The Friedrichs Extension for Elliptic Wedge Operators of Second Order

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Joint work with Gerardo Mendoza

References for this talk:

[1] *The Friedrichs extension for elliptic wedge operators of second order.* arXiv preprint 1509.01842.

[2] *Elliptic systems of variable order.* Rev. Mat. Iberoam. **31** (2015), 127–160.

[3] *The kernel bundle of a holomorphic Fredholm family.* Comm. Partial Differential Equations **38** (2013), 2107–2125.

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Some earlier work

Cheeger, Brüning-Seeley, Lesch, Gil-Mendoza, Mazzeo-Vertman, Melrose-Vasy-Wunsch (Friedrichs form-domain).

The Friedrichs extension

Let H be a complex Hilbert space, and let

$$A : \mathcal{D}_c \subset H \to H$$

be symmetric, densely defined. Let $A_{\max} := A^\star$ and $A_{\min} := A^{\star\star}$. In fact, both operators act as $A^\star$ with domains

$$\mathcal{D}_{\max} := \mathcal{D}(A^\star) = \{v \in H;\ \mathcal{D}_c \ni u \mapsto \langle Au, v\rangle \text{ is } H\text{-bounded}\},$$
$$\mathcal{D}_{\min} := \text{Closure of } \mathcal{D}_c \text{ with respect to } \|u\|_A^2 = \|u\|_H^2 + \|A_{\max}u\|_H^2 \text{ in } \mathcal{D}_{\max},$$

respectively. The *closed extensions* of A are restrictions of $A_{\max}$ to subspaces $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$ that are closed in $(\mathcal{D}_{\max}, \|\cdot\|_A)$. This is an abstract framework of (homogeneous) *boundary conditions*.

The Friedrichs extension

Now suppose that $\langle Au, u\rangle \geq k\langle u, u\rangle$ for some $k \in \mathbb{R}$ for all $u \in \mathcal{D}_c$. Define an inner product

$$[u, v] := \langle Au, v\rangle + K\langle u, v\rangle, \quad u, v \in \mathcal{D}_c,$$

where $K \gg 0$ is sufficiently large.
Let $(\mathscr{H}, [\cdot,\cdot]) \hookrightarrow H$ be the completion of $\mathcal{D}_c$ with respect to $[\cdot,\cdot]$.
The space $\mathscr{H}$ is independent of $K \gg 0$ involved in the definition of $[\cdot,\cdot]$. Then

$$A_F := A_{\max} : \mathcal{D}_F \subset H \to H$$

with domain $\mathcal{D}_F := \mathcal{D}_{\max} \cap \mathscr{H}$ is selfadjoint. A_F is the *Friedrichs extension* of A.
Note: If $A_{\mathcal{D}}$ is *any* selfadjoint extension of A with $\mathcal{D} \subset \mathscr{H}$, then $A_{\mathcal{D}} = A_F$.

Problem: How to describe domains/boundary conditions

1. Characterize $\mathcal{D}_{\text{min}}$.
2. Characterize $\mathcal{D}_{\text{max}}/\mathcal{D}_{\text{min}}$ via a 'suitable' complement $\mathcal{E}$ of $\mathcal{D}_{\text{min}}$ in $\mathcal{D}_{\text{max}}$.
3. Domains are of the form $\mathcal{D} = \mathcal{D}_{\text{min}} \oplus \mathcal{E}_{\mathcal{D}}$ with $\mathcal{E}_{\mathcal{D}} \subset \mathcal{E}$.

Regarding (2): Von Neumann formulas:

$$\mathcal{D}_{\text{max}} = \mathcal{D}_{\text{min}} \oplus \big(\ker(A_{\text{max}} + i) \oplus \ker(A_{\text{max}} - i)\big).$$

Issue: Generally impossible to determine kernels!

Problem: How to describe domains/boundary conditions

Now consider Riemannian manifold M, Hermitian bundle E, and $A \in \mathrm{Diff}^m(M;E)$, $m > 0$, elliptic and symmetric.

$$A : C_c^\infty(M;E) \subset L^2(M;E) \to L^2(M;E).$$

Elliptic regularity: $H^m_{\mathrm{comp}}(M;E) \subset \mathcal{D}_{\min} \subset \mathcal{D}_{\max} \subset H^m_{\mathrm{loc}}(M;E)$.

- $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$ differ at the singular locus (at infinity).
- Elliptic regularity (again): $\ker(A_{\max} \pm i) \subset C^\infty(M;E)$.

Goal: Find complement $\mathcal{E} \subset C^\infty(M;E)$ of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$ distinguished by the *asymptotic behavior* of functions towards the singular locus.

Leitmotif: *Specify boundary conditions/domains by specifying asymptotics.*

In case of the Friedrichs extension (abstractly defined boundary condition): Equivalent *characterization in terms of asymptotics.*

Example: Classical boundary value problems

Let $\Omega \Subset \mathbb{R}^{q+1}$,

$$\Delta : C_c^\infty(\Omega) \subset L^2(\Omega) \to L^2(\Omega).$$

Then:

- $\mathcal{D}_{\min} = H_0^2(\overline{\Omega})$ because $\|\cdot\|_\Delta \sim \|\cdot\|_{H_0^2}$ on C_c^∞.
- $\mathscr{H} = H_0^1(\overline{\Omega})$ (by definition + Poincaré Inequality). This leads to Dirichlet boundary condition $u\big|_{\partial\Omega} = 0$ if $\partial\Omega$ is smooth.
- By definition:

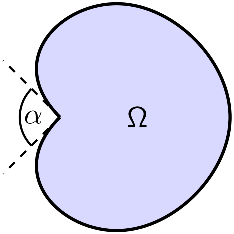
$$\mathcal{D}_F = \{u \in H_0^1(\overline{\Omega});\ \Delta u \in L^2(\Omega)\} = \mathcal{D}_{\max} \cap H_0^1(\overline{\Omega}).$$

Example: Classical boundary value problems

Theorem (Classical Boundary Regularity Theorem)

If $\partial\Omega \in C^\infty$ we have $\mathcal{D}_F = \mathcal{D}_{\max} \cap H_0^1(\overline{\Omega}) = H^2(\Omega) \cap H_0^1(\overline{\Omega})$.

- Result is 'delicate': Consider $\Omega \Subset \mathbb{R}^2$ as shown.



Let (r, θ) be polar coordinates centered at the singular point on the boundary. Then $\mathcal{D}_F$ contains C^∞-functions u with

$$u(r,\theta) \sim c(\theta) r^{\frac{\pi}{2\pi - \alpha}}$$

as $r \to 0$ with $c(\theta) \not\equiv 0$.
Note: $u \notin H^2(\Omega)$ for $0 < \alpha < \pi$.

Example: Classical boundary value problems

Theorem (Classical Boundary Regularity Theorem)

If $\partial\Omega \in C^\infty$ we have $\mathcal{D}_F = \mathcal{D}_{\max} \cap H_0^1(\overline{\Omega}) = H^2(\Omega) \cap H_0^1(\overline{\Omega})$.

- Even if $\partial\Omega \in C^\infty$, $\mathcal{D}_{\max} \neq H^2(\Omega)$ (for $q > 0$, $\Omega \subset \mathbb{R}^{q+1}$): $H^2(\Omega) \hookrightarrow L^2(\Omega)$ compact, but $\mathcal{D}_{\max} \hookrightarrow L^2(\Omega)$ is not compact:

$$\dim\{u \in C^\infty(\overline{\Omega});\ \Delta u = 0 \text{ in } \Omega\} = \infty.$$

- Standard proofs of the regularity theorem involve approximation by difference quotients/extension across the boundary.
  Methods not generalizable to singular spaces.

Example: Classical boundary value problems

There is a split-exact sequence:

$$0 \longrightarrow H_0^2(\overline{\Omega}) \xrightarrow{\ \iota\ } H^2(\Omega) \xrightarrow{T : u \mapsto \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}} \begin{matrix} H^{3/2}(\partial\Omega) \\ \bigoplus \\ H^{1/2}(\partial\Omega) \end{matrix} \longrightarrow 0$$

Taylor expansion of $u \in H^2(\Omega)$ at $\partial\Omega$ (x defining function for $\partial\Omega$):

$$u \sim u_0(y)x^0 + u_1(y)x^1 + \mathcal{O}(x^2), \quad y \in \partial\Omega.$$

Note:

$$\begin{matrix} H^{3/2}(\partial\Omega) \\ \bigoplus \\ H^{1/2}(\partial\Omega) \end{matrix} = H^{\mathfrak{g}}(\partial\Omega; \mathbb{C}^2), \quad \mathfrak{g} = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix} \in \mathrm{End}(\mathbb{C}^2).$$

Example: Classical boundary value problems

In local coordinates U of $\partial\Omega$: $H^{\mathfrak{g}}$-norm of $u \in C_c^\infty(U;\mathbb{C}^2)$ given by

$$\|u\|_{H^{\mathfrak{g}}}^2 = \int_{\mathbb{R}^q} \|\langle D_y\rangle^{\mathfrak{g}} u\|_{\mathbb{C}^2}^2 \, dy.$$

Here $\langle D_y\rangle^{\mathfrak{g}} = \mathcal{F}^{-1}\langle\eta\rangle^{\mathfrak{g}}\mathcal{F}$ with

$$\langle\eta\rangle^{\mathfrak{g}} = \langle\eta\rangle^{\begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}} = \begin{bmatrix} \langle\eta\rangle^{3/2} & 0 \\ 0 & \langle\eta\rangle^{1/2} \end{bmatrix} \in \mathrm{End}(\mathbb{C}^2).$$

For the Friedrichs extension we find:

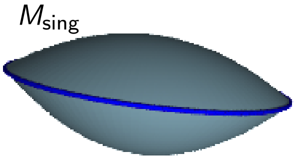
Theorem (Friedrichs Extension for Classical BVPs)

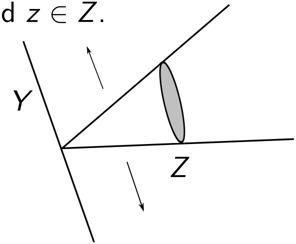
There is a split-exact sequence

$$0 \longrightarrow H_0^2(\overline{\Omega}) = \mathcal{D}_{\min} \xrightarrow{\ \iota\ } \mathcal{D}_F \xrightarrow{\ T\ } H^{1/2}(\partial\Omega) \longrightarrow 0.$$

Manifolds with edges

- M_{sing} compact manifold with edge Y.
- Near Y: M_{sing} is a cone bundle with link Z.
- After blow-up: Compact manifold M with boundary ∂M that is the total space of a fibration: $\begin{array}{ccc} Z & \hookrightarrow & \partial M \\ & & \downarrow \wp \\ & & Y \end{array}$
- Y and Z closed manifolds. Let x be a defining function for the boundary of M, $y \in Y$, and $z \in Z$.





w-metrics and geometric operators

- w-metric ${}^w g$: Any metric on ${}^w T^*M$ (wedge cotangent bundle of M).
  Near ∂M: ${}^w g$ given by positive definite 2-cotensor in the forms
  $$dx,\ dy_j,\ xdz_k$$
  with coefficient functions that are smooth up to ∂M.
  Example: $dy^2 + dx^2 + x^2 dz^2$, a model wedge.
- Incomplete metrics, with singular locus on the boundary.
- Geometric operators: Laplacians $\Delta_{{}^w g}$ give rise to wedge differential operators.
- Let ${}^w g$ be any w-metric. Then
  $${}^{{}^w g}L^2(M) = x^{-\frac{\dim Z}{2}} L^2(M) = x^{-\frac{1+\dim Z}{2}} L^2_b(M).$$
  Consider generally $H = x^{-\gamma} L^2_b$ as base Hilbert space, $\gamma \in \mathbb{R}$.

Wedge differential operators

Wedge differential operators of order m: $A \in \mathrm{Diff}^m(\mathring{M})$ that in adapted coordinates near the boundary take the form

$$A = x^{-m} \sum_{|\alpha|+|\beta|+k \le m} a_{\alpha,\beta,k}(x,y,z)(xD_x)^k D_z^{\beta}(xD_y)^{\alpha}$$

with coefficients $a_{\alpha,\beta,k}$ that are C^∞ up to $x = 0$.

Notation: $A \in x^{-m}\,\mathrm{Diff}_e^m(M)$ ($\mathrm{Diff}_e \to$ Mazzeo '91).

Example: Any regular differential operator $A \in \mathrm{Diff}^m(M)$ with coefficients smooth up to the boundary is an example for this with the trivial boundary fibration $Y = \partial M$ and $Z = \{\mathrm{pt}\}$.

Example: Cone operators correspond to the other extreme case: $Y = \{\mathrm{pt}\}$ and $Z = \partial M$.

Wedge differential operators

A near the boundary:

$$A = x^{-m} \sum_{|\alpha|+|\beta|+k \leq m} a_{\alpha,\beta,k}(x,y,z)(xD_x)^k D_z^\beta (xD_y)^\alpha$$

w-principal symbol ${}^w\sigma(A)$:

$${}^w\sigma(A) = \sum_{|\alpha|+|\beta|+k=m} a_{\alpha,\beta,k}(x,y,z)\xi^k \zeta^\beta \eta^\alpha$$

Is invariantly defined on ${}^wT^*M \setminus 0$.
w-ellipticity: Invertibility of ${}^w\sigma(A)$ on ${}^wT^*M \setminus 0$ (assumed henceforth)
Note: w-ellipticity reduces to standard ellipticity up to the boundary in the regular case.

Wedge differential operators

A near the boundary:

$$A = x^{-m} \sum_{|\alpha|+|\beta|+k \le m} a_{\alpha,\beta,k}(x,y,z)(xD_x)^k D_z^\beta (xD_y)^\alpha$$

Conormal symbol/indicial family:

$$\hat{A}(y,\sigma) = \sum_{|\beta|+k \le m} a_{0,\beta,k}(0,y,z)\sigma^k D_z^\beta : C^\infty(Z_y) \to C^\infty(Z_y)$$

for $y \in Y$ and $\sigma \in \mathbb{C}$.

This is, for each $y \in Y$, a family of differential operators on Z_y depending on $\sigma \in \mathbb{C}$.

Wedge differential operators

Proposition (Elliptic estimates and analytic Fredholm theory)

$\hat{A}(y,\sigma)$ is a holomorphic family of Fredholm operators on Z_y that is meromorphically invertible with finitely many poles in each horizontal strip of finite width, for every $y \in Y$.

Example: In the regular case of $A \in \mathrm{Diff}^m(M)$,

$$\hat{A}(y,\sigma) = \boldsymbol{\sigma}(A)(d_y x)\sigma(\sigma + i)\cdots(\sigma + i(m-1)).$$

Since this is fully governed by $\boldsymbol{\sigma}(A)$, the indicial family remains implicit in the classical theory of elliptic boundary value problems.
Note: The poles of $\hat{A}(y,\sigma)^{-1}$ encode important fiberwise information about the boundary behavior of functions. They generally vary with $y \in Y$, which gives rise to the fundamental problem of analyzing branching of poles.

The trace bundle

Indicial operator (quantized indicial family):

$$ {}^bA_y = x^{-m}\hat{A}(y, xD_x) : C^\infty(Z_y^\wedge) \to C^\infty(Z_y^\wedge), $$

where $Z_y^\wedge = \mathbb{R}_+ \times Z_y$.
Fix $\alpha < \beta$, and consider

$$ \mathcal{T}_y = \ker({}^bA_y) \cap \Big\{ \sum_{\substack{\alpha < \Im(\sigma) < \beta \\ \nexists \hat{A}(y,\sigma)^{-1}}} \sum_{j=0}^{m_\sigma} c_{\sigma,j}(z) \log^j(x) x^{i\sigma} : c_{\sigma,j} \in C^\infty(Z_y) \Big\}. $$

Example: In the regular case, ${}^bA_y = \sigma(A)(d_y x) D_x^m$, and $\mathcal{T}_y = \{\sum_{j=0}^{m-1} e_{j,y} x^j : e_{j,y} \in E_y\}$ if $\alpha < -(m-1)$ and $\beta > 0$.

The trace bundle

Theorem (K. & Mendoza '13)

Let $\alpha < \beta$, and suppose that $\hat{A}(y,\sigma)^{-1}$ exists for all $y \in Y$ and all $\Im(\sigma) = \alpha$ and $\Im(\sigma) = \beta$. Then

$$\mathcal{T} = \bigsqcup_{y \in Y} \mathcal{T}_y$$

is a C^∞ vector bundle over Y whose space of smooth sections are all $s(y,x,z)$ such that $s(y,\cdot,\cdot) \in \mathcal{T}_y$, and s is smooth in all variables.

The operator $x\partial_x$ restricts to a C^∞ bundle endomorphism on $\mathcal{T}$ and generates the radial action

$$\varrho^{x\partial_x} : s(y,x,z) \longmapsto s(y,\varrho x,z), \quad \varrho > 0.$$

$\mathcal{T}$ is the trace bundle associated with A and $\alpha < \Im(\sigma) < \beta$.

The trace bundle

Guiding Principle

Traces/Cauchy data of functions on M with respect to Y are generalized sections of (a certain) $\mathcal{T}$ over Y.

Example: Consider a regular elliptic operator $A \in \mathrm{Diff}^m(M)$.
Cauchy data of u are

$$(u_0, \dots, u_{m-1}) = (u\big|_Y, \partial_x u\big|_Y, \dots, \frac{1}{(m-1)!}\partial_x^{m-1} u\big|_Y).$$

By Taylor expansion near $Y = \partial M$

$$u \sim \sum_{j=0}^{m-1} u_j x^j + \mathcal{O}(x^m)$$

where $\tau = \sum_{j=0}^{m-1} u_j x^j$ is a (generalized) section of $\mathcal{T}$.

Friedrichs extension for cone operators

$A \in x^{-2}\,\mathrm{Diff}^2_b(M)$ near the boundary:

$$A = x^{-2} \sum_{k+|\alpha|\le 2} a_{k,\alpha}(x,z)(xD_x)^k D_z^\alpha.$$

$$\hat{A}(\sigma) = \sum_{k+|\alpha|\le 2} a_{k,\alpha}(0,z)\sigma^k D_z^\alpha : C^\infty(Z) \to C^\infty(Z), \quad \sigma \in \mathbb{C}.$$

Let $\mathrm{spec}_b(A) = \{\sigma \in \mathbb{C};\ \nexists \hat{A}(\sigma)^{-1}\}$. Consider semibounded

$$A : C_c^\infty(\mathring{M}) \subset x^{-\gamma}L^2_b(M) \to x^{-\gamma}L^2_b(M).$$

Theorem (Gil-Mendoza '03)

$$\mathcal{D}_{\min} = \mathcal{D}_{\max} \cap \bigcap_{\varepsilon>0} x^{-\gamma+2-\varepsilon}H^2_b(M).$$

$$\mathcal{D}_{\min} = x^{-\gamma+2}H^2_b(M) \iff \mathrm{spec}_b(A) \cap \{\Im(\sigma) = \gamma - 2\} = \emptyset.$$

Friedrichs extension for cone operators

Theorem (Lesch '97, Gil-Mendoza '03)

$\dim \mathcal{D}_{\max}/\mathcal{D}_{\min} < \infty$. *More precisely,* $\mathcal{D}_{\max} = \mathcal{D}_{\min} \oplus \omega\mathcal{E}$ *(*ω *is a cut-off function near the boundary), where*

$$\mathcal{E} = \bigoplus_{\sigma_0 \in \mathrm{spec}_b(A) \cap \{\sigma;\, \gamma-2 < \Im(\sigma) < \gamma\}} \mathcal{E}_{\sigma_0}$$

$$\mathcal{E}_{\sigma_0} \ni \tau = \sum_{\substack{q \in \{0,1\} \\ \Im(\sigma_0 - iq) \geq \gamma - 2}} \underbrace{\sum_{k=0}^{m_q} c_{q,k}(z) \log^k(x) x^{i(\sigma_0 - iq)}}_{=:\tau_q}$$

$\hat{A}(\sigma)\mathcal{M}(\omega\tau_0)(\sigma)$ *is holomorphic (at* $\sigma = \sigma_0$*), or* ${}^bA\tau_0 = 0$*, and* τ_1 *depends linearly on* τ_0*.*

Friedrichs extension for cone operators

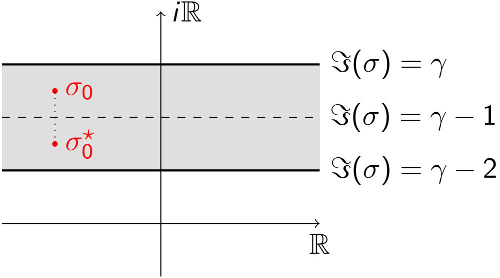
Critical strip associated with $A \in x^{-2}\,\mathrm{Diff}^2_b$ acting in $x^{-\gamma}L^2_b$:

$$\{\sigma \in \mathbb{C};\ \gamma - 2 < \Im(\sigma) < \gamma\}$$

- Base Hilbert space $x^{-\gamma}L^2_b$: Mellin transform holomorphic in $\Im(\sigma) > \gamma$.
- Minimal domain: Mellin transform holomorphic in $\Im(\sigma) > \gamma - 2$.
- Maximal domain: Mellin transform meromorphic in critical strip. Principal parts of Laurent expansions determine regular-singular asymptotics as $x \to 0$.
- Principal terms in the asymptotic behavior: Trace 'bundle' $\mathcal{T}$ associated with A and the critical strip.

Friedrichs extension for cone operators

Symmetry of A in $x^{-\gamma}L^2_b$: $\hat{A}(\sigma) = \left[\hat{A}(\sigma^\star)\right]^\star$, where $\sigma^\star$ is the reflection of σ about $\Im(\sigma) = \gamma - 1$. In particular, $\mathrm{spec}_b(A)$ is symmetric about $\Im(\sigma) = \gamma - 1$.



We have $\mathcal{D}_{\max}/\mathcal{D}_{\min} \cong \bigoplus_{\sigma_0 \in \mathrm{spec}_b(A) \cap \{\sigma;\, \gamma-2 < \Im(\sigma) < \gamma\}} \mathcal{E}_{\sigma_0}$, and Green Formula of Gil-Mendoza (AJM '03) shows $\mathcal{E}_{\sigma_0} \cong \mathcal{E}_{\sigma_0^\star}$.

Friedrichs extension for cone operators

Theorem (Gil-Mendoza '03 – special case)

Suppose $\Im(\sigma) = \gamma - 1$ is free of boundary spectrum. Let

$$\mathcal{E}_F = \bigoplus_{\sigma_0 \in \mathrm{spec}_b(A) \cap \{\sigma;\, \gamma - 2 < \Im(\sigma) < \gamma - 1\}} \mathcal{E}_{\sigma_0}.$$

Then $\mathcal{D}_F = \mathcal{D}_{\min} \oplus \omega \mathcal{E}_F$.

- Let $\mathcal{D}_0 = \mathcal{D}_{\min} \oplus \omega \mathcal{E}_F$. Because

$$\dim \mathcal{D}_0 / \mathcal{D}_{\min} = \frac{1}{2} \dim \mathcal{D}_{\max} / \mathcal{D}_{\min}$$

by Green Formula it is enough to show that $\mathcal{D}_0 \subset \mathcal{D}_F$.

Friedrichs extension for cone operators

Theorem (Gil-Mendoza '03 – special case)

Suppose $\Im(\sigma) = \gamma - 1$ is free of boundary spectrum. Let

$$\mathcal{E}_F = \bigoplus_{\sigma_0 \in \mathrm{spec}_b(A) \cap \{\sigma;\, \gamma-2<\Im(\sigma)<\gamma-1\}} \mathcal{E}_{\sigma_0}.$$

Then $\mathcal{D}_F = \mathcal{D}_{\min} \oplus \omega \mathcal{E}_F$.

- $A : x^{-\gamma+1} H_b^1 \to x^{-\gamma-1} H_b^{-1}$ continuous, and $x^{-\gamma} L_b^2$-inner product gives $[x^{-\gamma+1} H_b^1]' \cong x^{-\gamma-1} H_b^{-1}$. Thus

$$\left| \langle Au, u \rangle_{x^{-\gamma} L_b^2} \right| \lesssim \|Au\|_{x^{-\gamma-1} H_b^{-1}} \|u\|_{x^{-\gamma+1} H_b^1} \lesssim \|u\|^2_{x^{-\gamma+1} H_b^1}.$$

  Consequently, $x^{-\gamma+1} H_b^1 \hookrightarrow \mathscr{H}$.
- We have $\mathcal{D}_0 \subset x^{-\gamma+1} H_b^1 \cap \mathcal{D}_{\max} \subset \mathscr{H} \cap \mathcal{D}_{\max} = \mathcal{D}_F$.

Back to wedge differential operators...

A near the boundary ($m = 2$):

$$A = x^{-m} \sum_{|\alpha|+|\beta|+k \leq m} a_{\alpha,\beta,k}(x, y, z)(xD_x)^k D_z^\beta (xD_y)^\alpha$$

Normal family:

$$A_\wedge(y, \eta) : C_c^\infty(Z_y^\wedge) \subset x^{-\gamma} L_b^2(Z_y^\wedge) \to x^{-\gamma} L_b^2(Z_y^\wedge)$$

$$A_\wedge(y, \eta) = x^{-m} \sum_{|\alpha|+|\beta|+k \leq m} a_{\alpha,\beta,k}(0, y, z)(xD_x)^k D_z^\beta (x\eta)^\alpha$$

Operator family on $Z_y^\wedge = Z \times \mathbb{R}_+$, parametrized by $T^*Y \setminus 0$.
Note: In the classical case of regular BVPs, we recover the boundary symbol:

$$A_\wedge(y, \eta) = \varpi(D)(y, 0, \eta, D_x)$$

The normal family

Standing assumptions:

- $A \in x^{-2}\,\mathrm{Diff}^2_e(M)$ w-elliptic, bounded from below in $x^{-\gamma}L^2_b$.
- $\hat{A}(y,\sigma)$ invertible for all $y \in Y$ and $\Im(\sigma) = \gamma - 2$ and $\Im(\sigma) = \gamma - 1$.

Let $\mathcal{T}_F$ be the trace bundle associated with A and the strip $\gamma - 2 < \Im(\sigma) < \gamma - 1$.

Proposition

- $A_\wedge(y,\eta) : \mathcal{D}_{\wedge,\max/\min} \to x^{-\gamma}L^2_b(Z^\wedge_y)$ *is Fredholm on* $T^*Y \setminus 0$.
- $\dim \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} < \infty$, *and complement represented by singular functions.*
- $A_\wedge(y,\eta) \geq 0$ *on* $C^\infty_c(Z^\wedge_y) \subset x^{-\gamma}L^2_b(Z^\wedge_y)$.
- $\mathcal{D}_{\wedge,F,y} = \mathcal{D}_{\wedge,\min} \oplus \omega\mathcal{T}_{F,y}$ *(varies only with* y*).*

The main result

Theorem (Friedrichs Extension for Wedge Operators)

Suppose $A_\wedge(y,\eta) > 0$ *on* $C_c^\infty(Z_y^\wedge)$ *for* $(y,\eta) \in T^*Y \setminus 0$.

- $\mathcal{D}_{\min} = x^{-\gamma+2} H_e^2(M)$.
- *Let* $\mathfrak{g} = \gamma + (x\partial_x) \in \mathrm{End}(\mathcal{T}_F)$. *Then*

$$0 \longrightarrow \mathcal{D}_{\min} \xrightarrow{\ \iota\ } \mathcal{D}_F \xrightarrow{\ T\ } H^{2-\mathfrak{g}}(Y;\mathcal{T}_F) \longrightarrow 0$$

  is split-exact.

- For regular boundary value problems: $L^2(M) = x^{-1/2} L_b^2(M)$, so $\gamma = 1/2$.
- $\mathcal{T}_F = \mathbb{C} \cdot x$. So $x\partial_x$ acts as the identity on $\mathcal{T}_F$.
- $\mathfrak{g} = 3/2$, and $H^{2-\mathfrak{g}}(Y;\mathcal{T}_F) = H^{1/2}(Y)$.

Sobolev space of sections of variable order

Idea of the space $H^{2-\mathfrak{g}}(Y; \mathcal{T}_F)$

"Eigensections of $\mathcal{T}_F$ w.r.t. the eigenvalue λ of $2-\mathfrak{g}$ (at y) are measured with Sobolev regularity $\mathrm{Re}(\lambda)$ (at y)."

Note that

$$\sum_{k=0}^{m_{\sigma_0}} c_k \log^k(x) x^{i\sigma_0}$$

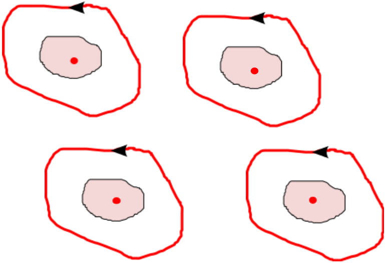
is a generalized eigensection of $2-\mathfrak{g} = 2-(\gamma + x\partial_x)$ associated with the eigenvalue $\lambda = 2-\gamma-i\sigma_0$.

Formal construction:

- Pair (E, a) consisting of a complex vector bundle $E \to Y$ and an endomorphism $a \in \mathrm{End}(E)$.
- For $y \in Y$ consider $\varrho^{a|_{E_y}} : E_y \to E_y$, $\varrho > 0$. Then $\varrho^a \in \mathrm{End}(E)$.

Sobolev space of sections of variable order

Fix y_0 and enclose eigenvalues of a at y_0 with contours $\Gamma_1, \ldots, \Gamma_N$:



Eigenvalues spread locally in clusters enclosed by these contours. The spectral projection onto $\tilde{U}_k$ ($=$ direct sum of generalized eigenspaces associated with eigenvalues in k-th cluster) is

$$P_{k,y} = \frac{1}{2\pi i} \int_{\Gamma_k} (\sigma - a(y))^{-1} \, d\sigma.$$

The formula depends smoothly on y (near y_0).

Sobolev space of sections of variable order

Definition

Fix $0 < \delta < 1$. A local trivialization $\phi : E|_\Omega \to \Omega \times \mathbb{C}^L$ is called δ-admissible for (E, a) if the following holds:

(a) $\exists$ open $W_1, \ldots, W_N \subset \mathbb{C}$, $W_i \cap W_j = \emptyset$ for $i \neq j$, $\operatorname{diam}(W_k) < \delta$, and $S_k \Subset W_k$ s.t. $\operatorname{spec}(a|_{E_y}) \cap W_k \subset S_k$ and

$$\operatorname{spec}(a|_{E_y}) = \bigcup_{k=1}^{N} \operatorname{spec}(a|_{E_y}) \cap W_k \text{ for all } y \in \Omega.$$

(b) ϕ is a direct sum of trivializations of the subbundles

$$\tilde{U}_k = \bigsqcup_{y \in \Omega} \Big(\bigoplus_{\lambda \in \operatorname{spec}(a|_{E_y}) \cap W_k} \ker(a|_{E_y} - \lambda)^{\dim E_y} \Big)$$

over Ω, $k = 1, \ldots, N$.

Sobolev space of sections of variable order

In a δ-admissible trivialization $\Omega \times \mathbb{C}^L$ over a chart, $a \in \mathrm{End}(E|_\Omega)$ is represented by $a(y) \in C^\infty(\Omega, \mathscr{L}(\mathbb{C}^L))$.

Definition

$u \in C_c^{-\infty}(\Omega; \mathbb{C}^L)$ belongs to $H^a_{\mathrm{comp}}(\Omega; \mathbb{C}^L)$ iff

$$\langle D_y \rangle^{a(y)} u = \mathrm{op}[\langle \eta \rangle^{a(y)}] u \in L^2_{\mathrm{loc}}(\Omega; \mathbb{C}^L)$$

This local definition gives rise to a well-defined global space $H^a(Y; E)$.

Idea of Proof of the Main Theorem

- Use duality of edge Sobolev spaces and continuity to argue exactly as in the case of cone operators that
$$x^{-\gamma+1}H^1_e(M) \hookrightarrow \mathscr{H}.$$
- Construct explicitly $\mathcal{E} : H^{2-\mathfrak{g}}(Y;\mathcal{T}_F) \to \mathcal{D}_{\max} \cap x^{-\gamma+1}H^1_e(M)$ that provides a splitting (local construction + patching – technical part, requires subtle estimates, pseudodifferential techniques adapted from Schulze's calculus).
- Let $\mathcal{D}_0 = x^{-\gamma+2}H^2_e(M) + \mathcal{E}(H^{2-\mathfrak{g}}(Y;\mathcal{T}_F)) \subset \mathcal{D}_F$.
- Show that $A + \lambda^2 : \mathcal{D}_0 \to x^{-\gamma}L^2_b$ is invertible for $\lambda \gg 0$. Proof of this is based on parameter-dependent parametrix construction. Assumptions on the Friedrichs extension of the normal family $A_\wedge(y,\eta)$ are needed for this construction to work.

Thank you!