ALC manifolds with special holonomy

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ALC manifolds

- (Σ, g_Σ) closed connected $(n-2)$–dimensional smooth Riemannian manifold
- $\pi : N \to \Sigma$ a circle bundle.
- θ a connection on $\pi : N \to \Sigma$, $\ell > 0$ a constant

$\rightsquigarrow$ model metric on $M_\infty = \mathbb{R}^+ \times N$

$$g_\infty = dr^2 + r^2 \pi^* g_\Sigma + \ell^2 \theta^2$$

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Definition A complete Riemannian manifold (M^n, g) with only one end is an **ALC manifold** asymptotic to M_∞ **with rate** $\nu < 0$ if there exists a compact set $K \subset M$, a positive number $R > 0$ and a diffeomorphism $\phi : M_\infty \cap \{r > R\} \to M \setminus K$ such that for all $j \geq 0$

$$|\nabla^j_{g_\infty}(\phi^* g - g_\infty)|_{g_\infty} = O(r^{\nu - j}).$$

Remark: ALF vs. ALC

The 4–dimensional hyperkähler case

ALF gravitational instantons: hyperkähler 4–dimensional ALC manifolds

$N = S^3/\Gamma$ where Γ is a cyclic or binary dihedral group, $\Sigma = S^2$ or $\mathbb{RP}^2$

Examples: Taub–NUT, Atiyah–Hitchin, . . .

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Page (1981): consider the Kummer construction of K3 along a 1–parameter family of "split" 4-tori $\mathbb{T}^4 = \mathbb{T}^3 \times S^1_\ell$ with a circle of length $\ell \to 0$.

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Theorem (F., 2016) Every collection of 8 ALF spaces of dihedral type $M_1, \ldots, M_8$ and n ALF spaces of cyclic type $N_1, \ldots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

appears as the collection of "bubbles" forming in a sequence of Kähler Ricci-flat metrics on the K3 surface collapsing to the flat orbifold $T^3/\mathbb{Z}_2$ with bounded curvature away from $n+8$ points.

- Construct (incomplete) S^1–invariant hyperkähler metrics on circle bundles over a punctured 3–torus.

  Gibbons–Hawking Ansatz: $\pi : M \to U \subset \mathbb{R}^3$ principal circle bundle;

$$h\,\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2$$

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- Complete the resulting hyperkähler metrics by gluing in ALF spaces at the $n+8$ punctures:

  an ALF space of dihedral type at each of the 8 fixed points of τ,
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- Deform the resulting approximately hyperkähler metric using the Implicit Function Theorem.

By allowing clusters of punctures coalescing together at different rates could also obtain "bubble trees" of ALF and ALE spaces

ALC G_2 manifolds

(joint work with Mark Haskins and Johannes Nordström)

G_2 holonomy: the whole geometric structure (including the metric) is determined by a closed and coclosed 3–form φ.

The model M_∞ for an ALC G_2 manifold:

- Σ is a Sasaki–Einstein 5–manifold: the cone $C(\Sigma)$ is Calabi–Yau (CY).
- (ω_C, Ω_C) conical CY structure + Hermitian–Yang–Mills connection θ on a circle bundle $M_\infty \to C(\Sigma) \rightsquigarrow$ model 3–form

$$\varphi_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C$$

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Motivation:

- Construct compact G_2 manifolds collapsing to CY 3–folds with isolated conical singularities.
- More in general, ALC G_2 manifolds provide a source of examples to explore collapsing phenomena in G_2 geometry.
- Duality between Type IIA String theory and M theory.

Examples

ALC G_2	$\mathbb{B}_7$	$\mathbb{C}_7$	$\mathbb{D}_7$
AC CY	T^*S^3	$K_{\mathbb{P}^1\times\mathbb{P}^1}$	small resolution ODP

An explicit example in the $\mathbb{B}_7$ case, the other examples are numerical.

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- Existence of a cohomogeneity one ALC G_2 manifold with an isolated conical singularity modelled on $C(S^3 \times S^3)$
- Smoothing of the previous example by gluing in Bryant–Salamon's AC G_2 metric on $S^3 \times \mathbb{R}^4$: 3 different ways to smooth the singularity $\rightsquigarrow$ $\mathbb{B}_7$ and $\mathbb{D}_7$ examples.

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- A cohomogeneity one conically singular ALC G_2 manifold.
- Smoothing of the previous example $\rightsquigarrow$ $\mathbb{B}_7$ and $\mathbb{D}_7$ families.

- ALC G_2 manifolds with an S^1 action with small orbits.
  - Apostolov–Salamon (2004): reduction of the PDEs for G_2 holonomy in the presence of a Killing field.

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  - Pass to the adiabatic limit: collapsed limit endowed with a CY structure (ω_0, Ω_0) and a **CY monopole** (h, θ): $*dh = d\theta \wedge \mathrm{Re}\,\Omega_0$, $d\theta \wedge \omega_0^2 = 0$.
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    (iii) Deform to honest ALC G_2 metrics using Joyce's deformation results.

Deformation theory of ALC G_2 manifolds

Theorem For $-3 < \nu < -1$ the moduli space of ALC G_2 manifolds with rate ν is a smooth manifold of dimension

$$\dim \mathcal{H}^3_\nu = \dim \{\rho \in \Omega^3 \text{ such that } d\rho = 0 = d^*\rho,\ \rho = O(r^\nu)\}$$

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- Eigenvalue estimate for the Laplacian on 2–forms on regular SE 5–mnflds $\rightsquigarrow \nu = -3$ and $\nu = -2$ are the only indicial roots for $d + d^*$ in $[-3, -1]$.
- Hausel–Hunsicker–Mazzeo (2004): consider compactification X of M obtained by collapsing the circle at infinity: $M = X \setminus \Sigma$. L^2–cohomology:

$$\dim\, \mathcal{H}^3_{-3-\epsilon} = \dim \big(H^3_c(M) \to H^3(X) \big)$$

- Identify jump of dimension as we cross the indicial roots -3 and -2:

$$\dim\, \mathcal{H}^3_{-3+\epsilon} - \dim\, \mathcal{H}^3_{-3-\epsilon} = \dim \operatorname{im} \big(H^3(X) \to H^3(\Sigma) \big) + \dim \operatorname{im} \big(H^4(M) \to H^3(\Sigma) \big)$$

$$\dim\, \mathcal{H}^3_{-2+\epsilon} - \dim\, \mathcal{H}^3_{-2-\epsilon} = \dim \operatorname{im} \big(H^3(M) \to H^2(\Sigma) \big)$$

	dim $\mathcal{H}^3_{-3+\epsilon}$	dim $\mathcal{H}^3_{-2+\epsilon}$
$\mathbb{B}_7$	1	1
$\mathbb{C}_7$	1	1
$\mathbb{D}_7$	0	1

Comparison with the dimension of the moduli space of AC CY structures on the collapsed limit explains the origin of these deformations.

A second parameter in the $\mathbb{C}_7$ family forced to vary in a discrete way.