Fourier Integral Operators on Lie Groupoids

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Lie groupoids

A Lie groupoid is a pair of manifolds $(G, G^{(0)})$ (arrows, units) with:

$$G \overset{s,r}{\rightrightarrows} G^{(0)} \quad \text{(source, range)}; \quad \upsilon : G^{(0)} \longrightarrow G \quad \text{(inclusion of units)};$$

$$m : G^{(2)} := G \underset{s,r}{\times} G \longrightarrow G \quad \text{(multiplication)}; \quad \iota : G \longrightarrow G \quad \text{(inversion)};$$

- s, r, m submersions; all maps C^∞;
- $r(\upsilon(x)) = s(\upsilon(x)) = x$;
- $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$;
- $r(\gamma)\gamma = \gamma; \quad \gamma s(\gamma) = \gamma$;
- $r(\gamma^{-1}) = s(\gamma); \quad s(\gamma^{-1}) = r(\gamma)$;
- $r(\gamma_1\gamma_2) = r(\gamma_1); \quad s(\gamma_1\gamma_2) = s(\gamma_2)$;
- $\gamma\gamma^{-1} = r(\gamma); \quad \gamma^{-1}\gamma = s(\gamma)$.

(Consequences: $\iota^{-1} = \iota$, υ is an embedding)

Simplifying assumption : $G^{(0)}$ is compact.

Examples

- Lie groups, bundles of Lie groups are Lie groupoids.
- $X \times X \rightrightarrows X$, with $(x,y).(y,z) = (x,z)$, etc. . . .
- Let $\pi : H \to S$ be a submersion. This gives: $H \underset{S}{\times} H \rightrightarrows H$.
- G a Lie group acting on X. Transformation gpd: $G \times X \rightrightarrows X$ with

  $$s(g,x) = x,\ r(g,x) = g.x,\ (g,hx).(h,x) = (gh,x),\ \iota(g,x) = (g^{-1}, gx).$$

- X a (cpct) mfd with boundary H and $\pi : H \to S$ a submersion.

  $$G_\pi := \overset{\circ}{X} \times \overset{\circ}{X} \cup \widetilde{\pi}^*(TS \times \mathbb{R}) \rightrightarrows X,$$

  where $\widetilde{\pi} : H \underset{S}{\times} H \to S$. This arises as a sub-mfd of a suitable blow-up (twice) of X^2.
  Following these lines, one can define similar blow-up spaces and Lie groupoids associated with any mfd with iterated fibred corners (Debord-Rochon-L.), a companion category to stratified spaces.
  We will go back to this example (depth 1 only) later.

Convolution

$$f * g(\gamma) = \int_{m^{-1}(\gamma)} f(\gamma_1)g(\gamma_2) = m_*(f \otimes g|_{G^{(2)}}), \quad f, g \in C_c^\infty(G, \Omega^{1/2})$$

G-operator

Any continuous linear map $P : C_c^\infty(G) \to C^\infty(G)$ such that

$$P(f * g) = P(f) * g \qquad \text{for any } f, g \in C_c^\infty.$$

A G-op P has an adjoint if there exists a G-op Q such that

$$P(f)^\star * g = f^\star * Q(g)\ ; \quad \forall f, g \in C_c^\infty.$$

Questions

- Can one define convolution of distributions on G ?
- Relationship between G-ops and convolution ops. by distributions ?

Transversal distributions

Consider the submersion $s : G \to G^{(0)}$ and set:

$$\mathcal{D}'_s(G) = \{u \in \mathcal{D}'(G) \ ; \ \forall f \in C_c^\infty(G), \ \ s_*(u.f) \in C^\infty(G^{(0)})\}.$$

Thm (Schwartz Kernel Thms for groupoids):

$$\mathcal{D}'(G) \simeq \mathcal{L}_{C^\infty(M)}(C_c^\infty(G), \mathcal{D}'(M)),$$

$$\mathcal{D}'_s(G) \simeq \mathcal{L}_{C^\infty(M)}(C_c^\infty(G), C^\infty(M)) \simeq C_s^\infty(M, \mathcal{D}'(G)).$$

Here $M = G^{(0)}$ and densities are hidden, similar statement for $\mathcal{D}'_r$.

Theorem (LMV)

Convolution of functions extends to:

$$\mathcal{D}'_s(G) \times \mathcal{E}'_{(s)}(G) \xrightarrow{\ *\ } \mathcal{D}'_{(s)}(G); \qquad \mathcal{D}'_r(G) \times C_c^\infty(G) \xrightarrow{\ *\ } C^\infty(G).$$

In particular $\mathcal{E}'_s(G)$ is an algebra with unit: $\langle \delta, f \rangle = \int_{G^{(0)}} f$.
$\mathcal{E}'_{r,s}(G)$ is a unital subalgebra with involution: $u^\star = \overline{\iota^*(u)}$.

G-ops and convolution

We can make precise the statement "G-ops are convolution operators":

Theorem (LMV)

The map $u \mapsto u * \cdot$ gives

$$\mathcal{D}'_r(G) \simeq \mathrm{Op}_G \qquad \text{(space of G-operators)}$$

$$\mathcal{D}'_{r,s}(G) \simeq \mathrm{Op}^*_G \qquad \text{(subspace of G-operators with adjoints)}$$

Question

How to compute the Wave Front set of a convolution product ?

It brings in the cotangent symplectic groupoid T^*G of Coste-Dazord-Weinstein (CDW)

CDW groupoid

Lie Algebroid of G: $AG = T_{G^{(0)}}G/TG^{(0)} \longrightarrow G^{(0)}$.
Also

$$AG \simeq \ker ds|_{G^{(0)}} \simeq \ker dr|_{G^{(0)}}$$

Dual Lie Algebroid of G: $A^*G = N^*G^{(0)} \subset T^*G$.

Differentiating all structure maps of a Lie gpd G produces another Lie groupoid: $TG \rightrightarrows TG^{(0)}$.

Transposing everything in TG leads to a gpd structure on the cotangent space T^*G with unit space A^*G.

CDW groupoid

The cotangent groupoid $\Gamma = (T^*G \rightrightarrows A^*G)$ is given as follows.

$$\text{(Source)} \quad s_\Gamma(\gamma,\xi) = (s(\gamma), L_\gamma^*(\xi)) \in A^*_{s(\gamma)}G,$$
$$\text{(Range)} \quad r_\Gamma(\gamma,\xi) = (r(\gamma), R_\gamma^*(\xi)) \in A^*_{r(\gamma)}G,$$

ie, for $s_\Gamma(\gamma,\xi)$, you restrict the linear form ξ to the subspace $T_\gamma G^{r(\gamma)}$ and then transport it over $x = s(\gamma)$ (with the only natural operation available : the co-differential at x of left multiplication $L_\gamma : G^x \to G^{r(\gamma)}$). The result is in a canonical way a linear form on T_xG vanishing on $T_xG^{(0)}$, thus an element of A^*_xG.

When $s_\Gamma(\gamma_1,\xi_1) = r_\Gamma(\gamma_2,\xi_2) \in A^*G$ then

$$\text{(product)} \quad m_\Gamma(\gamma_1,\xi_1,\gamma_2,\xi_2) = (\gamma,\xi) \in T^*G$$

where

$$\gamma = \gamma_1\gamma_2 \quad \text{and} \quad \xi = ({}^t dm_{(\gamma_1,\gamma_2)})^{-1}(\xi_1,\xi_2).$$

Finally,

$$\text{(Inversion)} \quad (\gamma,\xi)^{-1} = (\gamma^{-1}, -{}^t(d\iota_\gamma)(\xi)).$$

CDW groupoid

- All structure maps of $T^*G \rightrightarrows A^*G$ are linear.
- $\Gamma = (T^*G, \omega)$ is a symplectic groupoid, which means that

$$\mathrm{Graph}(m_\Gamma) = \{(\delta, \delta_1, \delta_2) \in \Gamma^3 \; ; \; \delta = \delta_1\delta_2\}$$

is a Lagrangian submanifold of $(-\Gamma) \times \Gamma \times \Gamma$.

Follows from

$$\mathrm{Graph}(m_\Gamma) = \phi(N^*\mathrm{Graph}(m))$$

where $\phi = (-\mathrm{Id}, \mathrm{Id}, \mathrm{Id}) : \Gamma^3 \xrightarrow{\simeq} (-\Gamma) \times \Gamma \times \Gamma$.

Let us give two basic examples.

T^*G: Examples

If G is a Lie group then $A^*G = \mathfrak{g}^*$.
G acts on $\mathfrak{g}^*$ on the right by:

$$\mathrm{Ad}_g^*.\xi = L_g^* R_{g^{-1}}^* \xi.$$

This gives a Lie groupoid $\mathfrak{g}^* \rtimes G \rightrightarrows \mathfrak{g}^*$ (transformation gpd).
Then the map

$$\begin{aligned} \Phi : T^*G &\longrightarrow \mathfrak{g}^* \rtimes G \\ (g, \xi) &\longmapsto (g, R_g^* \xi) \end{aligned}$$

is a Lie groupoid isomorphism.

T^*G: Examples

If $G = X \times X \times Z \rightrightarrows X \times Z$, then

$$\begin{aligned}\Gamma^{(0)} &= A^*G = \{(x,x,z,\xi,-\xi,0)\ ;\ (x,\xi) \in T^*X, z \in Z\}\\ &\simeq N^*(\Delta_X) \times Z\\ &\simeq T^*X \times Z.\end{aligned}$$

We get

$$s_\Gamma(x,y,z,\xi,\eta,\sigma) = (y,-\eta,z)\ ;\ r_\Gamma(x,y,z,\xi,\eta,\sigma) = (x,\xi,z)$$

and

$$(x,y,z,\xi,\eta,\sigma).(y,x',z,-\eta,\xi',\sigma') = (x,x',z,\xi,\xi',\sigma+\sigma'),$$

$$(x,y,z,\xi,\eta,\sigma)^{-1} = (y,x,z,-\eta,-\xi,-\sigma).$$

Convolution, Wave Front and T^*G

Some notations:

$$\overset{\circ}{\Gamma} := r_\Gamma^{-1}(A^*G \setminus 0) \cap s_\Gamma^{-1}(A^*G \setminus 0) \subset T^*G$$ (admissible or no-zeros sub-gpd),

$$\mathcal{D}'_a(G) := \{u \in \mathcal{D}'(G) \; ; \; \mathrm{WF}(u) \subset \overset{\circ}{\Gamma}\}$$ (admissible distributions).

Easy facts: $\Psi(G) = I(G, G^{(0)}) \subset \mathcal{D}'_a(G) \subset \mathcal{D}'_{s,r}(G)$.

Theorem (LMV)

Let $u_j \in \mathcal{E}'_a(G), j = 1, 2$. Then: $\mathrm{WF}(u_1 * u_2) \subset \mathrm{WF}(u_1) \, . \, \mathrm{WF}(u_2)$.

Convolution is permitted under the weaker assumption:

$$\mathrm{WF}(u_1) \times \mathrm{WF}(u_2) \cap \ker m_\Gamma = \emptyset,$$

and then $\mathrm{WF}(u_1 * u_2) \subset m_\Gamma((\mathrm{WF}(u_1) \cup 0) \times (\mathrm{WF}(u_2) \cup 0) \setminus 0 \times 0)$.

A calculus for Lagrangians in T^*G

Definition: G-relations (a replacement of *canonical relations*)

Any conic Lagrangian submanifold of $\Gamma = T^*G$ contained in $\mathring{\Gamma}$.

If $G = X \times X$, these are the conic Lagrangian submanifolds of $T^*(X \times X)$ contained in $\subset T^*X \setminus 0 \times T^*X \setminus 0$.

Theorem (LV)

1. Let Λ_1, Λ_2 be two G-relations.
   If $\Lambda_1 \times \Lambda_2$ and $\Gamma^{(2)}$ intersect cleanly (cleanly composable), then

   $$\Lambda_1.\Lambda_2 \subset \Gamma$$

   is a local (= immersed) G-relation.
2. Let Λ be a G-relation. Then

   $$\Lambda^\star := \iota_\Gamma(\Lambda)$$

   is a G-relation.

A calculus for Lagrangians in T^*G

Theorem (LV)

Let Λ be a G-relation. The following conditions are equivalent:

1. There exists a G-relation Λ' cleanly composable with Λ such that
$$\Lambda.\Lambda' = r_\Gamma(\Lambda) \quad \text{and} \quad \Lambda'.\Lambda = s_\Gamma(\Lambda).$$
2. Λ is a bissection, that is, the maps
$$r_\Gamma : \Lambda \longrightarrow \Gamma^{(0)} \text{ and } s_\Gamma : \Lambda \longrightarrow \Gamma^{(0)}$$
are diffeomorphisms onto their images.
3. Λ and $\Lambda^\star$ are transversally composable, that is $\Lambda \times \Lambda^\star \pitchfork \Gamma^{(2)}$, and
$$\Lambda.\Lambda^\star = r_\Gamma(\Lambda) \quad \text{and} \quad \Lambda^\star.\Lambda = s_\Gamma(\Lambda).$$

In that case, we say that Λ is invertible.

Reminder: Lagrangian distributions

Let

- X be a C^∞ manifold of dimension n,
- Λ be a conic Lagrangian submanifold of $T^*X \setminus 0$.

The set $I^m(X, \Lambda)$, $m \in \mathbb{R}$, consists of distributions $u \in \mathcal{D}'(X)$ of the form:

$$u = \sum_{j \in J} \int e^{i\phi_j(x,\theta_j)} a_j(x, \theta_j) d\theta_j \mod C^\infty(X)$$

where for all j,

- $(x, \theta_j) \in \mathcal{V}_j \subset U_j \times \mathbb{R}^{N_j}$ (here U_j a coordinate patch and $\mathcal{V}_j$ an open cone);
- $\phi_j : \mathcal{V}_j \to \mathbb{R}$ is a non-degenerate phase function parametrizing Λ;
- $a_j(x, \theta_j) \in S^{m+(n_X - 2N_j)/4}(U_j \times \mathbb{R}^{N_j})$ and $\mathrm{supp}(a_j) \subset \mathcal{V}_j \setminus 0$.

Such distributions are called *Lagrangian distributions subordinated to* Λ.

G-FIOs: definition, composition

Definition

G-FIO are the Lagrangian distributions on G subordinated to G-relations.

Theorem (LV)

1. If Λ is a G-relation and $A \in I^m(G,\Lambda)$, then $A^\star \in I^m(G,\Lambda^\star)$.
2. If Λ_1, Λ_2 are closed G-relations, cleanly composable with excess e and $A_1 \in I_c^{m_1}(G,\Lambda_1)$, $A_2 \in I^{m_2}(G,\Lambda_2)$, then

$$A_1 * A_2 \in I^{m_1+m_2+e/2-(n-2n^{(0)})/4}(G,\Lambda_1.\Lambda_2).$$

Here n is the dimension of G and $n^{(0)}$ is the dimension of $G^{(0)}$.

- Observation : Λ being a G-relation, G-FIO are adjointable G-operators.
- Convention : $\Psi^m(G) := I^{m+(n-2n^{(0)})/4}(G, A^*G)$.

Principal symbols

Remember that densities have been hidden:

$$I(G,\Lambda) \subset \mathcal{D}'(G,\Omega^{1/2}) \quad \text{and } \Omega^{1/2} = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr).$$

It yields: $\sigma : I^m(G,\Lambda) \longrightarrow S^{[m+n/4]}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes \Omega^{1/2}(\ker ds_\Gamma))$.

When $\Lambda = A^*G$, the Maslov bundle M_Λ is trivial and

$$\Omega^{1/2}_{A^*G} \otimes \Omega^{1/2}(\ker ds_\Gamma) = (\Omega^{1/2}_{T^*G})|_{A^*G} \simeq A^*G \times \mathbb{C}.$$

The last trivialization decreases by $(n-n^{(0)})/2$ the degree of symbols, thus :

$$\sigma : \Psi^m(G) = I^{m+(n-2n^{(0)})/4}(G, A^*G) \longrightarrow S^{[m]}(A^*G).$$

If a_1, a_2 and a are principal symbols of A_1, A_2 and $A_1.A_2$ then

$$a(\delta) = \int_{\substack{\delta_1\delta_2=\delta,\\ \delta_j\in\Lambda_j}} a_1(\delta_1)a_2(\delta_2).$$

First consequences

Corollary : Module structure, Egorov thm

1. Any G-relation Λ is transversally composable with A^*G, and:

$$\Psi_c^*(G) * I(G,\Lambda) \subset I(G,\Lambda)$$

2. Assume that two composable G-relations Λ_1, Λ_2 satisfy $\Lambda_1.\Lambda_2 \subset A^*G$. Then

$$I_c(G,\Lambda_1) * \Psi(G) * I_c(G,\Lambda_2) \subset \Psi(G).$$

Corollary: C^*-continuity

Let Λ be an invertible closed G-rel. and $A \in I_c^{(n-2n^{(0)})/4}(G,\Lambda)$. Then

$$A \in \mathcal{M}(C^*(G)).$$

If $A \in I_c^m(G,\Lambda)$ with $m < (n-2n^{(0)})/4$ then $A \in C^*(G)$.

(Hint: if $A \in I_c^{(n-2n^{(0)})/4}(G,\Lambda)$ with Λ invertible, then $A^*A \in I_c^{(n-2n^{(0)})/4}(G, A^*G) = \Psi_c^0(G)$.)

Representations of G-FIOs

Remind: P is a G-op iff
P is a C^∞ equivariant family $P_x \in \mathcal{L}(C_c^\infty(G_x), C^\infty(G_x))$, $x \in G^{(0)}$.

(Which means that

1. $\forall x \in G^{(0)},\ \forall f \in C_c^\infty(G),\ f|_{G_x} = 0 \Rightarrow P(f)|_{G_x} = 0.$
2. $\forall \gamma \in G,\ \forall f \in C_c^\infty(G), \quad R_\gamma^* P_{s(\gamma)}(f) = P_{r(\gamma)}(R_\gamma^* f).$)

Also, if P is a (compactly supported) G-op, then the formula

$$r_\#(P)(f)(x) = P(r^* f)(x), \qquad f \in C^\infty(G^{(0)}), \quad x \in G^{(0)}$$

defines an operator $r_\#(P) \in \mathcal{L}(C^\infty(G^{(0)}))$.

Questions

When P is a G-FIO, nature of P_x ? nature of $r_\#(P)$?

Consider :

- The orbits in $G^{(0)}$: $\quad O_x = r(s^{-1}(x)), x \in G^{(0)}$,
- The orbits in G: $\quad L = G_O = r^{-1}(O), O \in \{\text{orbits of } G^{(0)}\}$.

(Orbits are immersed sub-mfds. Orbits in G are saturated sub-gpds.)

Representations of G-FIOs

G-FIOs represented in the fibers

Assume that Λ is a family G-relation, that is:

$$T^*_U G \pitchfork \Lambda \qquad \text{for any orbit } U \text{ in } G \qquad \text{(family G-relation)}.$$

Then Λ produces, by functorial operations, an equivariant family of canonical relations (*) $\Lambda_x \subset T^*G_x \times T^*G_x$ and

$$P \in I^m(G, \Lambda) \Rightarrow P_x \in I^{m-(n-2n^{(0)})/4}(G_x \times G_x, \Lambda_x), \quad \forall x.$$

Remarks:

- If Λ fails to be a family G-relation, then the P_x are still given by oscillatory integrals, the phases being possibly degenerated.
- If Λ is invertible, then it is a family G-relation.
- (*) means
    - $c^*_\gamma(\Lambda_y) = \Lambda_x$, for any $\gamma \in G^x_y$, where $c_\gamma : G_x \times G_x \longrightarrow G_y \times G_y$.
    - Setting $\pi : G \underset{s}{\times} G \to G^{(0)}, (\gamma_1, \gamma_2) \mapsto s(\gamma_1)$,
        - $\mathcal{L} = \cup_{x \in G^{(0)}} \Lambda_x \subset (\ker d\pi)^*$ is a C^∞ submanifold,
        - $\mathcal{L} \pitchfork \widetilde{\pi}$, where $\widetilde{\pi} : (\ker d\pi)^* \to G^{(0)}$ is the natural extension of π.

Representations of G-FIOs

There are converse statements:

Theorem

Let $(\Lambda_x)_{x\in G^{(0)}}$ be an equivariant C^∞ family of Lagrangians $\subset T^*G_x \setminus 0 \times T^*G_x \setminus 0$. Then there exists a unique (family) G-relation Λ "gluing" the family in the sense that

$$d_x^*(\Lambda) = \Lambda_x \quad \forall x \in G^{(0)}.$$

Here d_x is the map: $(\gamma_1, \gamma_2) \to \gamma_1\gamma_2^{-1}$.

Proposition

G-FFIOs are in one-to-one correspondence with G-op P such that for all x, the operator P_x is a FIO on G_x.

G-FFIO = G-FIO associated with a family G-rel.

Representations of G-FIOs

Now, consider $r_{\#}P \in \mathcal{L}(C^{\infty}(G^{(0)})$:

$$r_{\#}P(f)(x) = P(f \circ r)(x).$$

We take $P \in I_c(G, \Lambda)$.

Firstly, $r_{\#}(P)$ extends to $\mathcal{D}'(G^{(0)})$:

Indeed: $\forall u \in \mathcal{D}'(G^{(0)}), \mathrm{WF}(r^*u) \subset \ker s_{\Gamma}$. By assumption on Λ, we have $\Lambda \times \ker s_{\Gamma} \cap \ker m_{\Gamma} = \emptyset$ so that the convolution $P * (r^*u)$ is permitted and

$$WF(P * (r^*u)) \subset (\Lambda \cup 0). \ker s_{\Gamma} \subset \ker s_{\Gamma}.$$

Since $\ker s_{\Gamma} \cap A^*G = 0$, the following restriction is well defined:

$$r_{\#}P(u) = \upsilon^*(P * (r^*u)) \in \mathcal{D}'(G^{(0)}).$$

Representations of G-FIOs

Secondly, assume that the G-relation Λ of P satisfies

$$\Lambda \pitchfork \ker s_\Gamma + \ker r_\Gamma .$$

Then for any orbit O and $x \in O$, one obtains from Λ and functorial operations a Lagrangian

$$\Lambda_x \subset T^*O.$$

Gluing these Lagrangian produces a canonical relation $\mathcal{I} = \mathcal{I}_{x \in O}(\Lambda_x) \subset T^*O \times O$ and

$$r_{\#,O}P \in I^{m-(n-2n^{(0)})/4}(O \times O, \mathcal{I}).$$

Here $r_{\#,O}P \in \mathcal{L}(C^\infty(O))$ is obtained from $r_\# P$ in the obvious way.

Remarks:

- Without the extra transversality (or any weaker) assumption on Λ, $r_{\#,O}P$ is still given by oscillatory integrals, but the phases can be degenerated.
- invertible G relations satisfy the condition above.

Illustration: manifold with fibred boundary

Model case only: $X = [0,\infty) \times H$, $H = \mathbb{R}^k \times \mathbb{R}^{n-1-k}$, $\pi = p_1$. Recall

$$G = \mathring{X} \times \mathring{X} \cup \mathbb{R} \times TS \times Z \times Z = \mathring{G} \cup \partial G.$$

Write $m = (x, y, z) \in X$. Then the bijection $\psi : [0,\infty)_x \times \partial G \to G$ defined by:

$$\psi(x, t, y, v, z_1, z_2) = \begin{cases} (0, t, y, v, z_1, z_2) \in \partial G & \text{if } x = 0 \\ (x + x^2 t, y + xv, z_1, x, y, z_2) \in \mathring{X} \times \mathring{X} & \text{if } x > 0. \end{cases}$$

provides a C^∞ structure to G. In these coordinates, the Lie algebroid, at a unit point $m \in X$ is the linear span of $\partial_t, \partial_v, \partial_{z_1}$ and the anchor map

$$a = dr : AG \longrightarrow TX$$

sends them, respectively, to $x^2\partial_x$, $x\partial_y$, ∂_z. It induces an injective map

$$r_\# : \Gamma(AG) \longrightarrow \Gamma(TX)$$

with image $\mathcal{V}_\pi(X) = \{\chi \in \mathcal{V}_b(X),\ |\ \chi|_{\partial X} \in \ker d\pi \text{ and } \chi.x \in x^2 C^\infty(X)\}$.

Illustration: manifold with fibred boundary

In particular:

$$AG \simeq {}^{\pi}TX \quad \text{and} \quad A^*G \simeq {}^{\pi}T^*X.$$

This gives the compactly supported part of the Φ-pseudodifferential calculus (Mazzeo-Melrose) on X:

$$r_{\#} : I_c(G, A^*G) = \Psi_c(G) \longrightarrow \Psi_{\Phi}(X).$$

Let Λ be a conic Lagrangian submanifold of $T^*G \setminus 0$. Then Λ satisfy the no-zeros condition for Lagrangians if

- $\Lambda|_{\overset{\circ}{G}} \subset (T^*\overset{\circ}{X} \setminus 0) \times (T^*\overset{\circ}{X} \setminus 0)$.
- $\Lambda|_{\partial G}$ avoids
  - $\ker s_{\Gamma}$, which is characterized by vanishing coordinates on dt, dv, dz_2.
  - $\ker r_{\Gamma}$, which is characterized by vanishing coordinates on dt, dv, dz_1.

Asuming this holds for Λ, let us have a look to the extra transversality assumptions.

Illustration: manifold with fibred boundary

Recall that Λ is a family G-relation iff $T^*_F G \pitchfork \Lambda$ $(^*)$ for any orbit in G.
$((^*)$, equivalently : $TF + dp(T\Lambda) = TG$, where $p : T^*G \to G$.)
Here, G has the following orbits :

- $\overset{\circ}{X} \times \overset{\circ}{X}$: above condition empty.
- For all $y \in S$, $U_y = \mathbb{R} \times Z \times Z \times T_y S$: above condition non empty, can be tested easily on phase functions.

Assuming that $P \in I(G, \Lambda)$ with Λ as above, we get , for any $m \in X$:

- if $m \in \overset{\circ}{X}$, a single $\pi_m(P) = r_{\#, \overset{\circ}{X}}(P)$ FIO on the mfd $\overset{\circ}{X}$,
- if $m = (0, y, z) \in \partial X$, a family $\pi_m(P) = \pi_y(P)$ of FIOs on the mfds $\mathbb{R} \times T_y S \times Z$, commuting with the translation operators coming from $\mathbb{R} \times T_y S$.

$(\mathbb{R} \times G)$-FIO as a solution of a Cauchy problem

Going back to a general Lie gpd G, consider:

$$P \in \Psi^1_c(G) \text{ elliptic and positive.}$$

Let $p \in C^\infty(A^*G \setminus 0)$ be the principal symbol of P and χ_t be the hamiltonian flow of $r_\Gamma^* p \in C^\infty(\overset{\circ}{T}{}^*G)$ (commutes with right multiplication, complete). Then

$$\mathcal{C} = \{(t, \tau, \gamma, \xi) \in T^*(\mathbb{R} \times G) \mid \tau + p(r_\Gamma(\gamma, \xi)) = 0,\ (\gamma, \xi) \in \chi_t(A^*G \setminus 0)\}$$

is a family $\mathbb{R} \times G$-relation.

Theorem (in preparation, Vassout-L.)

The one parameter group $U : t \mapsto e^{-itP}$ satisfies $U \in I^{-1/4+(n-2n^{(0)})/4}(\mathbb{R} \times G, \mathcal{C})$.

The proof uses the fact that U satisfies the Cauchy problem

$$\begin{cases} (D_t + P)U = 0 \\ U(0) = \delta. \end{cases}$$

Historical Notes

1. Lie groups: *Invariant Fourier integral operators on Lie groups, Nielsen-Stetkær* (1974).
2. Mfds with boundary, b-geometry framework: *Boundary Transformation Problems, R. Melrose* (1981).
3. Foliations: *Functional calculus for tangentially elliptic operators on foliated manifolds, Y. Kordyukov* (1994).
4. Mfds with conical sing.: *The index of FIO on manifolds with conical singularities", V. Nazaikinskii, B. Schulze, B. Yu. Sternin* (2001). Also a chapter in *Elliptic theory on singular manifolds, Nazaikinskii-Savin-Schulze-Sternin*.
5. Mfds with boundary, Boutet de Monvel's framework: *Fourier Integral Operators of B-d-M Type*, *On a Class of Fourier Integral Operators on Mfds with Bdy* and *Fourier integral operators and the index of symplectomorphisms on Mfds with Bdy* by U. Battisti-S. Coriasco-E. Schrohe (2014-2015).