On the Hodge-Kodaira Laplacian on the canonical bundle of a compact Hermitian complex space

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Cheeger-Goresky-MacPherson's Conjecture, 1982

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This conjecture is still largely open.

MacPherson's conjecture, 1983

Conjecture

Let $V \subset \mathbb{CP}^n$ be a complex projective variety, $\pi : \tilde{V} \longrightarrow V$ a resolution of V and let g be the Kähler metric on $\mathrm{reg}(V)$ induced by the Fubini-Study metric. Then:

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$$\chi_2(\text{reg}(V), g) = \chi(\mathcal{O}_{\tilde{V}})$$

where

- $\chi_2(\text{reg}(V), g)) = \sum(-1)^q \dim(H^{0,q}_{2,\overline{\partial}}(\text{reg}(V), g))$
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Solved by Pardon and Stern in 1991 proving a stronger result:

$$H^{0,q}_{2,\overline{\partial}_{\min}}(\operatorname{reg}(V), g) \cong H^{0,q}_{\overline{\partial}}(\tilde{V}), \ q = 0, ..., v$$

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Consider now a compact and irreducible Hermitian complex space (X, h) of complex dimension m.

We are interested in the following operator

$$\overline{\partial}_{m,0} : L^2\Omega^{m,0}(\text{reg}(X), h) \to L^2\Omega^{m,1}(\text{reg}(X), h) \qquad (0.1)$$

with domain given by $\Omega_c^{m,0}(\text{reg}(X))$, the space of smooth $(m, 0)$-forms with compact support in $\text{reg}(X)$.

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the adjoint of (0.2). For example the operator (0.2) might be

$$\overline{\partial}_{m,0,\max/\min} : L^2\Omega^{m,0}(\text{reg}(X), h) \to L^2\Omega^{m,1}(\text{reg}(X), h)$$

respectively the maximal/minimal extension of (0.1).

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with domain

$$\mathcal{D}(\overline{D}^*_{m,0} \circ \overline{D}_{m,0}) = \{s \in \mathcal{D}(\overline{D}_{m,0}) : \ \overline{D}_{m,0}s \in \mathcal{D}(\overline{D}^*_{m,0})\}$$

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The operator $\overline{D}^*_{m,0} \circ \overline{D}_{m,0}$ is a self-adjoint extension of the Hodge-Kodaira Laplacian in bi-degree $(m, 0)$,

$$\Delta_{\overline{\partial},m,0} : \Omega_c^{m,0}(\text{reg}(X)) \to \Omega_c^{m,0}(\text{reg}(X))$$

The main result

Theorem

The operator

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Let $\{\lambda_k\}$ *be the eigenvalues of* $\overline{D}^*_{m,0} \circ \overline{D}_{m,0}$. *Then:*

$$\liminf \lambda_k k^{-\frac{1}{m}} > 0$$

as $k \to \infty$.

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as $k \to \infty$.

Equivalently there exists $c > 0$ and $n \in \mathbb{N}$ such that

$$\lambda_k \geq ck^{\frac{1}{m}}$$

for every $k \geq n$.

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- In particular the theorem holds true for

$$\overline{\partial}^t_{m,0,\min} \circ \overline{\partial}_{m,0,\max} : L^2\Omega^{m,0}(\operatorname{reg}(X), h) \to L^2\Omega^{m,0}(\operatorname{reg}(X), h)$$

and

$$\overline{\partial}^t_{m,0,\max} \circ \overline{\partial}_{m,0,\min} : L^2\Omega^{m,0}(\operatorname{reg}(X), h) \to L^2\Omega^{m,0}(\operatorname{reg}(X), h)$$

where the latter operator is the Friedrich extension of

$$\Delta_{\overline{\partial},m,0} : \Omega^{m,0}_c(\operatorname{reg}(X)) \to \Omega^{m,0}_c(\operatorname{reg}(X))$$

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Let $\pi : M \to X$ a resolution of X. Let $A := \pi^{-1}(\text{reg}(X))$ and let $\rho := \pi^* h$. ρ is a positive semi-definite Hermitian product on M strictly positive on M.

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Let

$$\overline{D}_{m,0} : L^2\Omega^{m,0}(A, \rho|_A) \to L^2\Omega^{m,1}(A, \rho|_A)$$

the operator unitarily equivalent to

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This follows because (M,g) is a compact Hermitian manifolds and

$$0 \to \Omega^{m,0}(M) \stackrel{\overline{\partial}_{m,0}}{\to} \ldots \stackrel{\overline{\partial}_{m,q-1}}{\to} \Omega^{m,q}(M) \stackrel{\overline{\partial}_{m,q}}{\to} \ldots \stackrel{\overline{\partial}_{m,m-1}}{\to} \Omega^{m,m}(M) \to 0.$$

is an elliptic complex.

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- $\overline{\partial}_m + \overline{\partial}_m^t : L^2\Omega^{m,\bullet}(A, g|_A) \to L^2\Omega^{m,\bullet}(A, g|_A)$ is essentially self-adjoint.

Using all these properties we can conclude that (0.4) is a continuous inclusion and finally this tells us that $\overline{D}_{m,0}^* \circ \overline{D}_{m,0}$ has discrete spectrum.

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with domain $\mathcal{D}(\Delta_{\overline{\partial},2,q,\text{abs}}) := \{\omega \in \mathcal{D}(\overline{\partial}_{2,q,\max}) \cap \mathcal{D}(\overline{\partial}^t_{2,q-1,\min}) :$
$\overline{\partial}^t_{2,q-1,\min}\omega \in \mathcal{D}(\overline{\partial}_{2,q-1,\max}),\ \overline{\partial}_{2,q,\max}\omega \in \mathcal{D}(\overline{\partial}^t_{2,q,\min})\}$.

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Combining our previous result with a theorem proved by Li and Tian 1995 and with another theorem proved by Pardon-Stern in 1991 we have the following result:

Theorem

Let $V \subset \mathbb{CP}^n$ be a complex projective surface and let h be the Kähler metric on $\text{reg}(V)$ induced by the Fubini-Study metric of $\mathbb{CP}^n$. Then, for each $q = 0, ..., 2$, the operator

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Sketch of the proof

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By Pardon-Stern we know that $\ker(\Delta_{\overline{\partial},2,1,abs})$ is finite dimensional and that $\mathrm{im}(\Delta_{\overline{\partial},2,1,abs})$ is closed.
Therefore $\Delta_{\overline{\partial},2,1,abs}$ has discrete spectrum if and only if the inclusion

$$\mathcal{D}(\Delta_{\overline{\partial},2,1,abs}) \cap \mathrm{im}(\Delta_{\overline{\partial},2,1,abs}) \hookrightarrow L^2\Omega^{2,1}(\mathrm{reg}(V), h)$$

is compact. Now this last point follows as a consequence of the next proposition

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Since we know that both $\Delta_{\overline{\partial},2,0,\mathrm{abs}}$ and $\Delta_{\overline{\partial},2,2,\mathrm{abs}}$ have discrete spectrum the proof is complete.

Mckean-Singer formula

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Corollary

$$\chi(\tilde{V}, \mathcal{K}_{\tilde{V}}) = \mathrm{ind}((\overline{\partial}_{2,\max} + \overline{\partial}^t_{2,\min})^+) = \sum_{q=0}^{2} (-1)^q \operatorname{Tr}(e^{-t\Delta_{\overline{\partial},2,q,\mathrm{abs}}}),$$

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where

- $\pi : \tilde{V} \to V$ *is any resolution of* V*,*
- $\mathcal{K}_{\tilde{V}}$ *is the sheaf of holomorphic* $(2,0)$*-forms on* $\tilde{V}$*,*
- $\chi(\tilde{V}, \mathcal{K}_{\tilde{V}}) = \sum_{q=0}^{2} (-1)^q \dim(H^q(\tilde{V}, \mathcal{K}_{\tilde{V}}))$*,*
- $\chi(\tilde{V}, \mathcal{O}_{\tilde{V}}) = \sum_{q=0}^{2} (-1)^q \dim(H^{0,q}_{\overline{\partial}}(\tilde{V}))$*.*

Thanks for your attention