

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THANK YOU !!!