The Neumann-Poisson Operator for Touching Hypersurfaces

Karsten Fritzsch[1]
(part of ongoing joint work with Daniel Grieser[2])

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[1]University College London, UK
[2]CvO Universität Oldenburg, Germany

The Neumann-Poincaré Operator

- $\Omega \subset \mathbb{R}^n$, $n \geq 3$, bounded with connected complement, exterior unit normal ν, $E = c_n \big| z - z' \big|^{2-n}$ fundamental solution for Δ on $\mathbb{R}^n$. Then:

$$K(z, z') = 2\partial_\nu(z') E(z, z') = c'_n \frac{< \nu(z'), z - z' >}{\big| z - z' \big|^n}$$

defines an operator $L^2(\Gamma) \longrightarrow L^2(\Gamma)$.

- Used in the method of layer potentials to solve Dir– and Neu– problems
- Can be used to study the DN–operator and related transmission problems across $\partial\Omega$

Earlier & Related Results

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The spectrum of K is of particular interest. One may define a scalar product on $H^{\frac{1}{2}}(\partial\Omega)$ with respect to which $K : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$ is self-adjoint. On this space:

- If $\partial\Omega$ is $C^{1,\alpha}$, $\alpha > 0$: $\quad \sigma_{ess}(K) = \{0\}$
- If Ω is a plane domain with unique corner with angle $0 < \vartheta < 2\pi$:

$$\sigma_{ess}(K) = \left\{ x \in \mathbb{R} \,:\, |x| \leq |1 - \tfrac{\vartheta}{\pi}| \right\}$$

  (Perfekt–Putinar, see also Chesnel–Claeys–Nazarov)
- **Goal:** If $\mathbb{R}^n \setminus \Omega$ has a certain type of cusp, show:

$$\sigma_{ess}(K) = [-1, 1]$$

Touching Hypersurfaces

Assume $\Omega = \Omega_- \cup \Omega_+$ is given by two touching domains in $\mathbb{R}^n$, $n \geq 3$, where $\partial\Omega_\pm$ smooth, $\overline{\Omega_-} \cap \overline{\Omega_+} = \{0\}$ and each has connected complement.

We resolve the singular stratum using two blow-ups: First, we blow up the intersection of $\partial\Omega_\pm$ and then the intersections of the their lifts.

(This resolves the exterior region $\mathbb{R}^n \setminus \Omega$ as well. We could also employ a quasi-homogeneous blow up.)

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A neighbourhood of the newly created face fibres trivially over the set $[0, \epsilon) \times \mathbb{S}^{n-2}$: This can be given explicitly using rescaled cylindrical coordinates $z = (\frac{x}{r^2}, r, \omega)$ by $\phi(z) = (r, \omega)$.

Behaviour near $\partial\Gamma$

How does this fibration relate to K? If

$$|< \nu(z'), z - z' >| \leq |z - z'|^2 ,$$

for all z, z' and as $|z - z'| \longrightarrow 0$, K defines a compact operator.

This fails when z, z' belong to the same fibre of ϕ but to different connected components of Γ, in which case we have $|< \nu(z'), z - z' >| \sim |z - z'|$ as $|z - z'| \to 0$.

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When restricted to $\partial\Gamma$, ϕ gives

$$\{\pm 1\} — \partial\Gamma \xrightarrow{\phi_\partial} \mathbb{S}^{n-2} ,$$

and we might consider K as a ϕ–ΨDO, as introduced by R. Mazzeo and R. Melrose. (Seems overly complicated, but might be useful when dealing with transmission problems.)

ϕ–Pseudodifferential Operators

Brief description: A ϕ–ΨDO (in the full calculus) is given by a $b\phi$–half density kernel which is polyhomogeneous conormal on the ϕ–double space of Γ with respect to the lifted diagonal:

$$\Gamma^2 \longleftarrow \Gamma_b^2 = \left[\Gamma^2; (\partial\Gamma)^2\right] \longleftarrow \Gamma_\phi^2 = \left[\Gamma_b^2; \Delta_\phi\right],$$

where $\Delta_\phi = \left\{ (h, h', 1) \in (\partial\Gamma)^2 \times [-1, 1] \cong \mathrm{bf} \,:\, \phi_\partial(h) = \phi_\partial(h') \right\}$ is the fibre diagonal of the b–face.

Mapping Properties of K

Theorem 1:

i. K *lifts to define an element of the full* ϕ*-calculus,* $K \in \Psi^{-1,(0,n-1,1,0)}(\Gamma)$

ii. If $\alpha > 1 - n$, $\beta < 0$ *and* $\beta \leq \alpha$, *then*

$$K : \rho^{\alpha} H_{\phi}^{s-1}(\Gamma; {}^{b}\Omega^{\frac{1}{2}}) \longrightarrow \rho^{\beta} H_{\phi}^{s}(\Gamma; {}^{b}\Omega^{\frac{1}{2}}) \qquad (*)$$

is bounded.

iii. If in addition $\beta < \alpha$, *then* $(*)$ *is compact.*

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Proof: Explicitly compute the index family for K, then apply a general theorem by D. Grieser and E. Hunsicker.

The Normal Operator

Theorem 1 is useful for the study of transmission problems, but does not directly help to compute $\sigma_{ess}(K)$. But we can use another object from the ϕ–calculus:

The kernel of the normal operator $N(P)$ of a ϕ–ΨDO P is given by restriction of the kernel of P to the ϕ–face.

In our case, with fibres being $\mathbb{S}^0$, it is a 2×2–matrix of functions on $\mathbb{S}^{n-2} \times \mathbb{R}^{n-1}$. Its symbol can be interpreted to act as a convolution operator on Fourier transforms of functions on a neighbourhood of $\partial\Gamma$.

The Normal Operator

Lemma 2:

i. The symbol of the normal operator of $\lambda - K$ is given by

$$(2\pi)^{1-n}\begin{pmatrix} -\lambda & \widehat{\chi}(\tau,\eta) \\ \widehat{\chi}(\tau,\eta) & -\lambda \end{pmatrix},$$

where $\chi(T,Y) = \frac{2}{|\mathbb{S}^{n-1}|}\frac{\kappa(\omega)}{(\kappa(\omega)^2+T^2+|Y|^2)^{\frac{n}{2}}}$.

ii. It is invertible if and only if $|\lambda| > 1$.

The Essential Spectrum

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The essential spectrum of K *is* $\sigma_{ess}(K) = [-1, 1]$.

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Proof:

- Given $\lambda \in [-1, 1]$, there is a zero (τ_0, η_0) of the symbol of $N(\lambda - K)$.
- Approximate $\delta_{(\tau_0, \eta_0)}$ by suitably normalised, compactly supported functions.
- These can be used to obtain a Weyl-sequence for $\lambda - K$: A sequence u_k so that $\|(\lambda - K)u_k\| \, \|u_k\|^{-1} \longrightarrow 0$ and $u_k \longrightarrow 0$ weakly.
- For more general symmetry reasons: $\sigma(K) \subset [-1, 1]$, whence $\sigma_{ess}(K) = [-1, 1]$.