The $\mathbb{L}^\bullet$-Homology Fundamental Class for Singular Spaces and the Stratified Novikov Conjecture

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June 2016

joint work with Gerd Laures, Jim McClure.

- $\mathbb{L}^\bullet = \mathbb{L}^\bullet\langle 0\rangle(\mathbb{Z})$ Ranicki's symmetric L-spectrum, $\pi_n(\mathbb{L}^\bullet) = L^n(\mathbb{Z})$
- **Objective:** For a (closed, oriented) singular space X^n, give a detailed construction of an $\mathbb{L}^\bullet$-*homology fundamental class*

$$[X]_{\mathbb{L}} \in \mathbb{L}^\bullet_n(X),$$

  using the formalism of "ad theories".
- Applications to stratified homotopy invariance of higher signatures, geometric cycle theory for $\mathbb{L}^\bullet$-homology.
- $[X]_{\mathbb{L}}$ appeared first in work of Cappell, Shaneson, Weinberger: $\Delta^G(X) \in \mathrm{KO}^G_*(X)[\frac{1}{2}]$, X Witt, G finite group; TOP inv.
- Th. Eppelmann: sheaf complexes in the derived category

The $\mathbb{L}$-Homology Fundamental Class for Manifolds

- **MSTop** Thom spectrum of oriented topological bordism.
- Ranicki: "symmetric signature map" $\mathbf{MSTop} \to \mathbb{L}^\bullet$.
- M: n-dim. closed oriented topological manifold.
- $[M]_{\mathrm{Top}} := [M \xrightarrow{\mathrm{id}} M] \in \Omega_n^{\mathrm{STop}}(M)$.
- **Def.**

$$\begin{array}{ccc} \Omega_n^{\mathrm{STop}}(M) & \longrightarrow & \mathbb{L}_n^\bullet(M) \\ [M]_{\mathrm{STop}} & \mapsto & [M]_{\mathbb{L}}. \end{array}$$

- Is an oriented homeomorphism invariant.

- $[M]_{\mathbb{L}}$ is an integral refinement of the Top. (Hirzebruch) L-class:

$$[M]_{\mathbb{L}} \otimes 1 = L^*(TM) \cap [M]_{\mathbb{Q}} \in \mathbb{L}^{\bullet}_n(M) \otimes \mathbb{Q} \cong \bigoplus_{j \geq 0} H_{n-4j}(M;\mathbb{Q}),$$

$L^*(TM) \in H^{4*}(M;\mathbb{Q})$,
stable tangent bundle $TM : M \to \mathrm{BSTop}$.

- Non-simply connected generalization of the Hirzebruch Signature Theorem: Image of $[M]_{\mathbb{L}}$ under assembly

$$\mathbb{L}^{\bullet}_n(M) \to L^n(\mathbb{Z}[\pi_1 M])$$

is the *symmetric signature* $\sigma^*(M)$ (Mishchenko, Ranicki).

Intersection Homology Poincaré Spaces

Def. An n-dimensional *PL pseudomanifold* is a PL space X for which some (and hence every) triangulation has the following properties.

1. Every simplex is contained in an n-simplex.
2. Every $(n-1)$-simplex is a face of exactly/at most two n-simplices.

Def. (Goresky, Siegel) An n-dimensional *Intersection homology Poincaré (IP-) space* is an n-dimensional PL pseudomanifold X such that:

1. $IH_k^{\bar{m}}(L^{2k};\mathbb{Z}) = 0$ for links L^{2k} and
2. $IH_k^{\bar{m}}(L^{2k+1};\mathbb{Z})$ is torsion free for links L^{2k+1}.

- **Thm.** (Goresky-Siegel.) If $(X^n, \partial X)$ is an oriented compact IP-space, then

$$\mathbf{IC}^\bullet_{\bar{m}}(X - \partial X; \mathbb{Z}) \cong \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{IC}^\bullet_{\bar{m}}(X - \partial X; \mathbb{Z}), \mathbb{D}^\bullet_{X-\partial X})[n]$$

(Verdier self-duality over $\mathbb{Z}$ in the derived category of sheaf complexes) and intersection of cycles induces a nonsingular pairing

$$IH_i(X, \partial X; \mathbb{Z})/\operatorname{Tors} \times IH_{n-i}(X; \mathbb{Z})/\operatorname{Tors} \longrightarrow \mathbb{Z}.$$

- W. Pardon: IP-bordism $\Omega^{\mathrm{IP}}_*(-)$, is a gen. homology theory,

$$\Omega^{\mathrm{IP}}_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n \equiv 0(4), \\ \mathbb{Z}/_2, & n \geq 5, n \equiv 1(4), \\ 0 & \text{otherwise.} \end{cases}$$

Note: very close to $L^n(\mathbb{Z})$.

The Symmetric Signature of IP-Spaces

- G. Friedman, J. McClure: for Witt spaces X^n, $\sigma^*_{Witt}(X) \in L^n(\mathbb{Q}[\pi_1 X])$.
- Their methods apply to yield

$$\sigma^*_{\mathrm{IP}}(X) \in L^n(\mathbb{Z}[\pi_1 X])$$

  for IP-spaces X.
- agrees with previous $\sigma^*(X)$ when X is a manifold.
- Def. A *stratified homotopy equivalence* $f : X \to Y$ is a homotopy equivalence with homotopy inverse $g : Y \to X$ such that for $H : gf \simeq \mathrm{id}_X$,

$$H^{-1}(\text{pure stratum of codim } k) = \bigcup \text{pure strata of codim } k,$$

  same condition for $H' : fg \simeq \mathrm{id}_Y$.
- $\sigma^*_{\mathrm{IP}}(X)$ is an oriented stratified homotopy invariant.
- $\sigma^*_{\mathrm{IP}}(X)$ is a bordism invariant over $B\pi$.

Ad Theories (Quinn; Buoncristiano-Rourke-Sanderson; Laures-McClure).

- **Def.** *Ball complex* K: like a finite simplicial complex, but as closed cells σ take PL balls $\subset \mathbb{R}^N$ instead of simplices. (Want $K \times I$ again to be a ball complex.)
- Morphisms of ball complexes: (subcomplex incl.)$\circ$(isom.)
- K' a subdivision of K. A subcomplex $R \subset K$ is *residual* if R is also a subcomplex of K'.

Target categories $\mathcal{A}$ of an ad-theory:
$\mathbb{Z}$-graded categories $\mathcal{A}$ (no morphisms that decrease dimension), with involution (will suppress). (Have inclusions of cells $\tau \subset \sigma$ only when $\dim \tau \leq \dim \sigma$.)

Ad Theories

To simplify exposition, will suppress orientation of cells.
Def. An *ad-theory* ad with target category $\mathcal{A}$ is an assignment

$$k \in \mathbb{Z}, \text{ ball complex pairs } (K, L) \mapsto \mathrm{ad}^k(K, L),$$

$$\mathrm{ad}^k(K, L) \subset \{\text{functors } F : K - L \to \mathcal{A} \mid F \text{ decr. dim. by } k\}$$

such that

- **Reindexing**: $(K_1, L_1) \cong (K_2, L_2)$ (decr. dim. by k) induces bij. $\mathrm{ad}^l(K_2, L_2) \cong \mathrm{ad}^{l+k}(K_1, L_1)$.
- **Gluing**: For every subdivision K' of K and $F' \in \mathrm{ad}^k(K')$ exists $F \in \mathrm{ad}^k(K)$ such that $F = F'$ on residual subcomplexes.
- **Cylinder**: Have natural transformation $J : ad^k(K) \to \mathrm{ad}^k(K \times I)$ such that $J(F)|_{K\times 0} = F = J(F)|_{K\times 1}$.

$F \in \mathrm{ad}^k(K, L)$ is called a (K, L)-*ad*.

Ad Theories: Bordism and Quinn Spectra

- A *morphism of ad theories* is a functor of target categories which takes ads to ads.
- $F, F' \in \mathrm{ad}^k(\mathrm{pt})$ are *bordant*, if exists I-ad G: $G|_0 = F$, $G|_1 = F'$. (Is an equivalence relation by axioms reindexing, gluing, cylinder.)
- bordism groups $\Omega_k :=$ bordism classes in $ad^{-k}(\mathrm{pt})$.
- Geometric realization $\mathbf{Q}_k := |Q_k|$ of semisimplicial sets Q_k with n-simplices $\mathrm{ad}^k(\Delta^n)$ gives associated *Quinn spectrum* $\mathbf{Q}$.
- $\pi_*(\mathbf{Q}) = \Omega_*$
- Morphism $\mathrm{ad}_1 \to \mathrm{ad}_2 \rightsquigarrow \mathbf{Q}_1 \to \mathbf{Q}_2$.

IP-ads:

- Target category $\mathcal{A}^{\mathrm{IP}}$:
  Objects: compact, oriented IP-spaces $(X, \partial X)$.
  Morphisms: orientation-preserving PL-homeomorphisms and stratum preserving PL-embeddings $\hookrightarrow$ boundary.
- $\mathrm{ad}^{\mathrm{IP},k}(K)$: all functors $F : K \to \mathcal{A}^{\mathrm{IP}}$, decr. dim. by k, s.t. for all cells $\sigma \in K$:

$$\operatorname*{colim}_{\tau \in \partial\sigma} F(\tau) \xrightarrow{\cong} \partial(F(\sigma)).$$

- **Prop.** $\mathrm{ad}^{\mathrm{IP}}$ is an ad theory.
- Get spectrum $\mathbf{Q}^{\mathrm{IP}}$ with $\pi_*(\mathbf{Q}^{\mathrm{IP}}) = \Omega_*^{\mathrm{IP}}(\mathrm{pt})$.

Suitable Model for the Symmetric $\mathbb{L}$-spectrum

Target category $\mathcal{A}^{\mathbb{L}}$:

Objects in deg. n: (C, D, β, φ)

- C chain complex, degreewise free abelian, $\simeq$ finite complex,
- D chain complex with $\mathbb{Z}/_2$-action,
- $\beta : C \otimes C \to D$ quasi-isom., $\mathbb{Z}/_2$-equivariant,
- $\varphi \in D_n^{\mathbb{Z}/_2}$.

Morphisms: $f = (f_C, f_D) : (C, D, \beta, \varphi) \to (C', D', \beta', \varphi')$

- $f_C : C \to C'$, $f_D : D \to D'$ chain maps,
- f_D $\mathbb{Z}/_2$-equivariant,
- $f_D \circ \beta = \beta' \circ (f_C \otimes f_C)$, $f_D(\varphi) = \varphi'$.

Key Example: $(M, \partial M)$ compact oriented n-manifold. $\xi \in S_n(M)$ representative of fundamental class. $C = S_*(M)$, $D = S_*(M \times M)$, $\beta =$ cross product, $\varphi = d_*(\xi)$, $d : M \to M \times M$ diagonal.

$\mathbb{L}$-ads:

- For a functor $F : K \to \mathcal{A}^{\mathbb{L}}$, put for $\sigma \in K$,

$$F(\sigma) =: (C_\sigma, D_\sigma, \beta_\sigma, \varphi_\sigma) \text{ and } C_{\partial\sigma} := \operatorname*{colim}_{\tau \in \partial\sigma} C_\tau .$$

- $\mathrm{ad}^{\mathbb{L},k}(K)$: all functors $F : K \to \mathcal{A}^{\mathbb{L}}$, decr. dim. by k, s.t. for all cells $\sigma \in K$:
  - $F(\tau \subset \sigma)_C : C_\tau \to C_\sigma$, $C_{\partial\sigma} \to C_\sigma$ degreewise split monomorphisms (sim. for D). (Thus

$$\beta_* : H_*(C_\sigma \otimes C_\sigma, (C \otimes C)_{\partial\sigma}) \longrightarrow H_*(D_\sigma, D_{\partial\sigma})$$

    is an isomorphism.)
  - The map

$$C_*^{\mathrm{cell}}(\sigma) \longrightarrow D_\sigma, \; \tau \mapsto F(\tau \subset \sigma)_D(\varphi_\tau)$$

    is a chain map. (Thus $[\varphi_\sigma] \in H_n(D_\sigma, D_{\partial\sigma})$, $n = \dim \sigma - k$.)
  - F is **nondegenerate**:

$$\beta_*^{-1}[\varphi_\sigma]/- : H^*(\mathrm{Hom}(C_\sigma), \mathbb{Z}) \xrightarrow{\cong} H_{\dim \sigma - k - *}(C_\sigma / C_{\partial\sigma}).$$

- **Prop.** (Laures, McClure) $\mathrm{ad}^{\mathbb{L}}$ is an ad theory.
- Get Quinn spectrum $\mathbf{Q}^{\mathbb{L}}$.
- Have canonical weak equivalence

$$\mathbb{L}^{\bullet} \xrightarrow{\simeq} \mathbf{Q}^{\mathbb{L}}$$

  induced by a morphism of ad theories.

Enriched IP-ads:

- Orientation of compact *IP*-space $X^n \rightsquigarrow$

$$[X] \in IH_n^{\bar{0}}(X, \partial X; \mathbb{Z}).$$

- Target category $\mathcal{A}^{[\mathrm{IP}]}$:
  Objects: pairs (X, ξ)
  - compact, oriented IP-spaces $(X, \partial X)$,
  - $\xi \in IS_n^{\bar{0}}(X; \mathbb{Z})$ representative for $[X]$.
    (Singular intersection chains, H. King, G. Friedman)

  Morphisms: dimension-preserving morphisms must respect ξ.
- For [IP]-ads $F : K \to \mathcal{A}^{[\mathrm{IP}]}$, require $\partial \xi_\sigma = \sum_{\tau \in \partial \sigma} \pm \xi_\tau$.
- **Prop.** $\mathrm{ad}^{[\mathrm{IP}]}$ is an ad theory.
- Forgetful functor $\mathcal{A}^{[\mathrm{IP}]} \to \mathcal{A}^{\mathrm{IP}}$
- Induces morphism $\mathrm{ad}^{[\mathrm{IP}]} \to \mathrm{ad}^{\mathrm{IP}}$
- Induces weak equivalence $\mathbf{Q}^{[\mathrm{IP}]} \xrightarrow{\simeq} \mathbf{Q}^{\mathrm{IP}}$. (Check iso. on bordism groups.)

- $\bar{n}$ upper middle perversity.
- On $X \times X$, for strata $S, T \subset X$, let

$$\bar{p}(S \times T) = \begin{cases} \bar{n}(S) + \bar{n}(T) + 2, & \operatorname{codim} S, \operatorname{codim} T > 0 \\ \bar{n}(S) + \bar{n}(T), & \text{otherwise} \end{cases}$$

- Diagonal $d : X \to X \times X$ induces

$$d_* : IS_*^{\bar{0}}(X) \longrightarrow IS_*^{\bar{p}}(X \times X).$$

- Have cross product

$$\beta : IS_*^{\bar{n}}(X) \otimes IS_*^{\bar{n}}(X) \xrightarrow{\simeq} IS_*^{\bar{p}}(X \times X).$$

(D. Cohen, M. Goresky, Lizhen Ji, G. Friedman)

- Functor $\mathrm{Sig} : \mathcal{A}^{[\mathrm{IP}]} \to \mathcal{A}^{\mathbb{L}}$:

$$(X, \xi) \mapsto (C, D, \beta, \varphi)$$

  - Give X the intrinsic stratification.
  - $C := IS_*^{\bar{n}}(X; \mathbb{Z})$, indeed $\simeq$ finite complex,
  - $D := IS_*^{\bar{p}}(X \times X; \mathbb{Z})$,
  - $\beta :=$ cross product,
  - $\varphi := d_*(\xi)$.

  A morphism $(X, \xi) \to (X', \xi')$ induces maps on intersection chains.
- **Prop.** If $F \in \mathrm{ad}^{[\mathrm{IP}]}(K)$, then $\mathrm{Sig} \circ F \in \mathrm{ad}^{\mathbb{L}}(K)$.
- Get morphism $\mathrm{Sig} : \mathrm{ad}^{[\mathrm{IP}]} \to \mathrm{ad}^{\mathbb{L}}$.
- On Quinn spectra $\mathrm{Sig} : \mathbf{Q}^{[\mathrm{IP}]} \to \mathbf{Q}^{\mathbb{L}}$.

- In the stable category, define $\mathrm{Sig} : \mathbf{Q}^{\mathrm{IP}} \to \mathbf{Q}^{\mathbb{L}}$ by

$$\mathbf{Q}^{\mathrm{IP}} \xleftarrow{\simeq} \mathbf{Q}^{[\mathrm{IP}]} \xrightarrow{\mathrm{Sig}} \mathbf{Q}^{\mathbb{L}}.$$

- Define

$$\boxed{\Omega_*^{\mathrm{IP}}(Y) \longrightarrow \mathbb{L}_*^{\bullet}(Y)}$$

to be

$$\begin{array}{ccc} \Omega_*^{\mathrm{IP}}(Y) & \dashrightarrow & \mathbb{L}_*^{\bullet}(Y) \\ \uparrow \cong & & \downarrow \cong \\ \mathbf{Q}_*^{\mathrm{IP}}(Y) & \xrightarrow{\mathrm{Sig}} & \mathbf{Q}_*^{\mathbb{L}}(Y) \end{array}$$

Thm.
The map $\Omega_n^{\mathrm{IP}}(\mathrm{pt}) \to \mathbb{L}_n^\bullet(\mathrm{pt}) = L^n(\mathbb{Z})$ is an isomorphism for all $n \neq 1$.

$(\Omega_1^{\mathrm{IP}}(\mathrm{pt}) = 0,\ L^1(\mathbb{Z}) = \mathbb{Z}/_2.)$

X^n a closed oriented IP-space.

- $[X]_{\mathrm{IP}} := [X \xrightarrow{\mathrm{id}} X] \in \Omega_n^{\mathrm{IP}}(X)$.
- **Def.**

$$\begin{array}{ccc} \Omega_n^{\mathrm{IP}}(X) & \longrightarrow & \mathbb{L}_n^\bullet(X) \\ [X]_{\mathrm{IP}} & \mapsto & [X]_{\mathbb{L}}. \end{array}$$

Thm. (B., Laures, McClure)
For an n-dimensional compact oriented IP-space X there is a fundamental class $[X]_{\mathbb{L}} \in \mathbb{L}^\bullet_n(X)$ with the following properties:

1. $[X]_{\mathbb{L}}$ is an oriented PL homeomorphism invariant,
2. The image of $[X]_{\mathbb{L}}$ under assembly is the symmetric signature:

$$\begin{array}{ccc} \mathbb{L}^\bullet_n(X) & \longrightarrow & L^n(\mathbb{Z}\pi_1(X)) \\ [X]_{\mathbb{L}} & \mapsto & \sigma^*_{\mathrm{IP}}(X) \end{array}$$

3. If X is a PL manifold, then $[X]_{\mathbb{L}}$ is the same as the fundamental class constructed by Ranicki.
4. Rationally, $[X]_{\mathbb{L}}$ agrees with the Goresky-MacPherson L-class of X ($\rightarrow$ next slide).

- An IP-space X has characteristic L-classes

$$L_j(X) \in H_j(X; \mathbb{Q}).$$

(Cheeger, Goresky, MacPherson, Siegel.)

- For X a smooth manifold: $L_j(X)$ are the Poincaré duals of the Hirzebruch L-classes of TX.
-
$$\mathbb{L}^\bullet \otimes \mathbb{Q} \simeq \prod_{j \geq 0} K(\mathbb{Q}, 4j),$$

- Induces natural isomorphisms

$$S_X : \mathbb{L}^\bullet_n(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{j \geq 0} H_{n-4j}(X; \mathbb{Q}).$$

- **Thm.** (B., Laures, McClure) $S_X([X]_{\mathbb{L}} \otimes 1) = L(X)$.

Application: Higher Signatures

- $G = \pi_1(X)$, $r : X \to BG$ a classifying map for the universal cover of X.
- $r_* : H_*(X; \mathbb{Q}) \longrightarrow H_*(BG; \mathbb{Q})$.
- The *higher signatures of X* are the rational numbers

$$\langle a, r_* L(X) \rangle, \ a \in H^*(BG; \mathbb{Q}).$$

- **Thm.** (B., Laures, McClure) Let X be an n-dimensional oriented closed IP-space such that the assembly map

$$\alpha : \mathbb{L}^\bullet_n(BG) \longrightarrow L^n(\mathbb{Z}[G])$$

is rationally injective. Then the higher signatures of X are (orient. pres.) stratified homotopy invariants.

Proof.

- $f : X' \to X$ an orient. pres. stratified homotopy equivalence.
- $r : X \to BG$, $r' = r \circ f : X' \to BG$.
- $$\begin{array}{ccc} \mathbb{L}^\bullet_n(X) & \longrightarrow & L^n(\mathbb{Z}[G]) \\ {\scriptstyle r_*}\downarrow & \nearrow_{\alpha} & \\ \mathbb{L}^\bullet_n(BG) & & \end{array}$$
- $\alpha r_*[X]_{\mathbb{L}} = \sigma^*_{\mathrm{IP}}(X) = \sigma^*_{\mathrm{IP}}(r) = \sigma^*_{\mathrm{IP}}(rf) = \sigma^*_{\mathrm{IP}}(X') = \alpha r'_*[X']_{\mathbb{L}}$.
- Injectivity assumption $\Rightarrow r_*[X]_{\mathbb{L}} = r'_*[X']_{\mathbb{L}} \in \mathbb{L}^\bullet_n(BG) \otimes \mathbb{Q}$.
- $$\begin{array}{ccc} \mathbb{L}^\bullet_n(X) \otimes \mathbb{Q} & \xrightarrow{r_*} & \mathbb{L}^\bullet_n(BG) \otimes \mathbb{Q} \\ {\scriptstyle S_X}\downarrow\cong & & \cong\downarrow{\scriptstyle S_{BG}} \\ \bigoplus_j H_{n-4j}(X;\mathbb{Q}) & \xrightarrow{r_*} & \bigoplus_j H_{n-4j}(BG;\mathbb{Q}) \end{array}$$
- $$\begin{aligned} r_* L(X) &= r_* S_X [X]_{\mathbb{L}} = S_{BG} r_* [X]_{\mathbb{L}} \\ &= S_{BG} r'_* [X']_{\mathbb{L}} = r'_* S_{X'} [X']_{\mathbb{L}} = r'_* L(X'). \end{aligned}$$

Analytic Approach.

- **Thm.** (Albin, Leichtnam, Mazzeo, Piazza) Let (X, g) be an n-dimensional oriented closed Cheeger space (e.g. smoothly stratified Witt space) such that the assembly map

$$\beta : K_*(BG) \longrightarrow K_*(C_r^* G)$$

  is rationally injective. Then the higher signatures of X are (orient. pres. smoothly) stratified homotopy invariants.
- **Pf.** Instead of $[X]_{\mathbb{L}} \in \mathbb{L}_n^\bullet(X)$, use K-homology class

$$[\eth_{\mathrm{sign}}] \in K_*(X) = KK_*(C(X), \mathbb{C})$$

  of a suitable signature operator $\eth_{\mathrm{sign}}$.
- Index gives analytic signature $\sigma_G^{\mathrm{an}} \in K_0(C_r^* G; \mathbb{Q})$, is stratified homotopy invariant, agrees with $\sigma^*_{\mathrm{Witt}}(X)$ under $L^*(\mathbb{Q}[G]) \to K_*(C_r^* G) \otimes \mathbb{Q}$.
- Instead of S_X, use $\mathrm{Ch} : K_*(X) \otimes \mathbb{Q} \to H_*(X; \mathbb{Q})$; $\mathrm{Ch}[\eth_{\mathrm{sign}}] = L(X)$ (Cheeger, Moscovici-Wu, ALMP).