# Long-time existence for the edge Yamabe flow

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1. Introduction
    - Yamabe problem
    - Yamabe flow

2. Incomplete edge metrics
    - The Yamabe problem for stratified spaces and related work

3. Recent work (joint work with Vertman)
    - Short-time existence and function spaces
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# Classical Yamabe problem

Let $(M^m, g_0)$ be a compact Riemannian manifold, $m \geq 3$.

**The Yamabe problem**

Does there exist a smooth positive function $u$ such that the conformal multiple

$$g = u^{\frac{4}{m-2}} g_0$$

has constant scalar curvature?

Yes! The classical proof [Yamabe; Trudinger; Aubin; Schoen] (1960s-1984) uses variational and elliptic PDE theory for the conformal factor:

$$-4\frac{m-1}{m-2}\Delta^{g_0} u + \mathrm{scal}(g_0)u = u^{\frac{m+2}{m-2}}\,\mathrm{scal}(g).$$

There is an alternative geometric flow approach.

# The Yamabe flow

R. Hamilton introduced the Yamabe flow (YF)

$$\begin{cases} \partial_t g(t) = -\operatorname{scal}(g(t)) \cdot g(t), \\ g(0) = g_0, \end{cases}$$

and a volume normalized YF (NYF)

$$\begin{cases} \partial_t g(t) = \Big( \rho(t) - \operatorname{scal}(g(t)) \Big) \cdot g(t), \\ g(0) = g_0, \end{cases}$$

where $\rho(t)$ is the average scalar curvature functional

$$\rho(t) = \frac{1}{\operatorname{vol}(g(t))} \int_M \operatorname{scal}(g(t)) \mathrm{dvol}_{g(t)}.$$

# Sign of a conformal class

Consider the total scalar curvature functional

$$\begin{aligned}
\mathfrak{s}(g) &:= \frac{1}{\mathrm{vol}(g)^{\frac{m-2}{m}}} \int_M \mathrm{scal}(g)\ \mathrm{dvol_g} \\
&= \frac{1}{\|u\|^2_{\frac{2m}{m-2}}} \int_M u \left( \underbrace{-4\frac{m-1}{m-2}\Delta^{g_0} u + \mathrm{scal}(g_0) u}_{:=\Box^{g_0} u} \right) \mathrm{dvol}_{g_0} \\
&= \frac{1}{\|u\|^2_{\frac{2m}{m-2}}} \int_M 4\frac{m-1}{m-2}|\nabla u|^2 + \mathrm{scal}(g_0) u^2 \mathrm{dvol}_{g_0}.
\end{aligned}$$

Define the Yamabe invariant:

$$\mathcal{Y}([g]) = \inf \left\{ \mathfrak{s}(\widetilde{g}) \mid \widetilde{g} = u^{\frac{4}{m-2}} g, u \in H^{1,2}(M), u > 0 \right\}.$$

# Sign of a conformal class II

The Yamabe invariant:

$$\mathcal{Y}([g]) = \inf \left\{ \mathfrak{s}(\widetilde{g}) \mid \widetilde{g} = u^{\frac{4}{m-2}} g, u \in H^{1,2}(M), u > 0 \right\}.$$

## Sign of a conformal class

A conformal class $[g]$ is positive, negative or zero if $\mathcal{Y}([g])$ is positive, negative or zero, respectively.

## Theorem (Folklore? Schoen)

*The following are equivalent.*

1. $[g]$ *is positive (resp. negative or zero).*
2. *First eigenvalue of* $\Box^g$ *is positive (resp. negative or zero).*
3. *There exists a metric* $\widetilde{g} = u^{\frac{4}{m-2}} g$ *such that* $\mathrm{scal}(\widetilde{g}) > 0$ *(resp.* $< 0$ *or* $= 0$*).*

# The Yamabe flow (YF) II

We may write the flow as an nonlinear equation for the conformal factor. For $m \geq 3$, let

$$g(t) = u(t)^{\frac{4}{m-2}} g_0.$$

The NYF becomes ($N = \frac{m+2}{m-2}$, $c(m) = \frac{m+2}{4}$)

$$\begin{cases} \partial_t u^N = N(m-1)\Delta^{g_0} u - c(m)\,\mathrm{scal}(g_0)u + c(m)\rho u^N, \\ u\big|_{t=0} = 1. \end{cases}$$

- For $(M, g_0)$ compact, the flow exists for all time (Hamilton).
- Convergence results:
  Chow, R. Ye, Schwetlick-Struwe, Brendle.
- Convergence for all data if $3 \leq m \leq 5$. For $m \geq 6$, requires technical assumption on Weyl curvature.

# Incomplete edge metrics

- We assume $M^m$ is a compact manifold with boundary, $\partial M$.
- $\partial M$ is the total space of a fibre bundle with compact fibre $F^n$ $(n \geq 1)$ and compact base $(B^b, g^B)$. Let $x$ be the radial coordinate.

$$\begin{array}{ccc} F^n & \longrightarrow & \partial M \\ & & \Big\downarrow \phi \\ & & B^b \end{array}$$

Model rigid incomplete edge metric

$$g_{\text{rigid}} = dx^2 + x^2 g^F + \phi^* g^B.$$

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Here

- $g^B$ is a Riemannian metric on $B$,
- $g^F$ is a symmetric 2-tensor that restricts to a metric on the fibres.
- $\phi : (\partial M, g^F + \phi^* g^B) \to (B, g^B)$ is a Riemannian submersion.

We then say $(M, g)$ is a *feasible* edge metric if

$$g = g_{\text{rigid}} + h,$$

and

- The $g^F$ are isospectral, and the lowest nonzero eigenvalue of $\Delta^F$ satisfies $\lambda_0 > n$.
- $|h|_{g_{\text{rigid}}} = O(x^2)$ as $x \to 0$.

# Related work

There has been work on the Yamabe problem on general stratified spaces, generalizing the variational approach.

- Local Yamabe invariant; existence of Yamabe minimizers for iterated edge spaces, (Akutagawa-Carron-Mazzeo, 2014). c.f. earlier work (Akutagawa-Botvinnik), and (Mondello, 2015).

  There is an obstruction. For an exact conic metric with $n$-dimensional link, if $g = dx^2 + x^2 k$, then

$$\mathrm{scal}(g) = \frac{\mathrm{scal}(k) - n(n-1)}{x^2}.$$

# Recent work with flows

There has been a lot of work on the Ricci flow on surfaces with conic singularities.

- Short-time existence for angle preserving flow with conic singularities (Mazzeo-Rubinstein-Sesum 2015; Yin 2010)
- Short-time existence for angle changing flow (Mazzeo-Rubinstein-Sesum)
- Long-time existence for angle-preserving flow and convergence under "Troyanov condition", (Mazzeo-Rubinstein-Sesum)
- Instantaneously complete Ricci flows (Giesen-Topping, 2011)
- Maximal regularity approaches (Shao, 2015)
- and others...

# Short-time existence

**Theorem (B. & Vertman 2014)**

*Let $g_0$ be a feasible incomplete edge metric such that $\mathrm{scal}(g_0) \in \mathcal{C}^{2+\sigma}_{\mathrm{ie}}(M)$ for $\sigma \in (0,1)$. Let $\Delta^{g_0}$ be the Friedrichs extension of the Laplacian. Then the equation*

$$\begin{cases} \partial_t u^N = N(m-1)\Delta^{g_0} u - c(m)\,\mathrm{scal}(g_0)u + c(m)\rho u^N, \\ u|_{t=0} = 1. \end{cases}$$

*(recall ($N = \frac{m+2}{m-2}$, $c(m) = \frac{m+2}{4}$)) admits a positive solution $u \in \mathcal{C}^{2+\alpha}_{\mathrm{ie}}(M \times [0,T])$ for some $\alpha \in (0,\sigma)$ for a short time $T > 0$.*

$g(t) = u^{\frac{4}{m-2}}(t)g_0$ is a solution to the NYF that remains an incomplete edge metric.

# Motivation for the function spaces

To solve

$$\begin{cases} \partial_t u^N = N(m-1)\Delta^{g_0} u - c(m)\,\mathrm{scal}(g_0)u + c(m)\rho u^N, \\ u|_{t=0} = 1. \end{cases}$$

Look for a solution of the form $u = 1 + v$ in parabolic Hölder space. After linearization, we can abstractly write

$$(\partial_t + L)v = I + Q(v),$$

Look for a fixed point of

$$v = H(I + Q(v)),$$

where $H$ is the appropriate convolution with the heat kernel.

# Function spaces

For many classical parabolic problems, say

$$\begin{cases} (\partial_t - \Delta) v(t,p) = f(t,p), \\ v(0,p) = 0. \end{cases}$$

one may use anisotropic Hölder spaces, $C^{k+\alpha,(k+\alpha)/2}(M \times [0,T])$.
Parabolic Hölder semi-norm:

$$[v]_{\alpha,\alpha/2} := \sup_{(p,t)\neq(p',t')} \left( \frac{|v(p,t) - v(p',t')|}{d(p,p')^\alpha + |t-t'|^{\alpha/2}} \right),$$

$$\|v\|_{k+\alpha,(k+\alpha)/2} := \sum_{2i+j\le k} \|\partial_t^i \nabla^j v\|_{L^\infty} + \sum_{2i+j=k} [\partial_t^i \nabla^j v]_{\alpha,\alpha/2},$$

Classical Schauder estimate: there is a constant $C > 0$ where

$$\|v\|_{2+\alpha,(1+\alpha)/2} \le C \|f\|_{\alpha,\alpha/2}.$$

# Function spaces

We adapt the function spaces for the geometric problem at hand.
For the NYF of an edge space, only need control of the Laplacian!
Parabolic Hölder semi-norm

$$[v]_{\alpha,\alpha/2} := \sup_{(p,t)\neq(p',t')} \left( \frac{|v(p,t) - v(p',t')|}{d(p,p')^{\alpha} + |t - t'|^{\alpha/2}} \right),$$

$$d(p,q) \approx \sqrt{|x - x'|^2 + |y - y'|^2 + (x + x')^2 (z - z')^2}.$$

Introduce $C^{2+\alpha}_{\mathrm{ie}}(M \times [0,T])$,

$$\|v\|_{2+\alpha} := \|v\|_{\alpha,\alpha/2} + \|\partial_t v\|_{\alpha,\alpha/2} + \sum_{X \in \mathcal{V}_e} \|x^{-1} X v\|_{\alpha,\alpha/2} + \|\Delta v\|_{\alpha,\alpha/2}.$$

# Heat kernel asymptotics

$\Delta$ = Friedrich's extension of the Laplacian of a feasible edge metric. Consider the inhomogeneous problem

$$\begin{cases} (\partial_t - \Delta)v(t,p) = f(t,p), \\ v(0,p) = 0. \end{cases}$$

**Theorem (Mazzeo-Vertman, 2012)**

*Let $(M, g)$ be an incomplete edge space with a feasible edge metric $g$.*

*Then the lift $\beta^* H$ of the heat kernel is a polyhomogeneous distribution on $\mathcal{M}_h^2$ with the index set $(-1+m, 0)$ at ff, $(-m+\mathbb{N}_0, 0)$ at td, vanishing to infinite order at tf, and with a discrete index set $(E, 0)$ at rf and lf, where $E \geq 0$.*

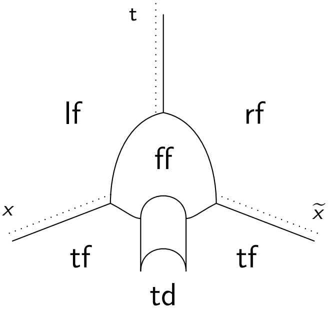


Figure: The heat-space $\mathcal{M}_h^2$.

From these asymptotics we obtain Schauder-type estimates on the spaces $\mathcal{C}_{\text{ie}}^{2+\alpha}(M)$.

# Long-time existence/convergence

## Theorem (B. & Vertman 2016)

*Let $g_0$ be a feasible incomplete edge metric such that $\mathrm{scal}(g_0) \in \mathcal{C}^{4+\sigma}_{\mathrm{ie}}(M)$ for $\sigma \in (0,1)$, and moreover $\mathrm{scal}(g_0) < 0$. Let $\Delta^{g_0}$ be the Friedrichs extension of the Laplacian. Then the equation*

$$\begin{cases} \partial_t u^N = N(m-1)\Delta^{g_0} u - c(m)\,\mathrm{scal}(g_0) u + c(m)\rho u^N, \\ u|_{t=0} = 1. \end{cases}$$

*admits a positive solution $u \in \mathcal{C}^{2+\alpha}_{\mathrm{ie}}(M \times [0,\infty))$ for some $\alpha \in (0,\sigma)$, and the NYF converges exponentially to a metric of constant negative curvature.*

# Outline of proof

The basic strategy:

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3. We may now speak of a maximum time of existence, $T_M$. Suppose for contradiction that $T_M < \infty$.

   We conclude the proof by showing that $u$ extends to $t = T_M$ and the flow can be restarted.

# Outline of proof

4. (Following R. Ye) Establish a uniform $L^\infty$ estimate for $u$.

$$u_{max}(t) = \max_p |u(p,t)|,$$

derive a differential inequality

$$\frac{du_{max}^N}{dt} \leq c(m) \max |\operatorname{scal}(g_0)| u_{max} + c(m) \rho u_{max}^N,$$

from the maximum principle. This can be estimated to obtain uniform upper bounds for $u$. Similar for lower bound. Here is where we use the *sign hypothesis on the scalar curvature.*

# Outline of proof

5. Return to the evolution of $u$, which can be rewritten

$$\partial_t u = \frac{m-2}{4}(\rho(t) - \mathrm{scal}(g(t))u.$$

The quantity $\rho(t) - \mathrm{scal}(g(t))$ decreases exponentially along the flow. This also uses the *sign hypothesis* of scalar curvature and *extra regularity* of $\mathrm{scal}(g_0)$. We obtain a uniform $L^\infty$ estimate for $\partial_t u$ up to $T_M$.

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6. Heat kernel estimates then allow us to conclude $u$ lies in $\mathcal{C}^{1+\alpha}_{\mathrm{ie}}(M \times [0, T_M])$.

# Outline of proof

7. In order to gain more regularity, we prove

**Theorem (B. & Vertman 2016)**

*Let $a \in \mathcal{C}^{1+\alpha}_{\text{ie}}(M \times [0,T])$ be positive and consider $P = \partial_t - a\Delta$. Then there is a bounded right inverse*

$$Q : \mathcal{C}^{\alpha}_{\text{ie}}(M \times [0,T]) \longrightarrow \mathcal{C}^{2+\alpha}_{\text{ie}}(M \times [0,T]),$$

*where $u = Qf$ solves*

$$(\partial_t - a\Delta)u = f, \quad u(p,0) = 0.$$

From the theorem we conclude that $u \in \mathcal{C}^{2+\alpha}_{\text{ie}}(M \times [0,T_M])$ and we may restart the flow at time $T_M$. This contradiction proves $T_M = \infty$.

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From the theorem we conclude that $u \in \mathcal{C}^{2+\alpha}_{\text{ie}}(M \times [0, T_M])$ and we may restart the flow at time $T_M$. This contradiction proves $T_M = \infty$.

8. Convergence is obtained by studying the evolution equations for scal and $\rho$.

# A final result

The $\nu$ Yamabe invariant:

$$\nu([g]) = \inf\left\{\mathfrak{s}(\widetilde{g}) \mid \widetilde{g} = u^{\frac{4}{m-2}} g, u \in \mathcal{C}^{2+\alpha}_{\mathrm{ie}}(M), u > 0\right\}.$$

**Theorem (B. & Vertman 2016)**

*The following are equivalent.*

1. *$[g]$ is positive (resp. negative or zero).*
2. *First eigenvalue of $\square^g$ is positive (resp. negative or zero).*
3. *There exists a metric $\widetilde{g} = u^{\frac{4}{m-2}} g$ such that $\mathrm{scal}(\widetilde{g}) > 0$ (resp. $< 0$ or $= 0$).*

Combined with the previous results, this gives a flow proof of the Yamabe problem in the negative case.

*Thank you!*

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