On higher Friedman's conjecture

Liang Yu
Joint with Yizheng Zhu

Nanjing University

June 20, 2016

Martin's theorem

Theorem (Martin)

Assume AD. Every set of Turing degrees either contains an upper cone or avoids an upper cone.

Proof.

Very simple. □

Applications of Martin's Theorem

- To recursion theory.
- To set theory.

Logic is about definability

Logic is about definability.

Definability v.s. computability

Computability implies definability.

Definability v.s. computability

Computability implies definability.

The reverse also holds, *in some sense*.

Some examples

Theorem (Gödel)

$A \subseteq \omega$ is Σ^0_1 if and only if it is computably enumerable.

Theorem (Spector, Gandy)

$A \subseteq \omega$ is Π^1_1 if and only if it is computably enumerable over $L_{\omega_1^{CK}}$.

Some examples

Theorem (Gödel)

$A \subseteq \omega$ is Σ^0_1 if and only if it is computably enumerable.

Theorem (Spector, Gandy)

$A \subseteq \omega$ is Π^1_1 if and only if it is computably enumerable over $L_{\omega_1^{CK}}$.

Can these results be lifted to higher levels?

Martin-Solovay's tree

Definition

Let u be the ω-th uniformly Silver-indiscernible. A Martin-Solovay's tree $T_2 \subseteq 2^{<\omega} \times u^{\omega}$ is a tree so that for any infinite path $(x, f) \in [T]$, x is a sharp of some real y and f is a witness of the sharpness of x.

Representing Π^1_3-set

Suppose that $A \subseteq \omega$ is a Π^1_3-set. Then there is a Σ^1_2-set $B \subseteq \omega \times 2^\omega$ so that $n \in A \leftrightarrow \forall y (n, y) \in B$.

Since B is a Σ^1_2-set, there is a truth-table functional Φ so that $(n, y) \notin B \leftrightarrow \Phi^{y^\sharp}(n) = 1$.

So $n \notin A$ if and only if $\exists x \exists f((x, f) \in [T_2] \wedge \Phi^x(n) = 1)$. In other words, $n \in A$ if and only if the tree $T_{2,n} = \{(\sigma, \tau) \mid (\sigma, \tau) \in T_2 \wedge \Phi^\sigma(n) = 1\}$ is well-founded.

Let $\omega_1^{T_2}$ be the least ordinal $\alpha > u$ so that $L_\alpha[T_2]$ is admissible. Then A is an r.e. set over $L_{\omega_1^{T_2}}[T_2]$.

Further results

Theorem (Zhu)

- $0^{\sharp,2n}$ *exists.*
- *There is a Martin-Solovay's tree* T_{2n}.
- $A \subseteq \omega$ *is* Π^1_{2n+1} *if and only if it is r.e. over* $L_{\omega_1^{T_{2n}}}[T_{2n}]$.

Higher degree determinacy

Given a kind of reduction $\leq_Q$, Q-degree determinacy says that every set of Q-degrees either contains an upper cone or avoids an upper cone. We also may restrict the sets to be nice.

For example, Δ^1_n-degrees, Q_{2n+1}-degrees.

Time v.s. Space

Classical recursion theorists think that time has the same scale as space.

Higher recursion theorists don't think so.

Friedman's conjecture

Conjecture (H.Friedman)

The Δ^1_1-equivalence closure of every uncountable Δ^1_1 set contains an upper cone of Δ^1_1-degrees.

Theorem (Martin)

The conjecture is true.

Why Q-theory?

Under PD, Π^1_{2n+1}-complete set is a minimal non-trivial Δ^1_{2n+1}-degree; and Gandy-basis theorem fails; and many other properties fail.

Harrington, Kechris, Martin, Solovay suggested Q_{2n+1}-theory to replace Δ^1_{2n+1}-theory.

Two open questions

Question (Kechris, Martin, Solovay)

Assume PD,

- *The Q_{2n+1}-equivalence closure of every uncountable Δ^1_{2n+1} set contains an upper cone of Δ^1_{2n+1}-degrees.*
- *Q_{2n+1} is the largest nontrivial Π^1_{2n+1}-set which are $\leq_{\Delta^1_{2n+1}}$-downward closed.*

A solution

Theorem (Y, Zhu)

Assume PD, both questions have positive answers.

A solution

Theorem (Y, Zhu)

Assume PD, both questions have positive answers.

Remark: Woodin also announced solutions to both questions (never written up).

Another question

The time-space trick sometimes really matters.

Question

Assume PD. Suppose that A and B are uncountable Σ^1_3-sets, then for any real z, are there reals $x^0 \in A$ and $x^1 \in B$ so that $x^0 \oplus x^1 \geq_{\Delta^1_3} z$?

Another question

The time-space trick sometimes really matters.

Question

Assume PD. Suppose that A and B are uncountable Σ^1_3-sets, then for any real z, are there reals $x^0 \in A$ and $x^1 \in B$ so that $x^0 \oplus x^1 \geq_{\Delta^1_3} z$?

Thanks